We consider again a homogeneous system of $n$ first order linear equations with constant, real coefficients,

$$
x_1' = a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n
$$

$$
x_2' = a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n
$$

\vdots

$$
x_n' = a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n,
$$

and thus the system can be written as $\mathbf{x}' = A\mathbf{x}$, where

$$
\mathbf{x}(t) = \begin{pmatrix}
x_1(t) \\
x_2(t) \\
\vdots \\
x_n(t)
\end{pmatrix}, \quad A = 
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{pmatrix}
$$
Conjugate Eigenvalues and Eigenvectors

- We know that $\mathbf{x} = \xi e^{rt}$ is a solution of $\mathbf{x}' = A\mathbf{x}$, provided $r$ is an eigenvalue and $\xi$ is an eigenvector of $A$.

- The eigenvalues $r_1, \ldots, r_n$ are the roots of $\det(A - rI) = 0$, and the corresponding eigenvectors satisfy $(A - rI)\xi = 0$.

- If $A$ is real, then the coefficients in the polynomial equation $\det(A - rI) = 0$ are real, and hence any complex eigenvalues must occur in conjugate pairs. Thus if $r_1 = \lambda + i\mu$ is an eigenvalue, then so is $r_2 = \lambda - i\mu$.

- The corresponding eigenvectors $\xi^{(1)}, \xi^{(2)}$ are conjugates also. To see this, recall $A$ and $I$ have real entries, and hence

$$ (A - r_1 I)\xi^{(1)} = 0 \Rightarrow (A - \bar{r}_1 I)\bar{\xi}^{(1)} = 0 \Rightarrow (A - r_2 I)\xi^{(2)} = 0 $$
Conjugate Solutions

It follows from the previous slide that the solutions

\[ \mathbf{x}^{(1)} = \xi^{(1)} e^{r_1 t}, \quad \mathbf{x}^{(2)} = \xi^{(2)} e^{r_2 t} \]

corresponding to these eigenvalues and eigenvectors are conjugates as well, since

\[ \mathbf{x}^{(2)} = \xi^{(2)} e^{r_2 t} = \bar{\xi}^{(1)} e^{\bar{r}_2 t} = \bar{\mathbf{x}}^{(1)} \]
Example 1: Direction Field  (1 of 7)

Consider the homogeneous equation \( \mathbf{x}' = \mathbf{A} \mathbf{x} \) below.

\[
\mathbf{x}' = \begin{pmatrix} -1/2 & 1 \\ -1 & -1/2 \end{pmatrix} \mathbf{x}
\]

A direction field for this system is given below.

Substituting \( \mathbf{x} = \xi e^{rt} \) in for \( \mathbf{x} \), and rewriting system as \( (\mathbf{A} - r \mathbf{I}) \xi = \mathbf{0} \), we obtain

\[
\begin{pmatrix} -1/2 - r & 1 \\ -1 & -1/2 - r \end{pmatrix} \begin{pmatrix} \xi \\ \xi_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]
Example 1: Complex Eigenvalues  (2 of 7)

We determine $r$ by solving $\det(A-rI) = 0$. Now

$$
\begin{vmatrix}
-1/2 - r & 1 \\
-1 & -1/2 - r
\end{vmatrix}
= (r + 1/2)^2 + 1 = r^2 + r + \frac{5}{4}
$$

Thus

$$
r = \frac{-1 \pm \sqrt{1^2 - 4(5/4)}}{2} = \frac{-1 \pm 2i}{2} = -\frac{1}{2} \pm i
$$

Therefore the eigenvalues are $r_1 = -1/2 + i$ and $r_2 = -1/2 - i$. 
Example 1: First Eigenvector  (3 of 7)

Eigenvector for \( r_1 = -1/2 + i \): Solve

\[
(A - rI)\xi = 0 \iff \begin{pmatrix} -1/2 - r & 1 \\ -1 & -1/2 - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

\[
\iff \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{pmatrix} 1 & i \\ -1 & -i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

by row reducing the augmented matrix:

\[
\begin{pmatrix} 1 & i & 0 \\ -1 & -i & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & i & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \xi^{(1)} = \begin{pmatrix} -i \xi_2 \\ \xi_2 \end{pmatrix} \rightarrow \text{choose } \xi^{(1)} = \begin{pmatrix} 1 \\ i \end{pmatrix}
\]

Thus

\[
\xi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]
Example 1: Second Eigenvector (4 of 7)

**Eigenvector for** $r_1 = -1/2 - i$: Solve

$$(A - rI)\xi = 0 \iff \begin{pmatrix} -1/2 - r & 1 \\ -1 & -1/2 - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\iff \begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{pmatrix} 1 & -i \\ -1 & i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

by row reducing the augmented matrix:

$$\begin{pmatrix} 1 & -i & 0 \\ -1 & i & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \xi^{(2)} = \begin{pmatrix} i \xi_2 \\ \xi_2 \end{pmatrix} \rightarrow \text{choose } \xi^{(2)} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

**Thus**

$$\xi^{(2)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$
Example 1: General Solution (5 of 7)

The corresponding solutions \( \mathbf{x} = \xi e^{rt} \) of \( \mathbf{x}' = \mathbf{A}\mathbf{x} \) are

\[
\mathbf{u}(t) = e^{-t/2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} = e^{-t/2} \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}
\]

\[
\mathbf{v}(t) = e^{-t/2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} = e^{-t/2} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}
\]

The Wronskian of these two solutions is

\[
W[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}](t) = \begin{vmatrix} e^{-t/2} \cos t & e^{-t/2} \sin t \\ -e^{-t/2} \sin t & e^{-t/2} \cos t \end{vmatrix} = e^{-t} \neq 0
\]

Thus \( \mathbf{u}(t) \) and \( \mathbf{v}(t) \) are real-valued fundamental solutions of \( \mathbf{x}' = \mathbf{A}\mathbf{x} \), with general solution \( \mathbf{x} = c_1 \mathbf{u} + c_2 \mathbf{v} \).
Example 1: Phase Plane (6 of 7)

- Given below is the phase plane plot for solutions $\mathbf{x}$, with

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} e^{-t/2} \cos t \\ -e^{-t/2} \sin t \end{pmatrix} + c_2 \begin{pmatrix} e^{-t/2} \sin t \\ e^{-t/2} \cos t \end{pmatrix}$$

- Each solution trajectory approaches origin along a spiral path as $t \to \infty$, since coordinates are products of decaying exponential and sine or cosine factors.

- The graph of $\mathbf{u}$ passes through $(1,0)$, since $\mathbf{u}(0) = (1,0)$. Similarly, the graph of $\mathbf{v}$ passes through $(0,1)$.

- The origin is a **spiral point**, and is asymptotically stable.
Example 1: Time Plots (7 of 7)

The general solution is $\mathbf{x} = c_1 \mathbf{u} + c_2 \mathbf{v}$:

$$
\mathbf{x} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} c_1 e^{-t/2} \cos t + c_2 e^{-t/2} \sin t \\ -c_1 e^{-t/2} \sin t + c_2 e^{-t/2} \cos t \end{pmatrix}
$$

As an alternative to phase plane plots, we can graph $x_1$ or $x_2$ as a function of $t$. A few plots of $x_1$ are given below, each one a decaying oscillation as $t \to \infty$. 

![Graph showing $x_1$ as a function of $t$ with a decaying oscillation.](image-url)
General Solution

To summarize, suppose \( r_1 = \lambda + i\mu, \ r_2 = \lambda - i\mu, \) and that \( r_3, \ldots, r_n \) are all real and distinct eigenvalues of \( A \). Let the corresponding eigenvectors be

\[
\xi^{(1)} = a + ib, \quad \xi^{(2)} = a - ib, \quad \xi^{(3)}, \xi^{(4)},\ldots, \xi^{(n)}
\]

Then the general solution of \( x' = Ax \) is

\[
x = c_1 u(t) + c_2 v(t) + c_3 \xi^{(3)} e^{r_3 t} + \ldots + c_n \xi^{(n)} e^{r_n t}
\]

where

\[
u(t) = e^{\lambda t} (a \cos \mu t - b \sin \mu t), \quad v(t) = e^{\lambda t} (a \sin \mu t + b \cos \mu t)
\]
Real-Valued Solutions

Thus for complex conjugate eigenvalues \( r_1 \) and \( r_2 \), the corresponding solutions \( x^{(1)} \) and \( x^{(2)} \) are conjugates also.

To obtain real-valued solutions, use real and imaginary parts of either \( x^{(1)} \) or \( x^{(2)} \). To see this, let \( \xi^{(1)} = a + ib \). Then

\[
x^{(1)} = \xi^{(1)} e^{(\lambda + i\mu)t} = (a + ib) e^{\lambda t} (\cos \mu t + i \sin \mu t)
\]

\[
= e^{\lambda t} (a \cos \mu t - b \sin \mu t) + ie^{\lambda t} (a \sin \mu t + b \cos \mu t)
\]

\[
= u(t) + iv(t)
\]

where

\[
u(t) = e^{\lambda t} (a \cos \mu t - b \sin \mu t), \quad v(t) = e^{\lambda t} (a \sin \mu t + b \cos \mu t),
\]

are real valued solutions of \( x' = Ax \), and can be shown to be linearly independent.
Spiral Points, Centers, Eigenvalues, and Trajectories

In previous example, general solution was

\[ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} e^{-t/2} \cos t \\ -e^{-t/2} \sin t \end{pmatrix} + c_2 \begin{pmatrix} e^{-t/2} \sin t \\ e^{-t/2} \cos t \end{pmatrix} \]

The origin was a **spiral point**, and was asymptotically stable.

If real part of complex eigenvalues is positive, then trajectories spiral away, unbounded, from origin, and hence origin would be an unstable spiral point.

If real part of complex eigenvalues is zero, then trajectories circle origin, neither approaching nor departing. Then origin is called a **center** and is stable, but not asymptotically stable. Trajectories periodic in time.

The direction of trajectory motion depends on entries in \( \mathbf{A} \).
Example 2: 
Second Order System with Parameter  (1 of 2)

The system \( \mathbf{x}' = \mathbf{A}\mathbf{x} \) below contains a parameter \( \alpha \).

\[
\mathbf{x}' = \begin{pmatrix} \alpha & 2 \\ -2 & 0 \end{pmatrix} \mathbf{x}
\]

Substituting \( \mathbf{x} = \xi e^{rt} \) in for \( \mathbf{x} \) and rewriting system as \( (\mathbf{A} - r\mathbf{I})\xi = 0 \), we obtain

\[
\begin{pmatrix} \alpha - r & 2 \\ -2 & -r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

Next, solve for \( r \) in terms of \( \alpha \):

\[
\begin{vmatrix} \alpha - r & 2 \\ -2 & -r \end{vmatrix} = r(r - \alpha) + 4 = r^2 - \alpha r + 4 \Rightarrow r = \frac{\alpha \pm \sqrt{\alpha^2 - 16}}{2}
\]
Example 2: Eigenvalue Analysis (2 of 2)

The eigenvalues are given by the quadratic formula above.
For \( \alpha < -4 \), both eigenvalues are real and negative, and hence origin is asymptotically stable node.
For \( \alpha > 4 \), both eigenvalues are real and positive, and hence the origin is an unstable node.
For \(-4 < \alpha < 0\), eigenvalues are complex with a negative real part, and hence origin is asymptotically stable spiral point.
For \( 0 < \alpha < 4 \), eigenvalues are complex with a positive real part, and the origin is an unstable spiral point.
For \( \alpha = 0 \), eigenvalues are purely imaginary, origin is a center. Trajectories closed curves about origin & periodic.
For \( \alpha = \pm 4 \), eigenvalues real & equal, origin is a node (Ch 7.8)
Second Order Solution Behavior and Eigenvalues: Three Main Cases

For second order systems, the three main cases are:

- Eigenvalues are real and have opposite signs; \( \mathbf{x} = 0 \) is a saddle point.
- Eigenvalues are real, distinct and have same sign; \( \mathbf{x} = 0 \) is a node.
- Eigenvalues are complex with nonzero real part; \( \mathbf{x} = 0 \) a spiral point.

Other possibilities exist and occur as transitions between two of the cases listed above:

- A zero eigenvalue occurs during transition between saddle point and node. Real and equal eigenvalues occur during transition between nodes and spiral points. Purely imaginary eigenvalues occur during a transition between asymptotically stable and unstable spiral points.

\[
r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]
Example 3: Multiple Spring-Mass System (1 of 6)

The equations for the system of two masses and three springs discussed in Section 7.1, assuming no external forces, can be expressed as:

\[ m_1 \frac{d^2 x_1}{dt^2} = -(k_1 + k_2)x_1 + k_2x_2 \quad \text{and} \quad m_2 \frac{d^2 x_2}{dt^2} = k_2x_1 - (k_2 + k_3)x_2 \]

or

\[ m_1 y_3' = -(k_1 + k_2)y_1 + k_2y_2 \quad \text{and} \quad m_2 y_4' = k_2y_1 - (k_2 + k_3)y_2 \]

where \( y_1 = x_1 \), \( y_2 = x_2 \), \( y_3 = x_1' \), and \( y_4 = x_2' \)

Given \( m_1 = 2 \), \( m_2 = 9/4 \), \( k_1 = 1 \), \( k_2 = 3 \), and \( k_3 = 15/4 \), the equations become

\[ y_1' = y_3, \quad y_2' = y_4, \quad y_3' = -2y_1 + 3/2 \quad y_2, \quad \text{and} \quad y_4' = 4/3 \quad y_1 - 3y_2 \]
Example 3: Multiple Spring-Mass System (2 of 6)

Writing the system of equations in matrix form:

\[
y' = \begin{pmatrix}
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1 \\
    -2 & 3/2 & 0 & 0 \\
    4/3 & -3 & 0 & 0
\end{pmatrix} y = Ay
\]

Assuming a solution of the form \( y = \xi e^{rt} \), where \( r \) must be an eigenvalue of the matrix \( A \) and \( \xi \) is the corresponding eigenvector, the characteristic polynomial of \( A \) is

\[r^4 + 5r^2 + 4 = (r^2 + 1)(r^2 + 4)\]

yielding the eigenvalues: \( r_1 = i, r_2 = -i, r_3 = 2i, \) and \( r_4 = -2i \)
Example 3: Multiple Spring-Mass System (3 of 6)

For the eigenvalues \( r_1 = i, r_2 = -i, r_3 = 2i, \) and \( r_4 = -2i \) the corresponding eigenvectors are

\[
\xi^{(1)} = \begin{pmatrix} 3 \\ 2 \\ 3i \\ 2i \end{pmatrix}, \quad \xi^{(2)} = \begin{pmatrix} 3 \\ 2 \\ -3i \\ -2i \end{pmatrix}, \quad \xi^{(3)} = \begin{pmatrix} 3 \\ -4 \\ 6i \\ -8i \end{pmatrix}, \quad \text{and} \quad \xi^{(4)} = \begin{pmatrix} 3 \\ -4 \\ -6i \\ 8i \end{pmatrix}
\]

The products \( \xi^{(1)} e^{it} \) and \( \xi^{(3)} e^{2it} \) yield the complex-valued solutions:

\[
\xi^{(1)} e^{it} = \begin{pmatrix} 3 \\ 2 \\ 3i \\ 2i \end{pmatrix} (\cos t + i \sin t) = 3 \cos t \begin{pmatrix} 3 \cos t \\ 2 \cos t \\ -3 \sin t \\ -2 \sin t \end{pmatrix} + i 3 \sin t \begin{pmatrix} 3 \sin t \\ 2 \sin t \\ 3 \cos t \\ 2 \cos t \end{pmatrix} = \mathbf{u}^{(1)}(t) + i \mathbf{v}^{(1)}(t)
\]

\[
\xi^{(3)} e^{2it} = \begin{pmatrix} 3 \\ -4 \\ 6i \\ -8i \end{pmatrix} (\cos 2t + i \sin 2t) = 3 \cos 2t \begin{pmatrix} 3 \cos 2t \\ -4 \cos 2t \\ -6 \sin 2t \\ 8 \sin 2t \end{pmatrix} + i 3 \sin 2t \begin{pmatrix} -4 \cos 2t \\ -4 \sin 2t \\ 6 \cos 2t \\ -8 \cos 2t \end{pmatrix} = \mathbf{u}^{(2)}(t) + i \mathbf{v}^{(2)}(t)
\]
\[ y_1' = y_3, \quad y_2' = y_4, \quad y_3' = -2y_1 + \frac{3}{2}y_2, \quad \text{and} \quad y_4' = 4/3y_1 - 3y_2 \]

**Example 3: Multiple Spring-Mass System (4 of 6)**

After validating that \( u^{(1)}(t), v^{(1)}(t), u^{(2)}(t), v^{(2)}(t) \) are linearly independent, the general solution of the system of equations can be written as

\[
\mathbf{y} = c_1 \begin{pmatrix} 3\cos t \\ 2\cos t \\ -3\sin t \\ -2\sin t \end{pmatrix} + c_2 \begin{pmatrix} 3\sin t \\ 2\sin t \\ 3\cos t \\ 2\cos t \end{pmatrix} + c_3 \begin{pmatrix} 3\cos 2t \\ -4\cos 2t \\ -6\sin 2t \\ 8\sin 2t \end{pmatrix} + c_4 \begin{pmatrix} 3\sin 2t \\ -4\sin 2t \\ 6\cos 2t \\ -8\cos 2t \end{pmatrix}
\]

where \( c_1, c_2, c_3, c_4 \) are arbitrary constants.

Each solution will be periodic with period \( 2\pi \), so each trajectory is a closed curve. The first two terms of the solution describe motions with frequency 1 and period \( 2\pi \) while the second two terms describe motions with frequency 2 and period \( \pi \). The motions of the two masses will be different relative to one another for solutions involving only the first two terms or the second two terms.
\(y_1\) and \(y_2\) represent the motion of the masses and \(y_3 = y_1', y_4 = y_1''\)

**Example 3: Multiple Spring-Mass System (5 of 6)**

- To obtain the fundamental mode of vibration with frequency 1
  \(c_3 = c_4 = 0 \rightarrow \text{occurs when } 3y_2(0) = 2y_1(0)\) and \(3y_4(0) = 2y_3(0)\)

- To obtain the fundamental mode of vibration with frequency 2
  \(c_1 = c_2 = 0 \rightarrow \text{occurs when } 3y_2(0) = -4y_1(0)\) and \(3y_4(0) = -4y_3(0)\)

- Plots of \(y_1\) and \(y_2\) and parametric plots \((y, y')\) are shown for a selected solution with frequency 1

\[
y(0) = \begin{pmatrix}
y_1(0) \\
y_2(0) \\
y_3(0) \\
y_4(0)
\end{pmatrix} = \begin{pmatrix}
3 \\
2 \\
0 \\
0
\end{pmatrix}
\]
$y_1$ and $y_2$ represent the motion of the masses and $y_3 = y_1'$, $y_4 = y_1''$

**Example 3: Multiple Spring-Mass System (6 of 6)**

- Plots of $y_1$ and $y_2$ and parametric plots $(y, y')$ are shown for a selected solution with frequency 2

![Plots of the solutions as functions of time](image1.png)

$$\mathbf{y}(0) = \begin{pmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \\ y_4(0) \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ 0 \\ 0 \end{pmatrix}$$

- Plots of $y_1$ and $y_2$ and parametric plots $(y, y')$ are shown for a selected solution with mixed frequencies satisfying the initial condition stated

![Plots of the solutions as functions of time](image2.png)

$$\mathbf{y}(0) = \begin{pmatrix} -1 \\ 4 \\ 1 \\ 1 \end{pmatrix}$$