QUALIFYING EXAM IN ALGEBRA
August 1998

1. There are 18 problems on the exam. Work and turn in 10 problems, in the following categories.

   I. Linear Algebra — 1 problem
   II. Group Theory — 3 problems
   III. Ring Theory — 2 problems
   IV. Field Theory — 3 problems
   Any of the four areas — 1 problem

2. Turn in only 10 problems. No credit will be given for extra problems. All problems are weighted equally.

3. Put each problem on a separate sheet of paper, and write only on one side. Put your name on each page.

4. If you feel there is a misprint or error in the statement of a problem, then interpret it in such a way that the problem is not trivial.
I. Linear Algebra

1. Let \( A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix} \).

Find the characteristic and minimal polynomials of \( A \) and determine the Jordan canonical form of \( A \).

2. Let \( V \) be a vector space over a field \( F \). A linear transformation \( T : V \to V \) is said to be \emph{idempotent} if \( T^2 = T \). Prove that if \( T \) is idempotent then \( V = V_0 \oplus V_1 \), where \( T(v_0) = 0 \) for all \( v_0 \in V_0 \) and \( T(v_1) = v_1 \) for all \( v_1 \in V_1 \).

3. Let \( V \) be a finite dimensional vector space and let \( W \) be a subspace. Show that \( \dim V/W = \dim V - \dim W \).

II. Group Theory

1. Show that if \( \sigma \in S_n \) is an \( (n-1) \)-cycle, where \( n \geq 3 \), then \( C_{S_n}(\sigma) = \langle \sigma \rangle \).

2. Let \( N \) be a normal subgroup of \( G \). Show that if \( N \cap G' = \langle 1 \rangle \), then \( N \) is contained in the center of \( G \).

3. Let \( G \) be a group acting on the set \( S \) and let \( H \) be a subgroup of \( G \) acting transitively on \( S \). Show that if \( t \in S \) then \( G = G_tH \), where \( G_t \) is the stabilizer of \( t \) in \( G \).

4. Show that a group of order \( 1998 = 2 \cdot 3^3 \cdot 37 \) must be solvable.

5. A subgroup \( H \) of a group \( G \) is subnormal if there exists a chain \( H = H_0 \leq H_1 \leq \cdots \leq H_k = G \) such that \( H_i \) is a normal subgroup of \( H_{i+1} \) for every \( i \). Prove that if \( P \) is a Sylow \( p \)-subgroup of a finite group \( G \) then \( P \) is a subnormal in \( G \) if and only if \( P \) is normal in \( G \).
III. Ring Theory

1. Let $R$ be a commutative ring with 1 and let $I$ and $J$ be ideals of $R$ such that $I + J = R$. Show that $R/(I \cap J) \cong R/I \oplus R/J$.

2. Let $R$ be a non-zero ring with identity. Show that every proper ideal of $R$ is contained in a maximal ideal.

3. Let $R$ be a non-zero commutative ring with 1. Show that if $I$ is an ideal of $R$ such that $1 + a$ is a unit in $R$ for all $a \in I$, then $I$ is contained in every maximal ideal of $R$.

4. Let $D$ be a unique factorization domain and $F$ its field of fractions. Prove that if $d$ is an irreducible element in $D$, then there is no $x \in F$ such that $x^2 = d$.

5. Let $R$ be an integral domain, $S$ a multiplicative set, and let $S^{-1}R = \{\frac{r}{s} \mid r \in R, s \in S\}$ (contained in the field of fractions of $R$). Show that if $P$ is a prime ideal of $R$ then, $S^{-1}P$ is either a prime ideal of $S^{-1}R$ or else equals $S^{-1}R$.

IV. Field Theory

1. Let $K$ be a finite degree extension of the field $F$ such that $[K : F]$ is relatively prime to 6. Show that if $u \in K$ then $F(u) = F(u^3)$.

2. Let $F$ be a field and $f(x) \in F[x]$ an irreducible polynomial. Prove that there is a prime $p$, an integer $a \geq 0$ and a separable polynomial $g(x) \in F[x]$ such that $f(x) = g(x^{p^a})$.

3. Show that the Galois group of $x^3 - 7$ over $\mathbb{Q}$ is $S_3$ and demonstrate the Galois correspondence between the subgroups of $S_3$ and the subfields of the splitting field. Which subfields are normal over $\mathbb{Q}$?

4. Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial of degree 4 with exactly 2 real roots. Show that the Galois group of $f$ is either $S_4$ or the dihedral group of order 8.

5. Let $\mathbb{F}_q$ be the field of $q$ elements and let $f(x)$ be a polynomial in $\mathbb{F}_q[x]$. Show that if $\alpha$ is a root of $f(x)$ in some extension of $\mathbb{F}_q$, then $\alpha^q$ is also a root of $f(x)$. 

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