Section 2.6: 26, 28

26. Find an integrating factor and solve it
\[ y' = e^{2x} + y - 1. \]

**Solution 1** The original equation could be rewritten as
\[(e^{2x} + y - 1)dx - dy = 0.\]

where
\[M(x, y) = e^{2x} + y - 1, M_y = 1,\]
\[N(x, y) = -1, N_x = 0.\]

we have
\[\frac{M_y - N_x}{N} = \frac{1 - 0}{-1} = -1\]

so the integrating factor satisfies
\[\frac{d\mu}{dx} = -\mu, \quad \mu = e^{-x}\]

Multiply the equation by \(\mu\), we have
\[(e^x + ye^{-x} - e^{-x})dx - e^{-x}dy = 0.\]

\[M_y = e^{-x}, N_x = e^{-x}\]

so the equation is exact. Thus there exists a function \(\psi(x, y)\) such that
\[\psi_x = M, \psi_y = N.\]

Integrate \(\psi_x = M\) with respect to \(x\), we have
\[\psi = e^x - ye^{-x} + e^{-x} + h(y)\]

set \(\psi_y = N\), we have
\[-e^{-x} + h'(y) = -e^{-x}\]
\[h'(y) = 0\]
\[h(y) = C.\]

Therefore the solution to the ODE is
\[e^x - ye^{-x} + e^{-x} = C.\]

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\[ydx + (2xy - e^{-2y})dy = 0.\]

**Solution 2** Since
\[M = y, M_y = 1\]
\[N = 2xy - e^{-2y}, N_x = 2y\]

we have
\[\frac{N_x - M_y}{M} = \frac{2y - 1}{y}\]
So the integrating factor satisfies

\[ \frac{d\mu}{dy} = \frac{2y - 1}{y} \mu \]
\[ \frac{d\mu}{\mu} = \frac{2y - 1}{y} dy \]
\[ \ln |\mu| = 2y - \ln |y| + C \]
\[ \mu = e^{2y/y}. \]

Multiply it to the equation, we have

\[ e^{2y} dx + (2xe^{2y} - 1/y) dy = 0. \]
\[ M_y = 2e^{2y}, N_x = 2e^{2y}. \]

So the equation is exact. Thus there exists a function \( \psi(x, y) \) such that

\[ \psi_x = M, \psi_y = N. \]

Integrate \( \psi_x = M \) with respect to \( x \), we have

\[ \psi(x, y) = xe^{2y} + h(y). \]

Set \( \psi_y = N \), we have

\[ 2xe^{2y} + h'(y) = 2xe^{2y} - 1/y \]
\[ h'(y) = -1/y \]
\[ h(y) = -\ln |y| + C. \]

Therefore the solution to the ODE is

\[ xe^{2y} - \ln |y| = C. \]

Also

\[ y = 0 \]

is a trivial solution.