A linear first order ODE has the general form
\[
\frac{dy}{dt} = f(t, y)
\]

where \( f \) is linear in \( y \). Examples include equations with constant coefficients, such as those in Chapter 1,
\[
y' = -ay + b
\]
or equations with variable coefficients:
\[
\frac{dy}{dt} + p(t)y = g(t)
\]
Constant Coefficient Case

For a first order linear equation with constant coefficients,

\[ y' = -ay + b, \]

recall that we can use methods of calculus to solve:

\[
\frac{dy}{dt} = -a \\
\int \frac{dy}{y-b/a} = -\int a \, dt \\
\ln |y - b/a| = -at + C \\
y = b/a + ke^{at}, \quad k = \pm e^C
\]
Variable Coefficient Case: Method of Integrating Factors

We next consider linear first order ODEs with variable coefficients:

\[
\frac{dy}{dt} + p(t) y = g(t)
\]

The method of integrating factors involves multiplying this equation by a function \( \mu(t) \), chosen so that the resulting equation is easily integrated.
Example 1: Integrating Factor  (1 of 2)

Consider the following equation:
\[ y' + \frac{1}{2} y = \frac{1}{2} e^{t/3} \]

Multiplying both sides by \( \mu(t) \), we obtain
\[ \mu(t) \frac{dy}{dt} + \frac{1}{2} \mu(t) y = \frac{1}{2} \frac{e^{t/3}}{t} \mu(t) \]

We will choose \( \mu(t) \) so that left side is derivative of known quantity. Consider the following, and recall product rule:
\[ \frac{d}{dt} [\mu(t) y] = \mu(t) \frac{dy}{dt} + \frac{d\mu(t)}{dt} y \]

Choose \( \mu(t) \) so that
\[ \mu'(t) = \frac{1}{2} \mu(t) \quad \Rightarrow \quad \mu(t) = e^{t/2} \]
Example 1: General Solution

With \( \mu(t) = e^{t/2} \), we solve the original equation as follows:

\[
y' + \frac{1}{2} y = \frac{1}{2} e^{t/3}
\]

\[
e^{t/2} \frac{dy}{dt} + \frac{1}{2} e^{t/2} y = \frac{1}{2} e^{5t/6}
\]

\[
\frac{d}{dt} \left[ e^{t/2} y \right] = \frac{1}{2} e^{5t/6}
\]

\[
e^{t/2} y = \frac{3}{5} e^{5t/6} + C
\]

general solution:

\[
y = \frac{3}{5} e^{t/3} + Ce^{-t/2}
\]
Method of Integrating Factors: Variable Right Side

In general, for variable right side \( g(t) \), the solution can be found as follows:

\[
y' + ay = g(t)
\]

\[
\mu(t) \frac{dy}{dt} + a\mu(t) y = \mu(t) g(t)
\]

\[
e^{at} \frac{dy}{dt} + ae^{at} y = e^{at} g(t)
\]

\[
\frac{d}{dt} [e^{at} y] = e^{at} g(t)
\]

\[
e^{at} y = \int e^{at} g(t) dt
\]

\[
y = e^{-at} \int e^{at} g(t) dt + Ce^{-at}
\]
Example 2: General Solution  (1 of 2)

We can solve the following equation

\[ y' - 2y = 4 - t \]

using the formula derived on the previous slide:

\[ y = e^{-at} \int e^{at} g(t) \, dt + Ce^{-at} = e^{2t} \int e^{-2t} (4 - t) \, dt + Ce^{2t} \]

Integrating by parts, \( \int e^{-2t} (4 - t) \, dt = \int 4e^{-2t} \, dt - \int te^{-2t} \, dt \)

\[ = -2e^{t/5} - \left[ -\frac{1}{2}te^{-2t} + \int \frac{1}{2}e^{-2t} \, dt \right] \]

\[ = -\frac{7}{4}e^{-2t} + \frac{1}{2}te^{-2t} \]

Thus

\[ y = e^{2t} \left( -\frac{7}{4}e^{-2t} + \frac{1}{2}te^{-2t} \right) + Ce^{2t} = -\frac{7}{4} + \frac{1}{2}t + Ce^{2t} \]
Example 2: Graphs of Solutions  (2 of 2)

The graph shows the direction field along with several integral curves. If we set $C = 0$, the exponential term drops out and you should notice how the solution in that case, through the point $(0, -7/4)$, separates the solutions into those that grow exponentially in the positive direction from those that grow exponentially in the negative direction.

\[ y = \frac{7}{4} + \frac{1}{2} t + Ce^{2t} \]
Method of Integrating Factors for General First Order Linear Equation

Next, we consider the general first order linear equation

\[ y' + p(t) y = g(t) \]

Multiplying both sides by \( \mu(t) \), we obtain

\[ \mu(t) \frac{dy}{dt} + p(t) \mu(t) y = g(t) \mu(t) \]

Next, we want \( \mu(t) \) such that \( \mu'(t) = p(t) \mu(t) \), from which it will follow that

\[ \frac{d}{dt} [\mu(t) y] = \mu(t) \frac{dy}{dt} + p(t) \mu(t) y \]
Integrating Factor for General First Order Linear Equation

Thus we want to choose \( \mu(t) \) such that \( \mu'(t) = p(t)\mu(t) \).

Assuming \( \mu(t) > 0 \), it follows that

\[
\int \frac{d\mu(t)}{\mu(t)} = \int p(t)dt \quad \Rightarrow \quad \ln \mu(t) = \int p(t)dt + k
\]

Choosing \( k = 0 \), we then have

\[
\mu(t) = e^{\int p(t)dt},
\]

and note \( \mu(t) > 0 \) as desired.
Solution for General First Order Linear Equation

Thus we have the following:

\[ y' + p(t)y = g(t) \]

\[ \mu(t) \frac{dy}{dt} + p(t)\mu(t)y = \mu(t)g(t), \quad \text{where} \quad \mu(t) = e^{\int p(t)dt} \]

Then

\[ \frac{d}{dt} [\mu(t)y] = \mu(t)g(t) \]

\[ \mu(t)y = \int \mu(t)g(t)dt + c \]

\[ y = \frac{\int \mu(t)g(t)dt + c}{\mu(t)}, \quad \text{where} \quad \mu(t) = e^{\int p(t)dt} \]
Example 3: General Solution  

To solve the initial value problem
\[ ty' + 2y = 4t^2, \quad y(1) = 2, \]
first put into standard form:
\[ y' + \frac{2}{t}y = 4t, \quad \text{for } t \neq 0 \]

Then
\[ \mu(t) = e^{\int p(t)dt} = e^{\int \frac{2}{t}dt} = e^{2\ln|t|} = e^{\ln|t^2|} = t^2 \]

and hence
\[ y = \frac{\int \mu(t)g(t)dt + C}{\mu(t)} = \frac{\int t^2(4t)dt + C}{t^2} = \frac{1}{t^2} \left[ \int 4t^3 dt + C \right] = t^2 + \frac{C}{t^2} \]
\[ ty' + 2y = 4t^2, \quad y(1) = 2, \]

**Example 3: Particular Solution**  (2 of 2)

- Using the initial condition \( y(1) = 2 \) and general solution
  \[ y = t^2 + \frac{C}{t^2}, \quad y(1) = 1 + C = 2 \implies C = 1 \]

  it follows that \[ y = t^2 + \frac{1}{t^2} \]

- The graphs below show solution curves for the differential equation, including a particular solution whose graph contains the initial point \((1,2)\). Notice that when \( C = 0 \), we get the parabolic solution \( y = t^2 \) (shown) and that solution separates the solutions into those that are asymptotic to the positive versus negative y-axis.
Example 4: A Solution in Integral Form (1 of 2)

To solve the initial value problem

\[ 2y' + ty = 2, \quad y(0) = 1, \]

first put into standard form:

\[ y' + \frac{t}{2} y = 1 \]

Then

\[ \mu(t) = e^{\int p(t) \, dt} = e^{\frac{t}{2}} = e^{\frac{t^2}{4}} \]

and hence

\[ y = e^{-t^2/4} \left( \int_0^t e^{s^2/4} \, ds + C \right) = e^{-t^2/4} \left( \int_0^t e^{s^2/4} \, ds \right) + Ce^{-t^2/4} \]
Example 4: A Solution in Integral Form (2 of 2)

Notice that this solution must be left in the form of an integral, since there is no closed form for the integral.

\[ y = e^{-t^2/4} \left( \int_0^t e^{s^2/4} \, ds \right) + Ce^{-t^2/4} \]

Using software such as Mathematica or Maple, we can approximate the solution for the given initial conditions as well as for other initial conditions.

Several solution curves are shown.