QUALIFYING EXAM IN ALGEBRA
JANUARY 1992

1. Work as many problems as you can. It is to your advantage to demonstrate a broad
   background.

2. If you feel there is a misprint or error in the statement of the problem, then interpret
   it in such a way that the problem is not trivial.
1. (a) Find the centralizer in $S_7$ of $(1\ 2\ 3)(4\ 5\ 6\ 7)$.
   (b) How many elements of order 12 are there in $S_7$?

2. Let $f : G \rightarrow H$ be a homomorphism of groups with kernel $K$ and image $I$.
   (a) Show that if $N$ is a subgroup of $G$ then $f^{-1}(f(N)) = KN$.
   (b) Show that if $L$ is a subgroup of $H$ then $f(f^{-1}(L)) = I \cap L$.

3. Let $G$ be a finite group.
   (a) Show that every proper subgroup of $G$ is contained in a maximal subgroup.
   (b) Show that the intersection of all maximal subgroups of $G$ is a normal subgroup.

4. Let $N$ be a group with trivial center such that all automorphisms of $N$ are inner automorphisms. Show that whenever $N$ occurs as a normal subgroup of a group $G$, there is a subgroup $H$ of $G$ such that $G = H \times N$.

5. Let $G$ be a subgroup of the symmetric group $S_n$. Show that if $G$ contains an odd permutation then $G \cap A_n$ is of index 2 in $G$.

6. Show that a simple group of order 168 must be isomorphic to a subgroup of the alternating group $A_8$. 

Group Theory
Ring Theory

1. Let $p$ be a prime and let $F_p = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \mid a, b \in \mathbb{Z}_p \right\}$.

   (a) Show that $F_p$, with the usual matrix operations, is a commutative ring with identity.
   (b) Show that $F_7$ is a field.
   (c) Show that $F_{13}$ is not a field.

2. Let $p$ be a prime.

   (a) Show that if $p \equiv 1 \pmod{4}$, then $x^2 + 1$ is not irreducible in $\mathbb{Z}_p[x]$.
   (b) Show that if $p \equiv 3 \pmod{4}$, then $x^2 + 1$ is irreducible in $\mathbb{Z}_p[x]$.

3. Let $R$ be a commutative ring with 1 such that for every $x$ in $R$ there is an integer $n > 1$ (depending on $x$) such that $x^n = x$. Show that every prime ideal of $R$ is maximal.

4. Let $D = \mathbb{Z}(\sqrt{13}) = \{ m + n\sqrt{13} \mid m, n \in \mathbb{Z} \}$ and $F = \mathbb{Q}(\sqrt{13})$ its field of fractions.

   Show the following:
   (a) $x^2 + 3x - 1$ is irreducible in $D[x]$ but not in $F[x]$.
   (b) $D$ is not a unique factorization domain.

5. Let $R$ be a ring.

   (a) Show that there is a unique smallest (with respect to inclusion) ideal $A$ such that $R/A$ is a commutative ring.
   (b) Give an example of a ring $R$ such that for every proper ideal $I$, $R/I$ is not commutative. Verify your example.
   (c) For the ring $R = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a, b, c \in \mathbb{Z} \right\}$ with the usual matrix operations, find the ideal $A$ of part (a).

6. Let $R$ be a non-zero commutative ring with 1.

   (a) Let $S$ be a multiplicative subset of $R$ not containing 0 and let $P$ be maximal in the set of ideals of $R$ not intersecting $S$. Show that $P$ is a prime ideal.
   (b) Show that the set of nilpotent elements of $R$ is the intersection of all prime ideals.
Field Theory

1. Let \( K \) be an extension field of the field \( F \) such that \([K:F]\) is odd. Show that if \( u \in K \) then \( F(u) = F(u^2) \).

2. Let \( F \subset E \subset K \) be a tower of fields such that \( K = F(\alpha) \) with \( \alpha \) algebraic over \( F \). Prove that if \( f(x) \in F[x] \) is the minimal polynomial of \( \alpha \) over \( F \) and \( F \neq E \), then \( f(x) \) is not irreducible in \( E[x] \).

3. Let \( f(x) \in \mathbb{Q}[x] \) be an irreducible polynomial of degree \( n \) with roots \( \alpha_1, \ldots, \alpha_n \). Show that \( \sum_{i=1}^{n} \frac{1}{\alpha_i} \) is a rational number.

4. Let \( f(x) = x^4 + x^3 + 4x - 1 \in \mathbb{Z}_5[x] \). Find the Galois group of the splitting field of \( f \) over \( \mathbb{Z}_5 \).

5. Let \( \eta \) be a complex primitive 11-th root of unity and let \( K = \mathbb{Q}(\eta) \). Find \( \text{Gal}(K/\mathbb{Q}) \) and express each intermediate field \( F \) between \( K \) and \( \mathbb{Q} \) as \( F = \mathbb{Q}(\beta) \) for some \( \beta \in K \).

6. Let \( f(x) \in \mathbb{Q}[x] \) be an irreducible polynomial of degree 4 with exactly 2 real roots. Show that the Galois group of \( f \) is either \( S_4 \) or the dihedral group of order 8.

Linear Algebra

1. Let \( V \) and \( W \) be finite dimensional vector spaces and let \( T : V \rightarrow W \) be a linear transformation. Show that \( \dim(\ker T) + \dim(\text{Im } T) = \dim(V) \).

2. Let \( V \) be a finite dimensional vector space over the field \( F \). Let \( V^* \) be the dual space of \( V \) (that is, \( V^* \) is the vector space of linear transformations \( T : V \rightarrow F \)). Show that \( V \cong V^* \).