13:

\[ y' - y = 2te^{2t}; \quad y(0) = 1. \]  

Soln: Step 1:

\[ \mu(t) \frac{dy}{dt} - \mu y = 2te^{2t}. \]  

Step 2: Compare lhs with \( \frac{d(\mu y)}{dt} = \mu \frac{dy}{dt} + \frac{d\mu}{dt} y \), we choose

\[ \frac{d\mu}{dt} = -\mu \Rightarrow \mu = e^{-t}, \quad \text{(we choose } c = 1) \]  

Step 3: equation in step 1 becomes

\[ \frac{d(e^{-t} y)}{dt} = 2te^{t}. \]  

Integrate the above equation both sides w.r.t \( t \), we obtain

\[ e^{-t} y = 2(te^{t} - e^{t}) + C, \]  

\[ y = 2(t - 1)e^{2t} + Ce^{3t}. \]  

Step 4: set \( t = 0, y = 1 \), we have \( C = 3 \).

So the solution for the given IVP is

\[ y = 2(t - 1)e^{2t} + 3e^{3t}. \]  

14:

\[ y' + 2y = te^{-2t}; \quad y(1) = 0. \]  

Soln: Step 1:

\[ \mu(t) \frac{dy}{dt} + 2\mu y = te^{-2t}. \]  

Step 2: Compare lhs with \( \frac{d(\mu y)}{dt} = \mu \frac{dy}{dt} + \frac{d\mu}{dt} y \), we choose

\[ \frac{d\mu}{dt} = 2\mu \Rightarrow \mu = e^{2t}, \quad \text{(we choose } c = 1) \]  

Step 3: equation in step 1 becomes

\[ \frac{d(e^{2t} y)}{dt} = t. \]  

Integrate the above equation both sides w.r.t \( t \), we obtain

\[ e^{2t} y = t^{2}/2 + C, \]  

\[ y = (t^{2}/2 + C)e^{-2t}. \]  

Step 4: set \( t = 1, y = 0 \), we have \( C = -1/2 \).

So the solution for the given IVP is

\[ y = \frac{(t^{2} - 1)e^{-2t}}{2}. \]  

18

\[ t\frac{dy}{dt} + 2y = \sin t, \quad y\left(\frac{\pi}{2}\right) = 1, \quad t > 0. \]  

Soln: Step 0: put in standard form

\[ y' + \frac{2}{t} y = \frac{\sin t}{t}. \]  

Step 1

\[ \mu(t) \frac{dy}{dt} + \frac{2}{t} \mu y = \mu \frac{\sin t}{t}. \]
Step 2: Compare lhs with \( \frac{d(\mu y)}{dt} = \mu \frac{dy}{dt} + \frac{dy}{dt} y \), we choose
\[
\frac{d\mu}{dt} = 2 \frac{\mu}{t} \Rightarrow \frac{d\mu}{\mu} = 2 \frac{dt}{t} \Rightarrow \ln |\mu| = 2 \ln |t| + C \Rightarrow \mu = t^2 \text{ (by choosing } C = 1). \tag{18}
\]
Step 3: equation in step 1 becomes
\[
\frac{d(t^2 y)}{dt} = t \sin t. \tag{19}
\]
Integrate the above equation both sides w.r.t \( t \), we obtain
\[
t^2 y = -t \cos t + \sin t + C, \tag{20}
\]
\[
y = \frac{\sin t - t \cos t + C}{t^2}. \tag{21}
\]
Step 4: set \( t = \frac{\pi}{2}, y = 1 \), we have \( C = \frac{\pi^2}{4} - 1 \).
So the solution for the given IVP is
\[
y = \frac{\sin t - t \cos t + \frac{\pi^2}{4} - 1}{t^2}. \tag{22}\]

20
\[
ty' + (t + 1)y = t, y(\ln 2) = 1, t > 0. \tag{23}
\]
Soln: Step0: put in standard form
\[
y' + \frac{t + 1}{t}y = 1. \tag{24}
\]
Step1
\[
\mu(t) \frac{dy}{dt} + \frac{t + 1}{t} \mu y = \mu. \tag{25}
\]
Step 2: Compare lhs with \( \frac{d(\mu y)}{dt} = \mu \frac{dy}{dt} + \frac{dy}{dt} y \), we choose
\[
\frac{d\mu}{at} = \frac{t + 1}{t} \mu \Rightarrow \frac{d\mu}{\mu} = t + 1 \frac{dt}{t} \Rightarrow \ln |\mu| = t + \ln t + C \Rightarrow \mu = te^t. \tag{26}
\]
since \( t > 0 \), we eliminate absolute value in term \( \ln |t| \).
Step 3: equation in step 1 becomes
\[
\frac{d(te^t y)}{dt} = te^t. \tag{27}
\]
Integrate the above equation both sides w.r.t \( t \), we obtain
\[
te^t y = te^t - e^t + C, \tag{28}
\]
\[
y = 1 - \frac{t}{e^t} + \frac{C}{te^t}. \tag{29}
\]
Step 4: set \( t = \ln 2, y = 1 \), we have \( C = 2 \).
So the solution for the given IVP is
\[
y = 1 - \frac{t}{e^t} + \frac{2}{te^t}. \tag{30}\]

32:
\[
\lim_{t \to \infty} y(t) = \lim_{t \to \infty} e^{-t^2/4} \int_0^t e^{s^2/4} ds + \lim_{t \to \infty} ce^{-t^2/4}
\]
\[
= \lim_{t \to \infty} \int_0^t e^{s^2/4} ds + 0 \tag{31}
\]
\[
(L'Hospital) = \lim_{t \to \infty} \frac{e^{t^2/4}}{\frac{1}{2} e^{t^2/4}} = \lim_{t \to \infty} \frac{2}{t^2} = 0. \tag{32}
\]
(recall that \( \frac{d}{dt} e^{t^2/4} = e^{t^2/4} \))