Recall the nonhomogeneous equation
\[ y'' + p(t)y' + q(t)y = g(t) \]
where \( p, q, g \) are continuous functions on an open interval \( I \).

The associated homogeneous equation is
\[ y'' + p(t)y' + q(t)y = 0 \]

In this section we will learn the **variation of parameters** method to solve the nonhomogeneous equation. As with the method of undetermined coefficients, this procedure relies on knowing solutions to the homogeneous equation.

Variation of parameters is a general method, and requires no detailed assumptions about solution form. However, certain integrals need to be evaluated, and this can present difficulties.
Example 1: Variation of Parameters  (1 of 6)

We seek a particular solution to the equation below.
\[ y'' + 4y = 3 \csc t \]

We cannot use the undetermined coefficients method since \( g(t) \) is a quotient of \( \sin t \) or \( \cos t \), instead of a sum or product.

Recall that the solution to the homogeneous equation is
\[ y_c(t) = c_1 \cos 2t + c_2 \sin 2t \]

To find a particular solution to the nonhomogeneous equation, we begin with the form
\[ y(t) = u_1(t) \cos 2t + u_2(t) \sin 2t \]

Then
\[ y'(t) = u_1'(t) \cos 2t - 2u_1(t) \sin 2t + u_2'(t) \sin 2t + 2u_2(t) \cos 2t \]

or
\[ y'(t) = -2u_1(t) \sin 2t + 2u_2(t) \cos 2t + u_1'(t) \cos 2t + u_2'(t) \sin 2t \]
Example: Derivatives, 2\textsuperscript{nd} Equation (2 of 6)

From the previous slide,
\[ y'(t) = -2u_1(t) \sin 2t + 2u_2(t) \cos 2t + u_1'(t) \cos 2t + u_2'(t) \sin 2t \]

Note that we need two equations to solve for \( u_1 \) and \( u_2 \). The first equation is the differential equation. To get a second equation, we will require
\[ u_1'(t) \cos 2t + u_2'(t) \sin 2t = 0 \]

Then
\[ y'(t) = -2u_1(t) \sin 2t + 2u_2(t) \cos 2t \]

Next,
\[ y''(t) = -2u_1'(t) \sin 2t - 4u_1(t) \cos 2t + 2u_2'(t) \cos 2t - 4u_2(t) \sin 2t \]
Example: Two Equations  (3 of 6)

Recall that our differential equation is
\[ y'' + 4y = 3\csc t \]

Substituting \( y'' \) and \( y \) into this equation, we obtain
\[
-2u_1'(t)\sin 2t - 4u_1(t)\cos 2t + 2u_2'(t)\cos 2t - 4u_2(t)\sin 2t \\
+ 4(u_1(t)\cos 2t + u_2(t)\sin 2t) = 3\csc t
\]

This equation simplifies to
\[
-2u_1'(t)\sin 2t + 2u_2'(t)\cos 2t = 3\csc t
\]

Thus, to solve for \( u_1 \) and \( u_2 \), we have the two equations:
\[
-2u_1'(t)\sin 2t + 2u_2'(t)\cos 2t = 3\csc t \\
u_1'(t)\cos 2t + u_2'(t)\sin 2t = 0
\]
Example: Solve for $u_1'$ (4 of 6)

To find $u_1$ and $u_2$, we first need to solve for $u_1'$ and $u_2'$

$$-2u_1'(t) \sin 2t + 2u_2'(t) \cos 2t = 3 \csc t$$

$$u_1'(t) \cos 2t + u_2'(t) \sin 2t = 0$$

From second equation,

$$u_2'(t) = -u_1'(t) \frac{\cos 2t}{\sin 2t}$$

Substituting this into the first equation,

$$-2u_1'(t) \sin 2t + 2\left[-u_1'(t) \frac{\cos 2t}{\sin 2t}\right] \cos 2t = 3 \csc t$$

$$-2u_1'(t) \sin^2(2t) - 2u_1'(t) \cos^2(2t) = 3 \csc t \sin 2t$$

$$-2u_1'(t)\left[\sin^2(2t) + \cos^2(2t)\right] = 3 \left[\frac{2 \sin t \cos t}{\sin t}\right]$$

$$u_1'(t) = -3 \cos t$$
Example: Solve for $u_1$ and $u_2$  \((5\text{ of }6)\)

From the previous slide,

$$u_1'(t) = -3\cos t, \quad u_2'(t) = -u_1'(t) \frac{\cos 2t}{\sin 2t}$$

Then

$$u_2'(t) = 3\cos t \left[ \frac{\cos 2t}{\sin 2t} \right] = 3\cos t \left[ \frac{1 - 2\sin^2 t}{2\sin t \cos t} \right] = 3\left[ \frac{1 - 2\sin^2 t}{2\sin t} \right]$$

$$= 3\left[ \frac{1}{2\sin t} - \frac{2\sin^2 t}{2\sin t} \right] = \frac{3}{2} \csc t - 3\sin t$$

Thus

$$u_1(t) = \int u_1'(t) \, dt = \int -3\cos tdtdt = -3\sin t + c_1$$

$$u_2(t) = \int u_2'(t) \, dt = \int \left( \frac{3}{2} \csc t - 3\sin t \right) \, dt = \frac{3}{2} \ln |\csc t - \cot t| + 3\cos t + c_2$$
Example: General Solution  (6 of 6)

Recall our equation and homogeneous solution $y_C$:

$$y'' + 4y = 3 \csc t, \quad y_C(t) = c_1 \cos 2t + c_2 \sin 2t$$

Using the expressions for $u_1$ and $u_2$ on the previous slide, the general solution to the differential equation is

$$y(t) = u_1(t) \cos 2t + u_2(t) \sin 2t + y_C(t)$$

$$= -3 \sin t \cos 2t + \frac{3}{2} \ln|\csc t - \cot t| \sin 2t + 3 \cos t \sin 2t + y_C(t)$$

$$= 3\left[\cos t \sin 2t - \sin t \cos 2t \right] + \frac{3}{2} \ln|\csc t - \cot t| \sin 2t + y_C(t)$$

$$= 3\left[2 \sin t \cos^2 t - \sin t (2 \cos^2 t - 1) \right] + \frac{3}{2} \ln|\csc t - \cot t| \sin 2t + y_C(t)$$

$$= 3 \sin t + \frac{3}{2} \ln|\csc t - \cot t| \sin 2t + c_1 \cos 2t + c_2 \sin 2t$$
\[ y'' + p(t)y' + q(t)y = g(t) \]

**Summary**

- Suppose \( y_1, y_2 \) are fundamental solutions to the homogeneous equation associated with the nonhomogeneous equation above, where we note that the coefficient on \( y'' \) is 1.
- To find \( u_1 \) and \( u_2 \), we need to solve the equations
  \[
  u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0 \\
  u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(t)
  \]
- Doing so, and using the Wronskian, we obtain
  \[
  u_1'(t) = -\frac{y_2(t)g(t)}{W(y_1, y_2)(t)}, \quad u_2'(t) = \frac{y_1(t)g(t)}{W(y_1, y_2)(t)}
  \]
- Thus
  \[
  u_1(t) = -\int \frac{y_2(t)g(t)}{W(y_1, y_2)(t)} \, dt + c_1, \quad u_2(t) = \int \frac{y_1(t)g(t)}{W(y_1, y_2)(t)} \, dt + c_2
  \]
Theorem 3.6.1

Consider the equations
\[ y'' + p(t)y' + q(t)y = g(t) \quad (1) \]
\[ y'' + p(t)y' + q(t)y = 0 \quad (2) \]

If the functions \( p, q \) and \( g \) are continuous on an open interval \( I \), and if \( y_1 \) and \( y_2 \) are fundamental solutions to Eq. (2), then a particular solution of Eq. (1) is

\[ Y(t) = -y_1(t) \int \frac{y_2(t)g(t)}{W(y_1, y_2)(t)} \, dt + y_2(t) \int \frac{y_1(t)g(t)}{W(y_1, y_2)(t)} \, dt \]

and the general solution is

\[ y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t) \]