An upper bound for $B_2[g]$ sets

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Abstract

Suppose $g$ is a fixed positive integer. For $N \geq 2$, a set $A \subset \mathbb{Z} \cap [1, N]$ is called a $B_2[g]$ set if every integer $n$ has at most $g$ distinct representations as $n = a + b$ with $a, b \in A$ and $a \leq b$. In this paper, we give an upper bound estimate for the size of such $A$, improving the existing results.

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1. Introduction

For positive integers $g$ and $N$, a set $A \subset [N]$ (where $[N] := \{1, 2, \ldots, N\}$) is called a $B_2[g]$ set if every integer $n$ has at most $g$ distinct representations as $n = a + b$ with $a, b \in A$ and $a \leq b$. (Thus $B_2[1]$ sets are the classical Sidon sets, or $B_2$ sets.)

Let $F(g, N)$ be the largest cardinality of a $B_2[g]$ set contained in $[N]$. The study for the asymptotic behavior of $F(g, N)$ has attracted a lot of attentions. (See O’Bryant’s excellent survey paper [11] for the complete up-to-date references.)

From the works of Singer [13] and Erdős and Turán [7] (observed by Erdős [6]), we have known that

$$F(1, N) = \sqrt{N} + O(N^{\alpha} + N^{1/4}),$$

(1.1)

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where $\alpha$ is the smallest positive number such that, for every sufficiently large $x$, the interval $[x, x + x^{2\alpha}]$ contains a prime. The only issue remained in the study of $F(1, N)$ is the improvement of the error term. It is generally believed that the error term in (1.1) should be $O(N^\epsilon)$ (but not $O(1)$ though) for any $\epsilon > 0$.

For $g \geq 2$, there has been no asymptotic formula for $F(g, N)$ similar to (1.1) known to us. There is still a big gap between the best known lower and upper bounds for $F(g, N)$, even for $g = 2$. For the lower bound, there have been various constructions of $B_2[g]$ sets with large cardinality (cf., [3–5,9,10]). Each of these constructions gives a result $F(g, N) \geq (c + o(1))\sqrt{gN}$ for some constant $c > 1$.

There have also been a number of results concerning the upper bound for $F(g, N)$. To list a few, let

$$
\sigma(g) := \limsup_{N \to \infty} \frac{F(g, N)}{\sqrt{gN}}.
$$

Using the technique of Fourier analysis, Cilleruelo, Ruzsa and Trujillo [5] showed that $\sigma(g) \leq \sqrt{3.4745}$. Combining the idea of [5] with the consideration of the fourth moment of the Fourier transform of a $B_2[g]$ set, Green [8] showed that, among other things,

$$
\sigma(g) < \sqrt{3.4}.
$$

(1.2)

By a more careful analysis of the test function involved in Green’s study, Martin and O’Bryant improved Green’s result to $\sigma(g) \leq \sqrt{3.3819}$, which seems to be nearly the limit of what Green’s method can give.

It should be remarked that another result Green proved in [8], $\sigma(g) \leq \sqrt{3.5 - 1.75/g}$, has significantly improved the previous results for small $g$. In particular, it gives $\sigma(g) \leq \sqrt{2.625}$ for $g = 2$, the most interesting case that has been studied by many people (cf., [1,9,12]).

In this paper, we are interested in giving an upper bound for $\sigma(g)$. The result we shall prove gives an improvement for (1.2). More precisely, we shall prove

**Theorem 1.** We have

$$
\sigma(g) < \sqrt{3.2}.
$$

(1.3)

We can actually strengthen (1.3) a little bit, with the upper bound replaced by $\sqrt{3.2} - \kappa(g)$ for some positive $\kappa(g)$ which tends to 0 as $g \to \infty$. But we shall not do so since such a result does not yield any improvement for $\sigma(g)$ when $g$ is small.

The new ingredient involved in the proof of Theorem 1 is based on the following observation: for any set $B \subset [N]$, either the difference set $B - B$ or the shifted sum set $B + B - N$ has a large concentration around 0.

**Notation.** We shall frequently use the notion $A \gtrsim B$ (respectively $A \lesssim B$) to mean that $A \geq (1 + o(1))B$ (respectively $A \leq (1 + o(1))B$), and $A \sim B$ to mean that $A = (1 + o(1))B$. 


2. Two lemmas

For a set $\mathcal{B} \subset [N]$, we define the generating function of $\mathcal{B}$ as

$$f_{\mathcal{B}}(\beta) = \sum_{b \in \mathcal{B}} e(\beta b), \quad \text{where } e(t) = \exp(2\pi it).$$

We also define that, for any $n \in \mathbb{Z}$,

$$r_{\mathcal{B}}(n) := \# \{(a, b) \in \mathcal{B} \times \mathcal{B}: a + b = n\}$$

and

$$d_{\mathcal{B}}(n) := \# \{(a, b) \in \mathcal{B} \times \mathcal{B}: a - b = n\}.$$

With our notation, a $B_2[g]$ set $\mathcal{A}$ thus satisfies $r_{\mathcal{A}}(n) \leq 2g$ for any $n$. And our observation, as stated at the end of last section, essentially says that $d_{\mathcal{B}}(n) + r_{\mathcal{B}}(N + n)$ has an average value on short intervals around 0 larger than its overall average on $[-N, N]$. To precisely describe this phenomenon, we need introduce a weight function $w(x)$ into play.

Let $u(x) \in C^2[0, 1]$ be a real-valued function satisfying $u(x) \geq 0$ for all $x \in [0, 1]$ and

$$\int_0^1 u(x) \, dx = 1.$$  

For $x \in [-1, 1]$, let

$$w(x) := \int_{0}^{1-|x|} u(t)u(t+|x|) \, dt. \quad (2.1)$$

We see that $w(x)$, as an even function on $[-1, 1]$, is non-negative and twice differentiable on $[0, 1]$. Moreover, $w(\pm 1) = 0$ and $\int_{-1}^{1} w(x) \, dx = 1$.

**Lemma 2.** Suppose $w(x)$ is a function satisfying the given conditions. For $\mathcal{B} \subset [N]$, let

$$D(L, w, \mathcal{B}) = \sum_{0 \leq |m| \leq L} w(m/L)d_{\mathcal{B}}(m)$$

and

$$R(L, w, \mathcal{B}) = \sum_{0 \leq |m| \leq L} w(m/L)r_{\mathcal{B}}(N + m).$$

Then for any positive integer $L \leq N$, we have

$$D(L, w, \mathcal{B}) + R(L, w, \mathcal{B}) \geq \frac{2|\mathcal{B}|^2L}{N+L} + O(|\mathcal{B}|^2/L). \quad (2.2)$$
Proof. For \( m \in \mathbb{Z} \cap [-L, L] \), let

\[
\phi(m/L) := \frac{1}{L} \sum_{1 \leq k \leq L - |m|} u(k/L)u((k + |m|)/L).
\]

Since \( u(x) \) is differentiable on \([0, 1]\), we have

\[
w(m/L) = \phi(m/L) + O\left( L^{-1} \right). \tag{2.3}
\]

From this and the definitions of \( D(L, w, B) \) and \( R(L, w, B) \), we thus have

\[
D(L, w, B) + R(L, w, B) = \sum_{0 \leq |m| \leq L} \phi(m/L)(d_B(m) + r_B(N + m)) + O\left( |B|^2 / L \right). \tag{2.4}
\]

Let \( K = N + L \). Note that for \( |m| < L \),

\[
r_B(N + m) = \frac{1}{K} \sum_{h=0}^{K-1} \left| f_B \left( \frac{h}{K} \right) \right|^2 e\left( -\frac{(N + m)h}{K} \right) \tag{2.5}
\]

and

\[
d_B(m) = \frac{1}{K} \sum_{h=0}^{K-1} \left| f_B \left( \frac{h}{K} \right) \right|^2 e\left( -\frac{mh}{K} \right). \tag{2.6}
\]

From (2.4)–(2.6), and the fact that \( \phi(1) = 0 \), we get

\[
D(L, w, B) + R(L, w, B) = \frac{1}{K} \sum_{h=0}^{K-1} \widehat{\phi}_L\left( -\frac{h}{K} \right) \left( \left| f_B \left( \frac{h}{K} \right) \right|^2 + \left( f_B \left( \frac{h}{K} \right) \right)^2 e\left( -\frac{Nh}{K} \right) \right) + O\left( |B|^2 / L \right), \tag{2.7}
\]

where

\[
\widehat{\phi}_L(\beta) = \sum_{0 \leq |m| \leq L} \phi(m/L)e(\beta m).
\]

It is easy to check that, for any \( \beta \in \mathbb{R} \),

\[
\widehat{\phi}_L(\beta) = \frac{1}{L} \left| \sum_{1 \leq k \leq L} u(k/L)e(\beta k) \right|^2 \geq 0. \tag{2.8}
\]

Note that

\[
\Re\left( \left| f_B \left( \frac{h}{K} \right) \right|^2 + \left( f_B \left( \frac{h}{K} \right) \right)^2 e\left( -\frac{Nh}{K} \right) \right) \geq 0. \tag{2.9}
\]
(2.8) and (2.9) together imply that the real part of each term in the sum in (2.7) is non-negative. Since $D(L, w, B) + R(L, w, B)$ is real, we thus have

$$D(L, w, B) + R(L, w, B) \geq \frac{2|B|^2 \hat{\phi}_L(0)}{K} + O(|B|^2 / L).$$  \hspace{1cm} (2.10)

Recall (2.3), and that $w(x)$ is even and differentiable on $[0, 1]$, then we have

$$\hat{\phi}_L(0) = \int_{-L}^{L} w(t/L) dt + O(1) = L + O(1),$$

which, put into (2.10), proves the lemma. \Box

**Lemma 3.** Suppose $\epsilon \in (0, \frac{1}{2})$ is any fixed number. For any $B_{2g}$ set $A \subset \mathbb{N}$, we have

$$\sum_{1 \leq n \leq N^\epsilon} \left| f_A \left( \frac{n}{2N} \right) \right|^4 \lesssim (2g - 1)N|A|^2 - \frac{1}{2}|A|^4.$$  \hspace{1cm} (2.11)

**Proof.** A proof has essentially been included in [8, §8]. Note that the sum

$$S(A) := \frac{1}{2N} \sum_{n=-N}^{N-1} \left( \left| f_A \left( \frac{n}{2N} \right) \right|^2 - |A| \right)^2$$

represents the number of solutions of the equation

$$a - b = c - d, \quad a, b, c, d \in A, \quad a \neq b.$$  

For any integer $n \geq 1$, there are at most $2g$ pairs $(b, c)$ with $b, c \in A$ such that $b + c = n$. Thus the equation $a + d = b + c$ with $a, b, c, d \in A$ has at most $2g|A|^2$ solutions. This implies

$$S(A) \leq (2g - 1)|A|^2.$$  \hspace{1cm} (2.12)

We also note that

$$\left| f_A \left( \frac{n}{2N} \right) \right|^4 = \left( \left| f_A \left( \frac{n}{2N} \right) \right|^2 - |A| \right)^2 + O(|A|^3).$$

Hence,

$$\sum_{1 \leq n \leq N^\epsilon} \left| f_A \left( \frac{n}{2N} \right) \right|^4 \leq \sum_{1 \leq n \leq N^\epsilon} \left( \left| f_A \left( \frac{n}{2N} \right) \right|^2 - |A| \right)^2 + O(N^\epsilon |A|^3)$$

$$\leq \frac{1}{2} \left( 2N \cdot S(A) - (f_A(0))^4 \right) + O(N^\epsilon |A|^3).$$  \hspace{1cm} (2.13)

The lemma then follows from (2.12) and (2.13). \Box
3. The function \(u(x)\)

The weight function \(w(x)\) involved in Lemma 2 is defined by (2.1) with \(u(x) \in C^2[0, 1]\) satisfying the conditions given there. In the proof for Theorem 1 that we will give in the next section, the upper bound for \(\sigma(g)\) is determined by the second moment of \(w(x)\)

\[
M(w) := \int_{-1}^{1} w^2(x) dx. \tag{3.1}
\]

Roughly speaking, the smaller \(M(w)\) is, the better an upper bound for \(\sigma(g)\) follows.

**Lemma 4.** There is such a function \(w(x)\) satisfying

\[
M(w) < 0.5771. \tag{3.2}
\]

**Proof.** Let

\[
h(x) = 1 + 0.0000028 \exp(60(x - 1/2)^2) + 3.4(x - 1/2)^2
\]

and

\[
u(x) = \frac{h(x)}{\int_{0}^{1} h(x) dx}.
\]

Then it is clear that \(u(x)\) is non-negative and twice differentiable on \([0, 1]\), and \(\int_{0}^{1} u(x) dx = 1\). For the \(w(x)\) given by (2.1), it can be checked by Maple that

\[
M(w) \leq 0.57706725 \ldots < 0.5771,
\]

which gives the lemma. \(\Box\)

**Remarks.** (1) The search for such \(u(x)\) (and thus \(w(x)\)) is closely related to a study of Green in [8]. He was, however, interested in proving a lower bound for \(M(w)\) for which \(w(x)\) is essentially defined by (2.1) with \(u(x)\) being a general (continuous) function supported on \([0, 1]\).

(2) There are many choices for \(u(x)\) which yield satisfactory results. For example, a simple choice of \(u(x) = \frac{6}{11}(1 + 10(x - 1/2)^2)\) (which has \(M(w) = 0.599776\ldots\)) implies \(\sigma(g) \leq \sqrt{3.2207}\). A more complicated choice of \(u(x)\) normalized from \(1 + a(x - 1/2)^2 + b(x - 1/2)^{12}\) for some constants \(a, b\) gives \(M(w) \approx 0.58\). The \(u(x)\) in Lemma 4 certainly is not a convenient choice, it however breaks the bound \(\sqrt{3.2}\) for \(\sigma(g)\).

(3) Though we do not know how to find the optimal value of \(M(w)\) (which exists and is unique following Green [8]), we can tell that the bound \(M(w) < 0.5771\) given in Lemma 4 cannot be improved too much. It has essentially been shown in [8] that

\[
M(w) > \frac{4}{7} = 0.5714\ldots,
\]

for any \(u(x)\).
4. Proof of Theorem 1

Let \( \mathcal{A} \subset [N] \) be a \( B_2[g] \) set with \(|\mathcal{A}| \gg \sqrt{N} \). Let \( c = \frac{|\mathcal{A}|^2}{gN} \). (We thus want to show that \( c < 3.2 \) when \( N \) is large.) Let \( L = \delta N \), where \( \delta \in (0, 1) \) is a parameter (independent of \( N \)) to be chosen later. Suppose \( w(x) \) is the function defined by (2.1) with \( u(x) \) given in the proof of Lemma 4.

From Lemma 2, we have

\[
D(L, w, \mathcal{A}) + R(L, w, \mathcal{A}) \gtrsim \frac{2\delta cgN}{1 + \delta}.
\] (4.1)

We want to give an upper bound for \( D(L, w, \mathcal{A}) + R(L, w, \mathcal{A}) \) which, along with (4.1), yields (1.3).

Since \( w(x) \) is non-negative and differentiable, we have

\[
R(L, w, \mathcal{A}) \leq 2g \sum_{0 \leq |m| \leq L} w(m/L) = 2g \int_{-L}^{L} w(t/L) dt + O(1) \sim 2g\delta N.
\] (4.2)

Let \( W(x) \) be the function of period 2 which takes value \( w(x/\delta) \) on \([-\delta, \delta]\) and 0 on the rest of its period \([-1, 1]\). Then \( W(x) \) is an even (and differentiable) function on \([-1, 1]\), thus has Fourier expansion

\[
W(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x).
\]

This yields

\[
D(L, w, \mathcal{A}) = \frac{1}{2}a_0|\mathcal{A}|^2 + \sum_{n=1}^{\infty} a_n \left| f_{\mathcal{A}} \left( \frac{n}{2N} \right) \right|^2,
\] (4.3)

which can be seen by expanding the \( |f_{\mathcal{A}}(n/2N)|^2 \) on the right-hand side and changing the order of summation. According to the conditions \( w(x) \) satisfying, we have

\[
\frac{1}{2}a_0 = \frac{1}{2}\delta \quad \text{and} \quad a_n = O_{\delta}(n^{-2}) \quad \text{for } n \geq 1.
\]

Then

\[
\sum_{n > N^\epsilon} a_n \left| f_{\mathcal{A}} \left( \frac{n}{2N} \right) \right|^2 = O(N^{-\epsilon}|\mathcal{A}|^2) = o(|\mathcal{A}|^2),
\] (4.4)

and by Cauchy’s inequality

\[
\sum_{1 \leq n \leq N^\epsilon} a_n \left| f_{\mathcal{A}} \left( \frac{n}{2N} \right) \right|^2 \leq \left[ \sum_{1 \leq n \leq N^\epsilon} a_n^2 \right] \left[ \sum_{1 \leq n \leq N^\epsilon} \left| f_{\mathcal{A}} \left( \frac{n}{2N} \right) \right|^4 \right],
\] (4.5)
where from Parseval’s identity
\[
\sum_{1 \leq n \leq N^\varepsilon} a_n^2 \leq \sum_{n=1}^\infty a_n^2 = \delta M(w) - \frac{\delta^2}{2}.
\] (4.6)

Recall that, for our choice of \( w(x) \), we have \( M(w) < 0.5771 \). Thus, from (4.3)–(4.6), we have
\[
D(L, w, A) \lesssim \frac{\delta |A|^2}{2} + \sqrt{(0.5771 \delta - 0.5 \delta^2) \left( (2g - 1)N|A|^2 - \frac{|A|^4}{2} \right)}.
\] (4.7)

An optimal choice of \( \delta \) (depending on \( g \)) based on (4.1), (4.2) and (4.7) gives a result which is a little stronger than Theorem 1. Just to prove (1.3), we use a worse bound for \( D(L, w, A) \) by replacing the \( 2g - 1 \) by \( 2g \) in (4.7). Then, combining (4.1), (4.2) and (4.7) together, we end up with
\[
\frac{2\delta c}{1 + \delta} \leq 2\delta + \frac{\delta c}{2} + \sqrt{(0.5771 \delta - 0.5 \delta^2)(2c - 0.5c^2)}.
\] (4.8)

Taking \( \delta = 0.237 \), we get from (4.8) that
\[
c \leq 3.199992566 \ldots < 3.2,
\]
as required.

5. Further remarks

Previous works on the upper bound for \( \sigma(g) \) essentially rely on the irregular distribution of either the sumset \( A + A \) or the difference set \( A - A \). In this paper, we have combined them together and obtained an improvement for the previous results for large \( g \). In our study, it naturally arises the following question.

Question. For a set of integers \( B \subset [N] \), how dense can the sumset \( B + B \) (weighted by \( r_B(n) \)) be over a subinterval \( I \subset [1, 2N] \)?

Note that the average value of \( r_B(n) \) for integers \( n \in [2N] \) is asymptotically \( \frac{|B|^2}{2N} \). Thus, we have
\[
\text{avg} |(B + B) \cap I| \sim \frac{|I||B|^2}{2N}.
\] (5.1)

In view of the irregular distribution of \( B + B \) on \([1, 2N]\), it is not surprising that there are short intervals \( I \subset [1, 2N] \) with \( |(B + B) \cap I| \) larger than the right-hand side of (5.1). Actually, we have the following more precise conjecture which, besides its application to \( B_2[g] \) sets, is of independent interest.
Conjecture 1. Suppose $\epsilon \in (0, 1)$ is a fixed number. For any $B \subset [N]$ satisfying $|B| = o(N)$, there is a subinterval $I \subset [1, 2N]$ with $|I| = L \gg N^\epsilon$ such that

$$\sum_{n \in I \cap \mathbb{Z}} r_B(n) \geq (2 + o(1)) \frac{|B|^2 L}{2N}.$$  \hspace{1cm} (5.2)

It is generally believed that the best upper bound one can expect for $B_2[g]$ ($g \geq 2$) sets would be $\sigma(g) \leq \sqrt{2}$ which, by a simple counting argument, directly follows from Conjecture 1.

We remark that the constant 2 in this conjecture cannot be improved. It is easy to check that, when $B$ is uniformly distributed over $[1, N]$, $(2 + o(1)) \frac{|B|^2}{2N}$ is the maximal local density of $B + B$, attained over short intervals $I$ around $N$. By “uniformly distributed,” here we mean that, for any $I \subset [1, N]$ with $|I| \gg N^{1+\epsilon} |B|^{-1}$, we have $|B \cap I| \sim \frac{|I| |B|}{N}$.

Martin and O’Bryant [10] conjectured that any $B_2[g]$ set $A \subset [N]$ with maximal size should be uniformly distributed over $[1, N]$. \(^1\) This then obviously yields the expected upper bound $\sigma(g) \leq \sqrt{2}$ for $g \geq 2$. Their conjecture, however, has only been proved for $g = 1$ (see Cilleruelo [2]).

While we do not know whether Conjecture 1 is true in general, it has actually been proved indirectly in [8] that, for any $B \subset [N]$, (5.2) holds with the 2 replaced by $\frac{8}{7}$. Also our proof of Theorem 1 is essentially consisting of two parts, in accordance with whether (5.2) holds with the constant 2 replaced by a number around $\frac{5}{4}$.

We remark that the expected estimate $\sigma(g) \leq \sqrt{2}$ also follows from the following weak version of Conjecture 1.

Conjecture 2. For a polynomial $f(x)$ with coefficients $\in \{0, 1\}$ and $f(1) = o(\deg(f))$, the polynomial $f^2(x)$ has a coefficient $\gg \frac{f^2(1)}{\deg(f)}$.

In estimating the upper bound for $\sigma(g)$, while it is not clear how far one can go with the techniques currently involved in the studies, it seems to us that any significant improvement may require non-trivial information about the distribution of the $B_2[g]$ set itself on $[1, N]$.

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References


\(^1\) It should be remarked that the more dense $B_2[g]$ sequences known to us are far to be well distributed.