We continue the discussion of the last section, and now consider the presence of a periodic external force:

\[ m u''(t) + \gamma u'(t) + k u(t) = F_0 \cos \omega t \]
Forced Vibrations with Damping

Consider the equation below for damped motion and external forcing function \( F_0 \cos \omega t \).

\[
m u''(t) + \gamma u'(t) + ku(t) = F_0 \cos \omega t
\]

The general solution of this equation has the form

\[
u(t) = c_1 u_1(t) + c_2 u_2(t) + A \cos(\omega t) + B \sin(\omega t) = u_c(t) + U(t)
\]

where the general solution of the homogeneous equation is

\[
u_c(t) = c_1 u_1(t) + c_2 u_2(t)
\]

and the particular solution of the nonhomogeneous equation is

\[
U(t) = A \cos(\omega t) + B \sin(\omega t)
\]
Homogeneous Solution

The homogeneous solutions \( u_1 \) and \( u_2 \) depend on the roots \( r_1 \) and \( r_2 \) of the characteristic equation:

\[
mr^2 + \gamma r + kr = 0 \implies r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}
\]

Since \( m, \gamma, \) and \( k \) are all positive constants, it follows that \( r_1 \) and \( r_2 \) are either real and negative, or complex conjugates with negative real part. In the first case,

\[
\lim_{t \to \infty} u_c(t) = \lim_{t \to \infty} \left( c_1 e^{r_1 t} + c_2 e^{r_2 t} \right) = 0,
\]

while in the second case

\[
\lim_{t \to \infty} u_c(t) = \lim_{t \to \infty} \left( c_1 e^{\lambda t} \cos \mu t + c_2 e^{\lambda t} \sin \mu t \right) = 0.
\]

Thus in either case,

\[
\lim_{t \to \infty} u_c(t) = 0
\]
Transient and Steady-State Solutions

Thus for the following equation and its general solution,

\[ m u''(t) + \gamma u'(t) + ku(t) = F_0 \cos \omega t \]

\[ u(t) = c_1 u_1(t) + c_2 u_2(t) + A \cos(\omega t) + B \sin(\omega t), \]

we have

\[ \lim_{t \to \infty} u_C(t) = \lim_{t \to \infty} (c_1 u_1(t) + c_2 u_2(t)) = 0 \]

Thus \( u_C(t) \) is called the **transient solution**. Note however that

\[ U(t) = A \cos(\omega t) + B \sin(\omega t) \]

is a steady oscillation with same frequency as forcing function.

For this reason, \( U(t) \) is called the **steady-state solution**, or **forced response**.
Transient Solution and Initial Conditions

For the following equation and its general solution,

\[ m u''(t) + \gamma u'(t) + ku(t) = F_0 \cos \omega t \]

\[ u(t) = c_1 u_1(t) + c_2 u_2(t) + A \cos(\omega t) + B \sin(\omega t) \]

the transient solution \( u_c(t) \) enables us to satisfy whatever initial conditions might be imposed.

With increasing time, the energy put into system by initial displacement and velocity is dissipated through damping force. The motion then becomes the response \( U(t) \) of the system to the external force \( F_0 \cos \omega t \).

Without damping, the effect of the initial conditions would persist for all time.
Example 1  (1 of 2)

Consider a spring-mass system satisfying the differential equation and initial condition
\[ u'' + u' + 1.25u = 0, \quad u(0) = 2, \quad u'(0) = 3 \]

Begin by finding the solution to the homogeneous equation

The methods of Chapter 3.3 yield the solution
\[ u_c(t) = c_1 e^{-t/2} \cos t + c_2 e^{-t/2} \sin t \]

A particular solution to the nonhomogeneous equation will have the form \( U(t) = A \cos t + B \sin t \) and substitution gives \( A = 12/17 \) and \( B = 48/17 \). So
\[ U(t) = 12/17 \cos t + 48/17 \sin t \]
Example 1  (2 of 2)

The general solution for the nonhomogeneous equation is
\[ u(t) = c_1 e^{-t/2} \cos t + c_2 e^{-t/2} \sin t + 12/17 \cos t + 48/17 \sin t \]

Applying the initial conditions yields
\[
\begin{align*}
  u(0) &= c_1 + 12/17 = 2 \\
  u'(0) &= -1/2 \ c_1 + c_2 + 48/17 = 3
\end{align*}
\]
\[ \Rightarrow c_1 = 22/17 \ , \ c_2 = 14/17 \]

Therefore, the solution to the IVP is
\[ u(t) = 22/17 e^{-t/2} \cos t + 14/17 e^{-t/2} \sin t + 12/17 \cos t + 48/17 \sin t \]

The graph breaks the solution into its steady state \((U(t))\) and transient \((u_{C}(t))\) components.
Rewriting Forced Response

Using trigonometric identities, it can be shown that
\[ U(t) = A \cos(\omega t) + B \sin(\omega t) \]
can be rewritten as
\[ U(t) = R \cos(\omega t - \delta) \]

It can also be shown that
\[
R = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2}},
\]
\[
\cos \delta = \frac{m(\omega_0^2 - \omega^2)}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2}}, \quad \sin \delta = \frac{\gamma \omega}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2}}
\]

where
\[ \omega_0^2 = k / m \]
Amplitude Analysis of Forced Response

The amplitude $R$ of the steady state solution

$$R = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2}},$$

depends on the driving frequency $\omega$. For low-frequency excitation we have

$$\lim_{\omega \to 0} R = \lim_{\omega \to 0} \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2}} = \frac{F_0}{m\omega_0^2} = \frac{F_0}{k}$$

where we recall $(\omega_0)^2 = k/m$. Note that $F_0/k$ is the static displacement of the spring produced by force $F_0$.

For high frequency excitation,

$$\lim_{\omega \to \infty} R = \lim_{\omega \to \infty} \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2}} = 0$$
Maximum Amplitude of Forced Response

Thus
\[
\lim_{\omega \to 0} R = F_0/k, \quad \lim_{\omega \to \infty} R = 0
\]

At an intermediate value of \( \omega \), the amplitude \( R \) may have a maximum value. To find this frequency \( \omega \), differentiate \( R \) and set the result equal to zero. Solving for \( \omega_{\text{max}} \), we obtain
\[
\omega_{\text{max}}^2 = \omega_0^2 - \frac{\gamma^2}{2m^2} = \omega_0^2 \left( 1 - \frac{\gamma^2}{2mk} \right)
\]

where \( (\omega_0)^2 = k/m \). Note \( \omega_{\text{max}} < \omega_0 \), and \( \omega_{\text{max}} \) is close to \( \omega_0 \) for small \( \gamma \). The maximum value of \( R \) is
\[
R_{\text{max}} = \frac{F_0}{\gamma \omega_0 \sqrt{1 - (\gamma^2/4mk)}}
\]
Maximum Amplitude for Imaginary \( \omega_{\text{max}} \)

We have

\[
\omega_{\text{max}}^2 = \omega_0^2 \left( 1 - \frac{\gamma^2}{2mk} \right)
\]

and

\[
R_{\text{max}} = \frac{F_0}{\gamma \omega_0 \sqrt{1 - (\gamma^2/4mk)}} \approx \frac{F_0}{\gamma \omega_0} \left( 1 + \frac{\gamma^2}{8mk} \right)
\]

where the last expression is an approximation for small \( \gamma \). If \( \gamma^2/(mk) > 2 \), then \( \omega_{\text{max}} \) is imaginary. In this case, \( R_{\text{max}} = F_0/k \), which occurs at \( \omega = 0 \), and \( R \) is a monotone decreasing function of \( \omega \). Recall from Section 3.8 that critical damping occurs when \( \gamma^2/(mk) = 4 \).
Resonance

From the expression

\[ R_{\text{max}} = \frac{F_0}{\gamma \omega_0 \sqrt{1 - (\gamma^2 / 4mk)}} \approx \frac{F_0}{\gamma \omega_0} \left( 1 + \frac{\gamma^2}{8mk} \right) \]

we see that \( R_{\text{max}} \equiv F_0 / (\gamma \omega_0) \) for small \( \gamma \).

Thus for lightly damped systems, the amplitude \( R \) of the forced response is large for \( \omega \) near \( \omega_0 \), since \( \omega_{\text{max}} \equiv \omega_0 \) for small \( \gamma \).

This is true even for relatively small external forces, and the smaller the \( \gamma \) the greater the effect.

This phenomena is known as \textbf{resonance}. Resonance can be either good or bad, depending on circumstances; for example, when building bridges or designing seismographs.
Graphical Analysis of Quantities

To get a better understanding of the quantities we have been examining, we graph the ratios $R/(F_0/k)$ vs. $\omega/\omega_0$ for several values of $\Gamma = \gamma^2/(mk)$, as shown below.

Note that the peaks tend to get higher as damping decreases.

As damping decreases to zero, the values of $R/(F_0/k)$ become asymptotic to $\omega = \omega_0$. Also, if $\gamma^2/(mk) > 2$, then $R_{\text{max}} = F_0/k$, which occurs at $\omega = 0$. 

![Graph of R/F0 vs. w/w0 for different values of Gamma]
Analysis of Phase Angle

Recall that the phase angle $\delta$ given in the forced response

$$U(t) = R \cos(\omega t - \delta)$$

is characterized by the equations

$$\cos \delta = \frac{m(\omega_0^2 - \omega^2)}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2}}, \quad \sin \delta = \frac{\gamma \omega}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2}}$$

If $\omega \equiv 0$, then $\cos \delta \equiv 1$, $\sin \delta \equiv 0$, and hence $\delta \equiv 0$. Thus the response is nearly in phase with the excitation.

If $\omega = \omega_0$, then $\cos \delta = 0$, $\sin \delta = 1$, and hence $\delta \equiv \pi/2$. Thus response lags behind excitation by nearly $\pi/2$ radians.

If $\omega$ large, then $\cos \delta \equiv -1$, $\sin \delta = 0$, and hence $\delta \equiv \pi$. Thus response lags behind excitation by nearly $\pi$ radians, and hence they are nearly out of phase with each other.
Example 2: Forced Vibrations with Damping

Consider the initial value problem

\[ u''(t) + 0.125u'(t) + u(t) = 3 \cos \omega t, \quad u(0) = 2, \quad u'(0) = 0 \]

Then \( \omega_0 = 1, \quad F_0 = 3, \) and \( \Gamma = \frac{\gamma^2}{I(mk)} = 1/64 = 0.015625. \)

The unforced motion of this system was discussed in Ch 3.7, with the graph of the solution given below, along with the graph of the ratios \( R/(F_0/k) \) vs. \( \omega/\omega_0 \) for different values of \( \Gamma. \)
Example 2:
Forced Vibrations with Damping  (2 of 4)

Recall that \( \omega_0 = 1 \), \( F_0 = 3 \), and \( \Gamma = \gamma^2/(mk) = 1/64 = 0.015625 \).

The solution for the low frequency case \( \omega = 0.3 \) is graphed below, along with the forcing function.

After the transient response is substantially damped out, the steady-state response is essentially in phase with excitation, and response amplitude is larger than static displacement.

Specifically, \( R \cong 3.2939 > F_0/k = 3 \), and \( \delta \cong 0.041185 \).
Example 2: Forced Vibrations with Damping

Recall that $\omega_0 = 1$, $F_0 = 3$, and $\Gamma = \gamma^2/(mk) = 1/64 = 0.015625$.

The solution for the resonant case $\omega = 1$ is graphed below, along with the forcing function.

The steady-state response amplitude is eight times the static displacement, and the response lags excitation by $\pi/2$ radians, as predicted. Specifically, $R = 24 > F_0/k = 3$, and $\delta = \pi/2$. 
Example 2: Forced Vibrations with Damping (4 of 4)

Recall that $\omega_0 = 1$, $F_0 = 3$, and $F = \gamma^2/(mk) = 1/64 = 0.015625$.

The solution for the relatively high frequency case $\omega = 2$ is graphed below, along with the forcing function.

The steady-state response is out of phase with excitation, and response amplitude is about one third the static displacement.

Specifically, $R \approx 0.99655 \approx F_0/k = 3$, and $\delta \approx 3.0585 \approx \pi$. 

![Graph of the solution and forcing function](image)
Undamped Equation:  
General Solution for the Case $\omega_0 \neq \omega$

Suppose there is no damping term. Then our equation is

$$mu''(t) + ku(t) = F_0 \cos \omega t$$

Assuming $\omega_0 \neq \omega$, then the method of undetermined coefficients can be used to show that the general solution is

$$u(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t$$
Undamped Equation: 
Mass Initially at Rest  (1 of 3)

If the mass is initially at rest, then the corresponding initial value problem is

\[ m u''(t) + ku(t) = F_0 \cos \omega t, \quad u(0) = 0, \quad u'(0) = 0 \]

Recall that the general solution to the differential equation is

\[ u(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t \]

Using the initial conditions to solve for \( c_1 \) and \( c_2 \), we obtain

\[ c_1 = -\frac{F_0}{m(\omega_0^2 - \omega^2)}, \quad c_2 = 0 \]

Hence

\[ u(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} \left( \cos \omega t - \cos \omega_0 t \right) \]
Undamped Equation: Solution to Initial Value Problem  

Thus our solution is
\[ u(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} (\cos \omega t - \cos \omega_0 t) \]

To simplify the solution even further, let \( A = (\omega_0 + \omega)/2 \) and \( B = (\omega_0 - \omega)/2 \). Then \( A + B = \omega_0 t \) and \( A - B = \omega t \). Using the trigonometric identity
\[ \cos(A \pm B) = \cos A \cos B \mp \sin A \sin B, \]
it follows that
\[ \cos \omega t = \cos A \cos B + \sin A \sin B \]
\[ \cos \omega_0 t = \cos A \cos B - \sin A \sin B \]
and hence
\[ \cos \omega t - \cos \omega_0 t = 2 \sin A \sin B \]
Undamped Equation: Beats  \((3 \text{ of } 3)\)

Using the results of the previous slide, it follows that

\[
u(t) = \left[ \frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin \left( \frac{\omega_0 - \omega}{2} \right) \right] \sin \left( \frac{\omega_0 + \omega}{2} \right)
\]

When \(|\omega_0 - \omega| \approx 0\), \(\omega_0 + \omega\) is much larger than \(\omega_0 - \omega\), and \(\sin[(\omega_0 + \omega)t/2]\) oscillates more rapidly than \(\sin[(\omega_0 - \omega)t/2]\).

Thus motion is a rapid oscillation with frequency \((\omega_0 + \omega)/2\), but with slowly varying sinusoidal amplitude given by

\[
\frac{2F_0}{m|\omega_0^2 - \omega^2|} \left| \sin \left( \frac{\omega_0 - \omega}{2} \right) \right|
\]

This phenomena is called a **beat**.

Beats occur with two tuning forks of nearly equal frequency.
Example 3: Undamped Equation, Mass Initially at Rest

Consider the initial value problem

\[ u''(t) + u(t) = 0.5 \cos 0.8t, \quad u(0) = 0, \quad u'(0) = 0 \]

Then \( \omega_0 = 1 \), \( \omega = 0.8 \), and \( F_0 = 0.5 \), and hence the solution is

\[ u(t) = 2.77778 (\sin 0.1t) (\sin 0.9t) \]

The displacement of the spring–mass system oscillates with a frequency of 0.9, slightly less than natural frequency \( \omega_0 = 1 \).

The amplitude variation has a slow frequency of 0.1 and period of \( 20\pi \).

A half-period of \( 10\pi \) corresponds to a single cycle of increasing and then decreasing amplitude.
Example 3: Increased Frequency (2 of 2)

- Recall our initial value problem
  \[ u''(t) + u(t) = 0.5 \cos 0.8t, \quad u(0) = 0, \quad u'(0) = 0 \]

- If driving frequency \( \omega \) is increased to \( \omega = 0.9 \), then the slow frequency is halved to 0.05 with half-period doubled to \( 20\pi \).

- The multiplier 2.77778 is increased to 5.2632, and the fast frequency only marginally increased, to 0.095.
Undamped Equation:
General Solution for the Case $\omega_0 = \omega$  \hspace{1cm} (1 of 2)  

Recall our equation for the undamped case:

$$mu''(t) + ku(t) = F_0 \cos \omega t$$

If forcing frequency equals natural frequency of system, i.e., $\omega = \omega_0$, then nonhomogeneous term $F_0 \cos \omega t$ is a solution of homogeneous equation. It can then be shown that

$$u(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{F_0}{2m\omega_0} t \sin \omega_0 t$$

Thus solution $u$ becomes unbounded as $t \to \infty$.

Note: Model invalid when $u$ gets large, since we assume small oscillations $u$. 

![Graph showing oscillations](image)
Undamped Equation: Resonance  (2 of 2)

- If forcing frequency equals natural frequency of system, i.e., \( \omega = \omega_0 \), then our solution is
  \[
  u(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{F_0}{2m\omega_0} t \sin \omega_0 t
  \]

- Motion \( u \) remains bounded if damping present. However, response \( u \) to input \( F_0 \cos \alpha t \) may be large if damping is small and \( |\omega_0 - \alpha| \approx 0 \), in which case we have resonance.