Boyce/DiPrima 9th ed, Ch1.3: Classification of Differential Equations

The main purpose of this course is to discuss properties of solutions of differential equations, and to present methods of finding solutions or approximating them.

To provide a framework for this discussion, in this section we give several ways of classifying differential equations.
Ordinary Differential Equations

When the unknown function depends on a single independent variable, only ordinary derivatives appear in the equation.

In this case the equation is said to be an ordinary differential equations (ODE).

The equations discussed in the preceding two sections are ordinary differential equations. For example,

$$\frac{dv}{dt} = 9.8 - 0.2v, \quad \frac{dp}{dt} = 0.5p - 450$$
Partial Differential Equations

When the unknown function depends on several independent variables, partial derivatives appear in the equation.

In this case the equation is said to be a partial differential equation (PDE).

Examples:

\[ \alpha^2 \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t} \]  (heat equation)

\[ \alpha^2 \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2} \]  (wave equation)
Systems of Differential Equations

Another classification of differential equations depends on the number of unknown functions that are involved.

If there is a single unknown function to be found, then one equation is sufficient. If there are two or more unknown functions, then a system of equations is required.

For example, predator-prey equations have the form

\[
\frac{du}{dt} = au - \alpha uv \\
\frac{dv}{dt} = -cv + \gamma uv
\]

where \( u(t) \) and \( v(t) \) are the respective populations of prey and predator species. The constants \( a, c, \alpha, \gamma \) depend on the particular species being studied.

Systems of equations are discussed in Chapter 7.
Order of Differential Equations

The order of a differential equation is the order of the highest derivative that appears in the equation.

Examples:

\[ y' + 3y = 0 \]
\[ y'' + 3y' - 2t = 0 \]
\[ \frac{d^4 y}{dt^4} - \frac{d^2 y}{dt^2} + 1 = e^{2t} \]
\[ u_{xx} + u_{yy} = \sin t \]

We will be studying differential equations for which the highest derivative can be isolated:

\[ y^{(n)}(t) = f\left(t, y, y', y'', y''', \ldots, y^{(n-1)}\right) \]
Linear & Nonlinear Differential Equations

An ordinary differential equation
\[ F(t, y, y', y'', y''', \ldots, y^{(n)}) = 0 \]
is linear if \( F \) is linear in the variables
\[ y, y', y'', y''', \ldots, y^{(n)} \]

Thus the general linear ODE has the form
\[ a_0(t) y^{(n)} + a_1(t) y^{(n-1)} + \cdots + a_n(t) y = g(t) \]

Example: Determine whether the equations below are linear or nonlinear.

1. \( y' + 3y = 0 \)
2. \( y'' + 3e^y y' - 2t = 0 \)
3. \( y'' + 3y' - 2t^2 = 0 \)
4. \( \frac{d^4 y}{dt^4} - t \frac{d^2 y}{dt^2} + 1 = t^2 \)
5. \( u_{xx} + uu_{yy} = \sin t \)
6. \( u_{xx} + \sin(u)u_{yy} = \cos t \)
Solutions to Differential Equations

- A solution $\phi(t)$ to an ordinary differential equation
  \[ y^{(n)}(t) = f(t, y, y', y'', \ldots, y^{(n-1)}) \]

- satisfies the equation:
  \[ \phi^{(n)}(t) = f(t, \phi, \phi', \phi'', \ldots, \phi^{(n-1)}) \]

- Example: Verify the following solutions of the ODE
  \[ y'' + y = 0; \quad y_1(t) = \sin t, \quad y_2(t) = -\cos t, \quad y_3(t) = 2\sin t \]
Solutions to Differential Equations

Three important questions in the study of differential equations:

- Is there a solution? (Existence)
- If there is a solution, is it unique? (Uniqueness)
- If there is a solution, how do we find it?
  (Analytical Solution, Numerical Approximation, etc)