1. Work as many problems as you can. It is to your advantage to demonstrate a broad background.

2. If you feel there is a misprint or error in the statement of the problem, then interpret it in such a way that the problem is not trivial.
Group Theory

1. (a) Find the centralizer in $S_5$ of $\sigma = (1 \ 2)(3 \ 4 \ 5)$.
   (b) How many elements of order 6 are there in $S_5$?

2. Let $G$ be a group and let $Z(G)$ be the center of $G$. Prove or disprove the following.
   (a) If $G/Z(G)$ is cyclic, then $G$ is abelian.
   (b) If $G/Z(G)$ is abelian, then $G$ is abelian.
   (c) If $G$ is of order $p^2$, where $p$ is a prime, then $G$ is abelian.

3. Let $G$ be a group. Show that if $G$ has a proper subgroup of finite index, then $G$ has a proper normal subgroup of finite index.

4. Let $\text{Inn}(G)$ be the group of inner automorphisms of the group $G$ and let $\text{Aut}(G)$ be the full automorphism group.
   (a) Show that $\text{Inn}(G) \leq \text{Aut}(G)$.
   (b) Show that if $Z(G)$ is the center of $G$, then $\text{Inn}(G) \cong G/Z(G)$.

5. Let $H$ be a subgroup of $G$ and suppose there is a normal subgroup $N$ of $G$ satisfying $HN = G$ and $H \cap N = \langle 1 \rangle$. Prove that if two elements of $H$ are conjugate in $G$, then they are conjugate in $H$.

6. Show that a group of order $1960 = 2^3 \cdot 5 \cdot 7^2$ cannot be simple.
Ring Theory

1. (a) Show that \( x^4 + x^3 + x^2 + x + 1 \) is irreducible in \( \mathbb{Z}_3[x] \).
   
   (b) Show that \( x^4 + 1 \) is not irreducible in \( \mathbb{Z}_3[x] \).

2. Let \( R \) be the ring of all \( 2 \times 2 \) matrices of the form \[
\begin{pmatrix}
a & b \\
2b & a
\end{pmatrix},
\] where \( a, b \in \mathbb{Z} \). Prove that \( R \) is isomorphic to \( \mathbb{Z}[\sqrt{2}] \).

3. Let \( D \) be a principal ideal domain. Prove that every nonzero prime ideal of \( D \) is a maximal ideal.

4. Let \( S \) be the ring of all bounded, continuous functions \( f : \mathbb{R} \to \mathbb{R} \), where \( \mathbb{R} \) is the set of real numbers. Let \( I \) be the set of functions \( f \) in \( S \) such that \( f(t) \to 0 \) as \( |t| \to \infty \).
   
   (a) Show that \( I \) is an ideal of \( S \).
   
   (b) Suppose \( x \in S \) is such that there is an \( i \in I \) with \( ix = x \). Show that \( x(t) = 0 \) for all sufficiently large \( |t| \).

5. Let \( R \) be a commutative ring with identity such that not every ideal is a principal ideal.
   
   (a) Show that there is an ideal \( I \) maximal with respect to the property that \( I \) is not a principal ideal.
   
   (b) If \( I \) is the ideal of part (a), show that \( R/I \) is a principal ideal ring.

6. Let \( R \) be a subring of a field \( F \) such that for each \( x \) in \( F \) either \( x \in R \) or \( x^{-1} \in R \). Prove that if \( I \) and \( J \) are two ideals of \( R \), then either \( I \subseteq J \) or \( J \subseteq I \).
Field Theory

1. Let $F$ be a field extension of the rational numbers.
   (a) Show that \{ $a + b\sqrt{2} \mid a, b \in F$ \} is a field.
   (b) Give necessary and sufficient conditions for \{ $a + b\sqrt{2} \mid a, b \in F$ \} to be a field.

2. (a) Find the Galois group of $x^3 - 5$ over $\mathbb{Q}$ and demonstrate the Galois correspondence
    between the subgroups of the Galois group and the subfields of the splitting field.
    (b) Find all automorphisms of $\mathbb{Q}(\sqrt[3]{5})$. Is there an $f \in \mathbb{Q}[x]$ with splitting field $\mathbb{Q}(\sqrt[3]{5})$?
    Explain.

3. Let $\alpha$ be a complex primitive $43^{rd}$ root of 1. Prove that there is an extension field $F$
    of the rational numbers such that $[F(\alpha) : F] = 14$.

4. Let $p$ be a prime. Show that the field of $p^m$ elements is contained in the field of $p^n$
   elements if and only if $m|n$.

5. Let $K = F(u)$ be a separable extension of $F$ with $u^m \in F$ for some positive integer $m$.
   Show that if the characteristic of $F$ is $p$ and $m = p^t r$, then $u^r \in F$.

6. Let $F$ be a field and let $f(x) \in F[x]$ be an irreducible polynomial of degree 4 with
   distinct roots $\alpha_1, \alpha_2, \alpha_3,$ and $\alpha_4$. Let $K$ be a splitting field for $f$ over $F$ and assume
   $Gal(K/F) \cong S_4$. Find $Gal(K/F(\beta))$, where $\beta = \alpha_1 \alpha_2 + \alpha_3 \alpha_4$.

Linear Algebra

1. Let $V$ be a finite dimensional vector space over a field $F$ and let $T : V \rightarrow V$ be a
   nilpotent linear transformation. Show that the trace of $T$ is 0.

2. (a) Show that two $3 \times 3$ complex matrices are similar if and only if they have the same
    characteristic and minimal polynomials.
(b) Is the conclusion of part (a) true for larger matrices? Prove or give a counterexample.