We consider again a homogeneous system of $n$ first order linear equations with constant real coefficients $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

If the eigenvalues $r_1, \ldots, r_n$ of $\mathbf{A}$ are real and different, then there are $n$ linearly independent eigenvectors $\mathbf{\xi}^{(1)}, \ldots, \mathbf{\xi}^{(n)}$, and $n$ linearly independent solutions of the form

$$\mathbf{x}^{(1)}(t) = \mathbf{\xi}^{(1)} e^{r_1 t}, \ldots, \mathbf{x}^{(n)}(t) = \mathbf{\xi}^{(n)} e^{r_n t}$$

If some of the eigenvalues $r_1, \ldots, r_n$ are repeated, then there may not be $n$ corresponding linearly independent solutions of the above form.

In this case, we will seek additional solutions that are products of polynomials and exponential functions.
Example 1: Eigenvalues (1 of 2)

We need to find the eigenvectors for the matrix:

\[ A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \]

The eigenvalues \( r \) and \( \xi \) eigenvectors satisfy the equation

\[ (A - rI) \xi = 0 \] or

\[ \begin{pmatrix} 1 - r & -1 \\ 1 & 3 - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \]

To determine \( r \), solve \( \det(A - rI) = 0 \):

\[ \begin{vmatrix} 1 - r & -1 \\ 1 & 3 - r \end{vmatrix} = (r - 1)(r - 3) + 1 = r^2 - 4r + 4 = (r - 2)^2 \]

Thus \( r_1 = 2 \) and \( r_2 = 2 \).
Example 1: Eigenvectors (2 of 2)

To find the eigenvectors, we solve

\[(A - rI)\xi = 0 \iff \begin{pmatrix} 1 & -2 & -1 \\ 1 & 3 & -2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\]

by row reducing the augmented matrix:

\[
\begin{pmatrix} -1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow 1\xi_1 + 1\xi_2 = 0 \quad 0\xi_2 = 0
\]

\[
\rightarrow \xi^{(1)} = \begin{pmatrix} -\xi_2 \\ \xi_2 \end{pmatrix} \rightarrow \text{choose } \xi^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}
\]

Thus there is only one linearly independent eigenvector for the repeated eigenvalue \(r = 2\).
Example 2: Direction Field  (1 of 10)

Consider the homogeneous equation $\mathbf{x}' = A\mathbf{x}$ below.

$$\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \mathbf{x}$$

A direction field for this system is given below.

Substituting $\mathbf{x} = \xi e^{rt}$ in for $\mathbf{x}$, where $r$ is $A$’s eigenvalue and $\xi$ is its corresponding eigenvector, the previous example showed the existence of only one eigenvalue, $r = 2$, with one eigenvector:

$$\xi = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
Example 2: First Solution; and Second Solution, First Attempt

The corresponding solution \( \mathbf{x} = \xi e^{rt} \) of \( \mathbf{x}' = \mathbf{Ax} \) is

\[
\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}
\]

Since there is no second solution of the form \( \mathbf{x} = \xi e^{rt} \), we need to try a different form. Based on methods for second order linear equations in Ch 3.5, we first try \( \mathbf{x} = \xi te^{2t} \).

Substituting \( \mathbf{x} = \xi te^{2t} \) into \( \mathbf{x}' = \mathbf{Ax} \), we obtain

\[
\xi e^{2t} + 2\xi te^{2t} = \mathbf{A}\xi te^{2t}
\]

or

\[
2\xi te^{2t} + \xi e^{2t} - \mathbf{A}\xi te^{2t} = 0
\]
Example 2: Second Solution, Second Attempt  

From the previous slide, we have

$$2\xi te^{2t} + \xi e^{2t} - A\xi te^{2t} = 0$$

In order for this equation to be satisfied for all $t$, it is necessary for the coefficients of $te^{2t}$ and $e^{2t}$ to both be zero.

From the $e^{2t}$ term, we see that $\xi = 0$, and hence there is no nonzero solution of the form $x = \xi te^{2t}$.

Since $te^{2t}$ and $e^{2t}$ appear in the above equation, we next consider a solution of the form

$$x = \xi te^{2t} + \eta e^{2t}$$
Example 2: Second Solution and its Defining Matrix Equations

Substituting $x = \xi t e^{2t} + \eta e^{2t}$ into $x' = Ax$, we obtain

$$\xi e^{2t} + 2\xi t e^{2t} + 2\eta e^{2t} = A(\xi t e^{2t} + \eta e^{2t})$$

or

$$2\xi t e^{2t} + (\xi + 2\eta)e^{2t} = A\xi t e^{2t} + A\eta e^{2t}$$

Equating coefficients yields $A\xi = 2\xi$ and $A\eta = \xi + 2\eta$, or

$$(A - 2I)\xi = \mathbf{0} \quad \text{and} \quad (A - 2I)\eta = \xi$$

The first equation is satisfied if $\xi$ is an eigenvector of $A$ corresponding to the eigenvalue $r = 2$. Thus

$$\xi = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
Example 2: Solving for Second Solution  (5 of 10)

Recall that

\[ A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}, \quad \xi = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \]

Thus to solve \((A - 2I)\eta = \xi\) for \(\eta\), we row reduce the corresponding augmented matrix:

\[
\begin{pmatrix} -1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \eta_2 = -1 - \eta_1
\]

\[
\rightarrow \eta = \begin{pmatrix} \eta_1 \\ -1 - \eta_1 \end{pmatrix} \rightarrow \eta = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + k \begin{pmatrix} 1 \\ -1 \end{pmatrix}
\]
Example 2: Second Solution (6 of 10)

Our second solution \( \mathbf{x} = \xi t e^{2t} + \eta e^{2t} \) is now

\[
\mathbf{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} t e^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{2t} + k \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}
\]

Recalling that the first solution was

\[
\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t},
\]

we see that our second solution is simply

\[
\mathbf{x}^{(2)}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} t e^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{2t},
\]

since the last term of third term of \( \mathbf{x} \) is a multiple of \( \mathbf{x}^{(1)} \).
Example 2: General Solution (7 of 10)

The two solutions of \( \mathbf{x}' = \mathbf{A}\mathbf{x} \) are

\[
\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}, \quad \mathbf{x}^{(2)}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{2t}
\]

The Wronskian of these two solutions is

\[
W[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}](t) = \begin{vmatrix} e^{2t} & te^{2t} \\ -e^{2t} & te^{2t} - e^{2t} \end{vmatrix} = -e^{4t} \neq 0
\]

Thus \( \mathbf{x}^{(1)} \) and \( \mathbf{x}^{(2)} \) are fundamental solutions, and the general solution of \( \mathbf{x}' = \mathbf{A}\mathbf{x} \) is

\[
\mathbf{x}(t) = c_1 \mathbf{x}^{(1)}(t) + c_2 \mathbf{x}^{(2)}(t)
\]

\[
= c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + c_2 \left[ \begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{2t} \right]
\]
Example 2: Phase Plane  (8 of 10)

The general solution is

\[ \mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} t e^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{2t} \]

Thus \( \mathbf{x} \) is unbounded as \( t \to \infty \), and \( \mathbf{x} \to \mathbf{0} \) as \( t \to -\infty \).

Further, it can be shown that as \( t \to -\infty \), \( \mathbf{x} \to \mathbf{0} \) asymptotic to the line \( x_2 = -x_1 \) determined by the first eigenvector.

Similarly, as \( t \to \infty \), \( \mathbf{x} \) is asymptotic to a line parallel to \( x_2 = -x_1 \).
Example 1: Phase Plane  (9 of 10)

- The origin is an improper node, and is unstable. See graph.
- The pattern of trajectories is typical for two repeated eigenvalues with only one eigenvector.
- If the eigenvalues are negative, then the trajectories are similar but are traversed in the inward direction. In this case the origin is an asymptotically stable improper node.
Example 2: Time Plots for General Solution

Time plots for $x_1(t)$ are given below, where we note that the general solution $\mathbf{x}$ can be written as follows.

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + c_2 \left[ \begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{2t} \right]$$

$$\iff \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} c_1 e^{2t} + c_2 t e^{2t} \\ -(c_1 + c_2) e^{2t} - c_2 t e^{2t} \end{pmatrix}$$
General Case for Double Eigenvalues

- Suppose the system \( \mathbf{x}' = \mathbf{A}\mathbf{x} \) has a double eigenvalue \( r = \rho \) and a single corresponding eigenvector \( \xi \).

- The first solution is
  \[
  \mathbf{x}^{(1)} = \xi e^{\rho t},
  \]
  where \( \xi \) satisfies \( (\mathbf{A} - \rho \mathbf{I})\xi = 0 \).

- As in Example 1, the second solution has the form
  \[
  \mathbf{x}^{(2)} = \xi te^{\rho t} + \eta e^{\rho t}
  \]
  where \( \xi \) is as above and \( \eta \) satisfies \( (\mathbf{A} - \rho \mathbf{I})\eta = \xi \).

- Since \( \rho \) is an eigenvalue, \( \det(\mathbf{A} - \rho \mathbf{I}) = 0 \), and \( (\mathbf{A} - \rho \mathbf{I})\eta = \mathbf{b} \) does not have a solution for all \( \mathbf{b} \). However, it can be shown that \( (\mathbf{A} - \rho \mathbf{I})\eta = \xi \) always has a solution.

- The vector \( \eta \) is called a generalized eigenvector.
Example 2 Extension: Fundamental Matrix $\Psi$ (1 of 2)

Recall that a fundamental matrix $\Psi(t)$ for $\mathbf{x}' = A\mathbf{x}$ has linearly independent solution for its columns.

In Example 1, our system $\mathbf{x}' = A\mathbf{x}$ was

$$
\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \mathbf{x}
$$

and the two solutions we found were

$$
\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}, \quad \mathbf{x}^{(2)}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} t e^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{2t}
$$

Thus the corresponding fundamental matrix is

$$
\Psi(t) = \begin{pmatrix} e^{2t} & t e^{2t} \\ -e^{2t} & -te^{2t} - e^{2t} \end{pmatrix} = e^{2t} \begin{pmatrix} 1 & t \\ -1 & -t -1 \end{pmatrix}
$$
Example 2 Extension:  
Fundamental Matrix $\Phi$  (2 of 2)

The fundamental matrix $\Phi(t)$ that satisfies $\Phi(0) = I$ can be found using $\Phi(t) = \Psi(t)\Psi^{-1}(0)$, where

$$\Psi(0) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \quad \Psi^{-1}(0) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix},$$

where $\Psi^{-1}(0)$ is found as follows:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 \end{pmatrix}$$

Thus

$$\Phi(t) = e^{2t} \begin{pmatrix} 1 & t \\ -1 & -t-1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} = e^{2t} \begin{pmatrix} 1-t & -t \\ t & t+1 \end{pmatrix}$$
Jordan Forms

If $A$ is $n \times n$ with $n$ linearly independent eigenvectors, then $A$ can be diagonalized using a similarity transform $T^{-1}AT = D$. The transform matrix $T$ consisted of eigenvectors of $A$, and the diagonal entries of $D$ consisted of the eigenvalues of $A$.

In the case of repeated eigenvalues and fewer than $n$ linearly independent eigenvectors, $A$ can be transformed into a nearly diagonal matrix $J$, called the Jordan form of $A$, with

$$T^{-1}AT = J.$$
Example 2 Extension: Transform Matrix (1 of 2)

In Example 2, our system $\mathbf{x}' = A \mathbf{x}$ was

$$\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \mathbf{x}$$

with eigenvalues $r_1 = 2$ and $r_2 = 2$ and eigenvectors

$$\xi = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \eta = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + k \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Choosing $k = 0$, the transform matrix $T$ formed from the two eigenvectors $\xi$ and $\eta$ is

$$T = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$$
Example 2 Extension: Jordan Form  (2 of 2)

The Jordan form $J$ of $A$ is defined by $T^{-1}AT = J$.

Now

$$T = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \quad T^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$$

and hence

$$J = T^{-1}AT = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 & 0 \\ 1 & 3 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

Note that the eigenvalues of $A$, $r_1 = 2$ and $r_2 = 2$, are on the main diagonal of $J$, and that there is a 1 directly above the second eigenvalue. This pattern is typical of Jordan forms.