Recall our 2\textsuperscript{nd} order linear homogeneous ODE
\[ ay'' + by' + cy = 0 \]
where \( a, b \) and \( c \) are constants.
Assuming an exponential soln leads to characteristic equation:
\[ y(t) = e^{rt} \implies ar^2 + br + c = 0 \]
Quadratic formula (or factoring) yields two solutions, \( r_1 \) & \( r_2 \):
\[ r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]
When \( b^2 - 4ac = 0 \), \( r_1 = r_2 = -b/2a \), since method only gives one solution:
\[ y_1(t) = ce^{-bt/2a} \]
Second Solution: Multiplying Factor $v(t)$

We know that

$$y_1(t) \text{ a solution } \Rightarrow y_2(t) = cy_1(t) \text{ a solution}$$

Since $y_1$ and $y_2$ are linearly dependent, we generalize this approach and multiply by a function $v$, and determine conditions for which $y_2$ is a solution:

$$y_1(t) = e^{-bt/2a} \text{ a solution } \Rightarrow \text{ try } y_2(t) = v(t)e^{-bt/2a}$$

Then

$$y_2(t) = v(t)e^{-bt/2a}$$

$$y'_2(t) = v'(t)e^{-bt/2a} - \frac{b}{2a}v(t)e^{-bt/2a}$$

$$y''_2(t) = v''(t)e^{-bt/2a} - \frac{b}{2a}v'(t)e^{-bt/2a} - \frac{b}{2a}v'(t)e^{-bt/2a} + \frac{b^2}{4a^2}v(t)e^{-bt/2a}$$
Finding Multiplying Factor \( v(t) \)

Substituting derivatives into ODE, we seek a formula for \( v \):

\[
e^{-bt/2a} \left\{ a \left[ v''(t) - \frac{b}{a} v'(t) + \frac{b^2}{4a^2} v(t) \right] + b \left[ v'(t) - \frac{b}{2a} v(t) \right] + cv(t) \right\} = 0
\]

\[
av''(t) - bv'(t) + \frac{b^2}{4a} v(t) + bv'(t) - \frac{b^2}{2a} v(t) + cv(t) = 0
\]

\[
av''(t) + \left( \frac{b^2}{4a} - \frac{b^2}{2a} + c \right) v(t) = 0
\]

\[
\begin{align*}
\frac{b^2}{4a} - \frac{2b^2}{4a} + \frac{4ac}{4a} & v(t) = 0 \\
\frac{b^2}{4a} - \frac{4ac}{4a} & v(t) = 0
\end{align*}
\]

\[
v''(t) = 0 \implies v(t) = k_3 t + k_4
\]
General Solution

To find our general solution, we have:

\[ y(t) = k_1 e^{-bt/2a} + k_2 v(t) e^{-bt/2a} \]

\[ = k_1 e^{-bt/2a} + (k_3 t + k_4) e^{-bt/2a} \]

\[ = c_1 e^{-bt/2a} + c_2 t e^{-bt/2a} \]

Thus the general solution for repeated roots is

\[ y(t) = c_1 e^{-bt/2a} + c_2 t e^{-bt/2a} \]
Wronskian

The general solution is

\[ y(t) = c_1 e^{-bt/2a} + c_2 te^{-bt/2a} \]

Thus every solution is a linear combination of

\[ y_1(t) = e^{-bt/2a}, \quad y_2(t) = te^{-bt/2a} \]

The Wronskian of the two solutions is

\[
W(y_1, y_2)(t) = \begin{vmatrix}
    e^{-bt/2a} & te^{-bt/2a} \\
    -\frac{b}{2a} e^{-bt/2a} & \left(1 - \frac{bt}{2a}\right)e^{-bt/2a}
\end{vmatrix}
\]

\[
= e^{-bt/a} \left(1 - \frac{bt}{2a}\right) + e^{-bt/a} \left(\frac{bt}{2a}\right)
\]

\[
= e^{-bt/a} \neq 0 \quad \text{for all } t
\]

Thus \( y_1 \) and \( y_2 \) form a fundamental solution set for equation.
Example 1  (1 of 2)

Consider the initial value problem
\[ y'' + 4y' + 4y = 0 \]

Assuming exponential soln leads to characteristic equation:
\[ y(t) = e^{rt} \quad \Rightarrow \quad r^2 + 4r + 4 = 0 \quad \Leftrightarrow \quad (r + 2)^2 = 0 \quad \Leftrightarrow \quad r = -2 \]

So one solution is \( y_1(t) = e^{-2t} \) and a second solution is found:
\[ y_2(t) = v(t)e^{-2t} \]
\[ y'_2(t) = v'(t)e^{-2t} - 2v(t)e^{-2t} \]
\[ y''_2(t) = v''(t)e^{-2t} - 4v'(t)e^{-2t} + 4v(t)e^{-2t} \]

Substituting these into the differential equation and simplifying yields \( v''(t) = 0 \), \( v'(t) = k_1 \), \( v(t) = k_1t + k_2 \)
where \( c_1 \) and \( c_2 \) are arbitrary constants.
Example 1 (2 of 2)

- Letting \( k_1 = 1 \) and \( k_2 = 0 \), \( v(t) = t \) and \( y_2(t) = t e^{-2t} \)
- So the general solution is
  \[
y(t) = c_1 e^{-2t} + c_2 t e^{-2t}
  \]
- Note that both \( y_1 \) and \( y_2 \) tend to 0 as \( t \to \infty \) regardless of the values of \( c_1 \) and \( c_2 \)
- Using initial conditions
  \[
y(0) = 1 \quad \text{and} \quad y'(0) = 3
  \]
  \[
  c_1 = 1 \quad \text{and} \quad -2c_1 + c_2 = 3 \\
  \Rightarrow \ c_1 = 1, \ c_2 = 5
  \]
- Therefore the solution to the IVP is \( y(t) = e^{-2t} + 5te^{-2t} \)
Example 2 (1 of 2)

Consider the initial value problem
\[ y'' - y' + 0.25y = 0, \quad y(0) = 2, \quad y'(0) = 1/3 \]

Assuming exponential soln leads to characteristic equation:
\[ y(t) = e^{rt} \implies r^2 - r + 0.25 = 0 \iff (r - 1/2)^2 = 0 \iff r = 1/2 \]

Thus the general solution is
\[ y(t) = c_1 e^{t/2} + c_2 te^{t/2} \]

Using the initial conditions:
\[
\begin{align*}
\frac{c_1}{2} + c_2 &= \frac{1}{3} \\
\frac{c_1}{2} + c_2 &= \frac{1}{3} \implies c_1 = 2, \quad c_2 = -\frac{2}{3}
\end{align*}
\]

Thus
\[ y(t) = 2e^{t/2} - \frac{2}{3}te^{t/2} \]
Example 2  (2 of 2)

- Suppose that the initial slope in the previous problem was increased
  \[ y(0) = 2, \quad y'(0) = 2 \]

- The solution of this modified problem is
  \[ y(t) = 2e^{t/2} + te^{t/2} \]

- Notice that the coefficient of the second term is now positive. This makes a big difference in the graph, since the exponential function is raised to a positive power: \( \lambda = 1/2 \quad > 0 \)
Reduction of Order

- The method used so far in this section also works for equations with nonconstant coefficients:
  \[y'' + p(t)y' + q(t)y = 0\]
- That is, given that \(y_1\) is solution, try \(y_2 = v(t)y_1\):
  \[y_2(t) = v(t)y_1(t)\]
  \[y'_2(t) = v'(t)y_1(t) + v(t)y'_1(t)\]
  \[y''_2(t) = v''(t)y_1(t) + 2v'(t)y'_1(t) + v(t)y''_1(t)\]
- Substituting these into ODE and collecting terms,
  \[y_1v'' + (2y'_1 + py_1)v' + (y''_1 + py'_1 + qy_1)v = 0\]
- Since \(y_1\) is a solution to the differential equation, this last equation reduces to a first order equation in \(v'\):
  \[y_1v'' + (2y'_1 + py_1)v' = 0\]
Example 3: Reduction of Order  (1 of 3)

Given the variable coefficient equation and solution \( y_1 \),
\[
2t^2 y'' + 3ty' - y = 0, \quad t > 0; \quad y_1(t) = t^{-1},
\]
use reduction of order method to find a second solution:
\[
y_2(t) = v(t) t^{-1}
\]
\[
y_2'(t) = v'(t) t^{-1} - v(t) t^{-2}
\]
\[
y_2''(t) = v''(t) t^{-1} - 2v'(t) t^{-2} + 2v(t) t^{-3}
\]
Substituting these into the ODE and collecting terms,
\[
2t^2 \left( v'' t^{-1} - 2v' t^{-2} + 2vt^{-3} \right) + 3t \left( v' t^{-1} - vt^{-2} \right) - vt^{-1} = 0
\]
\[
\iff 2v'' t - 4v' + 4vt^{-1} + 3v' - 3vt^{-1} - vt^{-1} = 0
\]
\[
\iff 2tv'' - v' = 0
\]
\[
\iff 2tu' - u = 0, \quad \text{where } u(t) = v'(t)
\]
Example 3: Finding $v(t)$ \hspace{1cm} (2 of 3)

To solve
\[ 2tu' - u = 0, \quad u(t) = v'(t) \]
for $u$, we can use the separation of variables method:

\[ 2t \frac{du}{dt} - u = 0 \Leftrightarrow \int \frac{du}{u} = \int \frac{1}{2t} dt \Leftrightarrow \ln|u| = \frac{1}{2} \ln|t| + C \]

\[ \Leftrightarrow \quad |u| = |t|^{1/2} e^C \Leftrightarrow \quad u = ct^{1/2}, \quad \text{since } t > 0. \]

Thus
\[ v' = ct^{1/2} \]
and hence
\[ v(t) = \frac{2}{3} \, ct^{3/2} + k \]
Example 3: General Solution

Since
\[ v(t) = \frac{2}{3} \, c \, t^{\frac{3}{2}} + k \]
\[ y_2(t) = \left( \frac{2}{3} \, c \, t^{\frac{3}{2}} + k \right) t^{-1} = \frac{2}{3} \, c \, t^{\frac{1}{2}} + k \, t^{-1} \]

Recall
\[ y_1(t) = t^{-1} \]

So we can neglect the second term of \( y_2 \) to obtain
\[ y_2(t) = t^{\frac{1}{2}} \]

The Wronskian of \( y_1(t) \) and \( y_2(t) \) can be computed
\[ W(y_1, y_2)(t) = \frac{3}{2} \, t^{-\frac{3}{2}} \neq 0, \quad t > 0 \]

Hence the general solution to the differential equation is
\[ y(t) = c_1 t^{-1} + c_2 t^{\frac{1}{2}} \]