Section 3.1: 12, 13, 20, 22

12. \( y'' + 3y' = 0, \ y(0) = -2, \ y'(0) = 3. \)

**Solution**

*The characteristic equation is*

\[
  r^2 + 3r = 0, \\
  r(r + 3) = 0, \\
  r_1 = 0, r_2 = -3.
\]

*So the general solution is*

\[
  y(t) = c_1 + c_2 e^{-3t}. 
\]

*The derivative is*

\[
  y'(t) = -3c_2 e^{-3t}. 
\]

*set \( t = 0, \) we have*

\[
  c_1 + c_2 = -2, \\
  -3c_2 = 3. 
\]

*So we have*

\[
  c_1 = -1, c_2 = -1. 
\]

*So the solution to the IVP is*

\[
  y(t) = -1 - e^{-3t}. 
\]

\[
  \lim_{t \to \infty} y(t) = -1. 
\]

13. \( y'' + 5y' + 3y = 0, y(0) = 1, y'(0) = 0. \)

**Solution**

*The characteristic equation is*

\[
  r^2 + 5r + 3 = 0, \\
  r_{1,2} = \frac{-5 \pm \sqrt{13}}{2}
\]

*So the general solution is*

\[
  y(t) = c_1 e^{\frac{-5 + \sqrt{13}}{2}t} + c_2 e^{\frac{-5 - \sqrt{13}}{2}t}. 
\]

*The derivative is*

\[
  y'(t) = \frac{-5 + \sqrt{13}}{2} c_1 e^{\frac{-5 + \sqrt{13}}{2}t} + \frac{-5 - \sqrt{13}}{2} c_2 e^{\frac{-5 - \sqrt{13}}{2}t}. 
\]

*set \( t = 0, \) we have*

\[
  c_1 + c_2 = 1, \\
  \frac{-5 + \sqrt{13}}{2} c_1 + \frac{-5 - \sqrt{13}}{2} c_2 = 0. 
\]

*So we have*
\[ c_1 = \frac{5\sqrt{3} + 13}{26}, \quad c_2 = \frac{-5\sqrt{3} + 13}{26}. \]

So the solution to the IVP is
\[ y(t) = \frac{5\sqrt{3} + 13}{26} e^{-\frac{5\sqrt{3}}{2} t} + \frac{-5\sqrt{3} + 13}{26} e^{\frac{5\sqrt{3}}{2} t}. \]

\[ \lim_{t \to \infty} y(t) = 0. \]

20. Find the solution and maximum value and zeros of the solution.

\[ 2y'' - 3y' + y = 0, \quad y(0) = 2, \quad y'(0) = \frac{1}{2}. \]

**Solution**

The characteristic equation is
\[ 2r^2 - 3r + 1 = 0, \]
\[ (2r - 1)(r - 1) = 0, \]
\[ r_1 = \frac{1}{2}, \quad r_2 = 1. \]

So the general solution is
\[ y(t) = c_1 e^{\frac{t}{2}} + c_2 e^t. \]

The derivative is
\[ y'(t) = \frac{1}{2} c_1 e^{\frac{t}{2}} + c_2 e^t. \]

Set \( t = 0 \), we have
\[ c_1 + c_2 = 2, \]
\[ \frac{1}{2} c_1 + c_2 = \frac{1}{2}. \]

So we have
\[ c_1 = 3, \quad c_2 = -1. \]

So the solution to the IVP is
\[ y(t) = 3e^{\frac{t}{2}} - e^t. \]

The maximum value occurs when \( y' = 0 \), and \( y'' < 0 \). so
\[ y'(t) = \frac{3}{2} e^{\frac{t}{2}} - e^t = 0, \]
\[ \ln \frac{3}{2} + \frac{t}{2} = t \]
\[ t = \ln \frac{9}{4}. \]

And
\[ y''(\ln \frac{9}{4}) = \frac{3}{4} e^{\ln \frac{3}{2}} - e^{\ln \frac{9}{4}} = \frac{3}{4} \cdot \frac{3}{2} - \frac{9}{4} < 0. \]

So the maximum value occurs at \( t = \ln \frac{9}{4} \), and
\[ y_{\text{max}} = 3e^{\ln \frac{3}{2}} - e^{\ln \frac{9}{4}} = \frac{9}{2} - \frac{9}{4} = \frac{9}{4}. \]

The zeros occurs when
22. Solve the initial value problem

\[ 4y'' - y = 0, y(0) = 2, y'(0) = \beta. \]

Then find \( \beta \) such that the solution approaches zero when \( t \to \infty \).

**Solution**

The characteristic equation is

\[ 4r^2 - 1 = 0, \]

\[ r^2 = \frac{1}{4}, \]

\[ r_1 = -\frac{1}{2}, r_2 = \frac{1}{2}. \]

So the general solution is

\[ y(t) = c_1 e^{-\frac{t}{2}} + c_2 e^{\frac{t}{2}}. \]

The derivative is

\[ y'(t) = -\frac{1}{2}c_1 e^{-\frac{t}{2}} + \frac{1}{2}c_2 e^{\frac{t}{2}}. \]

Set \( t = 0 \), we have

\[ c_1 + c_2 = 2, \]

\[ -\frac{1}{2}c_1 + \frac{1}{2}c_2 = \beta. \]

So we have

\[ c_1 = 1 - \beta, \]

\[ c_2 = 1 + \beta. \]

So the solution to the IVP is

\[ y(t) = (1 - \beta)e^{-\frac{t}{2}} + (1 + \beta)e^{\frac{t}{2}}. \]

In order to let the limit approaches zero as \( t \to \infty \) the term \( e^{\frac{t}{2}} \) has to vanish, i.e.,

\[ \beta = -1. \]

section 3.3: 17, 18, 27

17.

\[ y'' + 4y = 0, y(0) = 0, y'(0) = 1. \]

**Solution**

The characteristic equation is

\[ r^2 + 4 = 0, \]

\[ r_{1,2} = \pm 2i. \]

Thus the general solution is given by

\[ y(t) = c_1 \cos 2t + c_2 \sin 2t. \]

Take derivative
\( y'(t) = -2c_1 \sin 2t + 2c_2 \cos 2t. \)

plug in two initial conditions, we have
\[
\begin{align*}
    c_1 &= 0, \\
    c_2 &= 1/2.
\end{align*}
\]

Therefore the solution to the IVP is
\[
y(t) = 1/2 \sin 2t.
\]

As \( t \) increases, the solution behave like steady oscillation.

18.
\[
y'' + 4y' + 5y = 0, y(0) = 1, y'(0) = 0.
\]

Solution

The characteristic equation is
\[
r^2 + 4r + 5 = 0,
\]
\[
r_{1,2} = \frac{-4 \pm \sqrt{16 - 20}}{2} = -2 \pm i.
\]

Thus the general solution is given by
\[
y(t) = c_1 e^{-2t} \cos t + c_2 e^{-2t} \sin t.
\]

Take derivative
\[
y'(t) = c_1 (-2e^{-2t} \cos t - e^{-2t} \sin t) + c_2 (-2e^{-2t} \sin t + e^{-2t} \cos t).
\]

plug in two initial conditions, we have
\[
    c_1 = 1, \\
    -2c_1 + c_2 = 0.
\]

Therefore the solution to the IVP is
\[
y(t) = e^{-2t} \cos t + 2e^{-2t} \sin t.
\]

As \( t \) increases, the solution behave like decaying oscillation.

27. Show that
\[
W(e^{\lambda t} \cos \mu t, e^{\lambda t} \sin \mu t) = \mu e^{\lambda t}.
\]

Proof
\[
W(e^{\lambda t} \cos \mu t, e^{\lambda t} \sin \mu t) = \begin{vmatrix} e^{\lambda t} \cos \mu t & e^{\lambda t} \sin \mu t \\ \lambda e^{\lambda t} \cos \mu t - \mu e^{\lambda t} \sin \mu t & \lambda e^{\lambda t} \sin \mu t + \mu e^{\lambda t} \cos \mu t \end{vmatrix}
= e^{2\lambda t}(\lambda \cos \mu t \sin \mu y + \mu \cos^2 \mu t - \lambda \cos \mu t \sin \mu y + \mu \sin^2 \mu t)
= \mu e^{2\lambda t}.
\]

Section 3.4: 14, 15

14.
\[
y'' + 4y' + 4y = 0, y(-1) = 2, y'(-1) = 1.
\]

Solution

The characteristic equation is
\[ r^2 + 4r + 4 = 0, \]
\[ (r + 2)(r + 2) = 0 \]
\[ r = -2. \]

Thus the general solution is given by
\[ y(t) = c_1 e^{-2t} + c_2 te^{-2t}. \]

Its derivative is
\[ y'(t) = -2c_1 e^{-2t} + c_2 e^{-2t} - 2c_2 te^{-2t}. \]

Set \( t = -1 \), we have
\[ y(-1) = c_1 e^2 - c_2 e^2 = 2, \]
\[ y'(-1) = -2c_1 e^2 + 3c_2 e^2 = 1. \]
\[ c_1 = 7e^{-2}, c_2 = 5e^{-2}. \]

So the solution to the IVP is
\[ y(t) = 7e^{-2(t+1)} + 5te^{-2(t+1)} = (7 + 5t)e^{-2(t+1)}. \]

The graph has a zero at \( t = -7/5 \), has a critical points when
\[ 5e^{-2(t+1)} - 2(7 + 5t)e^{-2(t+1)} = 0 \]
\[ 5 - 14 - 10t = 0 \]
\[ t = -\frac{9}{10} \]

and decreases to 0 as \( t \) approaches infinity, so \( t = -\frac{9}{10} \) is a maximum.

15.

\[ 4y'' + 12y' + 9y = 0, y(0) = 1, y'(0) = -4. \]

Solution

a) The characteristic equation is
\[ 4r^2 + 12r + 9 = 0, \]
\[ (2r + 3)(2r + 3) = 0 \]
\[ r = -\frac{3}{2}. \]

Thus the general solution is given by
\[ y(t) = c_1 e^{-\frac{3}{2}t} + c_2 te^{-\frac{3}{2}t}. \]

Its derivative is
\[ y'(t) = -\frac{3}{2}c_1 e^{-\frac{3}{2}t} + c_2 e^{-\frac{3}{2}t} - \frac{3}{2}c_2 te^{-\frac{3}{2}t}. \]

Set \( t = 0 \), we have
\[ y(0) = c_1 = 1, \]
\[ y'(0) = -\frac{3}{2} + c_2 = -4, c_2 = -\frac{5}{2}. \]

So the solution to the IVP is
\[ y(t) = e^{-\frac{3}{2}t} - \frac{5}{2}te^{-\frac{3}{2}t}. \]

b) The solution has a zero at \( t = 2/5 \).

c) \[ y'(t) = -4e^{-2t} + \frac{15}{4}te^{-2t} = (-4 + \frac{15}{4}t)e^{-2t} = 0, \]
\[ t = 16/15, \]
\[ y(16/15) = e^{-\frac{3}{2} \cdot \frac{16}{15}} - \frac{5}{2} \cdot \frac{16}{15}e^{-\frac{3}{2} \cdot \frac{16}{15}} = -\frac{5}{3}e^{-\frac{4}{5}}. \]

d) If change the second condition to \( y'(0) = b, \) then we have
\[ c_1 = 1, \]
\[ c_2 = b + \frac{3}{2}, \]

So the solution to the IVP is
\[ y(t) = e^{-\frac{3}{2}t} + (b + \frac{3}{2})te^{-\frac{3}{2}t} = (1 - (b + \frac{3}{2})t)e^{-\frac{3}{2}t}. \]
\[ y(t) = 0 \text{ when } (1 - (b + \frac{3}{2})t) = 0 \]
\[ i.e., t_0 = \frac{1}{b + 1.5} \]

So if \( t_0 < 0, \text{i.e., } b > -1.5, \) then the solution will eventually become negative, otherwise, it stays positive.