Make the Math Club Great Again!
The Mathematics of Democratic Voting

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How do you become Math Club King, I mean, President?

King Arthur: I am your king.
Peasant Woman: Well, I didn't vote for you.
King Arthur: You don't vote for kings.
Peasant Woman: Well, how'd you become king, then?

Math Club Election: Plurality

- 4 Candidates: Ronald Drumpf, Sanford "Sandy" Burns, Natalie Cheatham Plimpton, Ned Luz
- 29 club members vote for their top choice
- Results:

Candidate	**Drumpf**	**Burns**	**Plimpton**	**Luz**
# votes	11	3	8	7
%	37.9%	10.3%	27.6%	24.1%

- Drumpf is declared the winner using the *Plurality Method*
- Note that no candidate earns a *Majority* of votes ($> 50\%$)
- Does this really reflect the will of the people?

Math Club Election: Antiplurality

- Burns wonders, “How can this be? Everyone I know *hates* Drumpf!”
- He suggests the club members vote *against* their bottom choice
- Results:

Candidate	**Drumpf**	**Burns**	**Plimpton**	**Luz**
# votes against	18	0	0	11
% against	62.1%	0%	0%	37.9%

- Burns and Plimpton are tied for president using the *Antiplurality Method*
- Plimpton is not happy with the tie

Math Club Election: Plurality with Elimination, Version I

- Plimpton suggests eliminating Drumpf and then revoting, removing the candidate with most last-place votes, etc., until one candidate remains
- Easiest to cast ballots with full rankings one time
- *Preference Schedule*:

# voters	11	7	7	3	1
1st place	Drumpf	Plimpton	Luz	Burns	Plimpton
2nd place	Burns	Burns	Plimpton	Luz	Luz
3rd place	Plimpton	Luz	Burns	Plimpton	Burns
4th place	Luz	Drumpf	Drumpf	Drumpf	Drumpf

- Drumpf has the most last-place votes, so he is eliminated

Math Club Election: Plurality with Elimination, Version I

- Drumpf is removed from the ballots and they are recounted
- Results:

# voters	11	7	7	3	1
1st place	Burns	Plimpton	Luz	Burns	Plimpton
2nd place	Plimpton	Burns	Plimpton	Luz	Luz
3rd place	Luz	Luz	Burns	Plimpton	Burns

- Now Luz has the most last-place votes (18), so he is eliminated

Math Club Election: Plurality with Elimination, Version I

- Luz is removed from the ballots and they are recounted
- Results:

# voters	**11**	**7**	**7**	**3**	**1**
1st place	Burns	Plimpton	Plimpton	Burns	Plimpton
2nd place	Plimpton	Burns	Burns	Plimpton	Burns

- Now Burns has the most last-place votes (15), so he is eliminated and Plimpton is the winner!

Math Club Election: Plurality with Elimination, Version II

- Not so fast, says Luz
- Instead of eliminating the candidate with the *most last-place* votes, we should eliminate the one with the *fewest first-place* votes
- Here that would be Burns with only 3 first-place votes

# voters	**11**	**7**	**7**	**3**	**1**
1st place	Drumpf	Plimpton	Luz	Burns	Plimpton
2nd place	Burns	Burns	Plimpton	Luz	Luz
3rd place	Plimpton	Luz	Burns	Plimpton	Burns
4th place	Luz	Drumpf	Drumpf	Drumpf	Drumpf

Math Club Election: Plurality with Elimination, Version II

- Remove Burns from the ballots and recount
- Results:

# voters	11	7	7	3	1
1st place	Drumpf	Plimpton	Luz	Luz	Plimpton
2nd place	Plimpton	Luz	Plimpton	Plimpton	Luz
3rd place	Luz	Drumpf	Drumpf	Drumpf	Drumpf

- Now Plimpton has fewest first-place votes (8)

Math Club Election: Plurality with Elimination, Version II

- Remove Plimpton from the ballots and recount
- Results:

# voters	**11**	**7**	**7**	**3**	**1**
1st place	Drumpf	Luz	Luz	Luz	Luz
2nd place	Luz	Drumpf	Drumpf	Drumpf	Drumpf

- Now Drumpf has fewest first-place votes (11), so he is eliminated and Luz is the winner!
- This method is sometimes called *Instant Run-Off Voting (IRV)*

Math Club Election: Borda Count

- Burns suggests using a point system.

	pts	11	7	7	3	1
1st	3	Drumpf	Plimpton	Luz	Burns	Plimpton
2nd	2	Burns	Burns	Plimpton	Luz	Luz
3rd	1	Plimpton	Luz	Burns	Plimpton	Burns
4th	0	Luz	Drumpf	Drumpf	Drumpf	Drumpf

- **Drumpf**: $11 \times 3 = 33$
- **Burns**: $(3 \times 3) + (18 \times 2) + (8 \times 1) = 53$
- **Plimpton**: $(8 \times 3) + (7 \times 2) + (14 \times 1) = 52$
- **Luz**: $(7 \times 3) + (4 \times 2) + (7 \times 1) = 36$
- So Burns is the winner!

Math Club Election: Pairwise Comparisons

- Plimpton notes that she would beat each of the other candidates in a head-to-head contest

# voters	**11**	**7**	**7**	**3**	**1**
1st place	Drumpf	Plimpton	Luz	Burns	Plimpton
2nd place	Burns	Burns	Plimpton	Luz	Luz
3rd place	Plimpton	Luz	Burns	Plimpton	Burns
4th place	Luz	Drumpf	Drumpf	Drumpf	Drumpf

- Plimpton beats Drumpf 18 to 11
- Plimpton beats Burns 15 to 14
- Plimpton beats Luz 19 to 10
- Plimpton is therefore a *Condorcet Winner*

"It's not the voting that's democracy, it's the counting."

– Dotty, in Tom Stoppard's play *Jumpers*

The crux of the matter:

How do we aggregate individual voters' preferences to produce a societal preference in the fairest way possible?

What is "fair"?

Fairness Criteria: The Majority Criterion

Definition (The Majority Criterion.)

If a candidate receives a majority ($> 50\%$) of the first-place votes, that candidate should be a winner of the election.

- Violated by Borda Count

	pts/vote	**3**	**2**
1st place	**2**	A	B
2nd place	**1**	B	C
3rd place	**0**	C	A

- A: $(3 \times 2) = 6$
- B: $(2 \times 2) + (3 \times 1) = 7$
- C: $(2 \times 1) = 2$
- A has a majority, but B wins under Borda Count

Fairness Criteria: The Condorcet Criterion

Definition (The Condorcet Criterion.)

If a candidate beats each other candidate in a pairwise comparison, that candidate should be a winner of the election.

- Violated by Plurality, Instant Run-Off Voting, and Borda Count
- Plimpton was Condorcet Candidate in Math Club Election, but lost using Plurality, Instant Run-off Voting, and Borda Count

Fairness Criteria: The Monotonicity Criterion

Definition (The Monotonicity Criterion.)

If candidate X is a winner, then X should remain a winner if a voter moves X (and only X) up on his/her ballot.

- Violated by Instant Run-Off Voting

	7	**8**	**10**	**2**
1st place	A	B	C	A
2nd place	B	C	A	C
3rd place	C	A	B	B

- C wins: B is eliminated in the first round and B's 8 votes get transferred to C (who now has 18/27)
- Now suppose the last two voters want to vote for the winner (C), so they change their votes, moving C up

Definition (The Monotonicity Criterion.)

If candidate X is a winner, then X should remain a winner if a voter moves X (and only X) up on his/her ballot.

- Now suppose the last two voters want to vote for the winner (C), so they change their votes, moving C up

	7	**8**	**10**	**2**
1st place	A	B	C	C
2nd place	B	C	A	A
3rd place	C	A	B	B

- B wins: A is eliminated in the first round and A's 7 votes get transferred to B, who beats C 15 to 12.

Fairness Criteria: The Independence of Irrelevant Alternatives Criterion

Definition (Independence of Irrelevant Alternatives Criterion.)

If candidate X is a winner, then X should remain a winner if any of the irrelevant (losing) candidates drops out of the race.

- All of the voting methods we've seen violate the Independence of Irrelevant Alternatives Criterion!
- Example to show Plurality violates IIAC:

	3	**2**	**2**
1st place	A	B	C
2nd place	B	C	B
3rd place	C	A	A

- Under Plurality, A wins.
- If C is declared ineligible and removed from the ballot, B wins 4 to 3.

Transitivity (or lack thereof)

Definition (Transitivity)

If I prefer P to R and R to S, it is reasonable to assume I prefer P to S. (Write $P > R > S$)

- Suppose there are two other voters with transitive preferences $R > S > P$ and $S > P > R$
- Preference schedule:

# voters	**1**	**1**	**1**
1st place	Paper	Rock	Scissors
2nd place	Rock	Scissors	Paper
3rd place	Scissors	Paper	Rock

- This is a tie, but it's worse than that— it's a *Cycle*.
- Pairwise comparison rankings are *Intransitive*
  - ($P > R$): Paper beats Rock 2 to 1
  - ($R > S$): Rock beats Scissors 2 to 1
  - ($S > P$): Scissors beats Paper 2 to 1

Theorem (Arrow's Impossibility Theorem)

Any transitive voting method that satisfies all of these fairness criteria is a dictatorship.

Conclusions

Count de Money: Your majesty, it is said that the people are revolting.

King Louis XVI: You said it. They stink on ice!

Math 49995
Mathematics of Social Choice
MWF 2:15–3:05

"I enjoyed this course as a Math major and someone who is interested in politics. Voting theory is way more interesting/ controversial than I realized."

"Delivery of the material was very well done! I really found this course interesting and enjoyable!"

"I really enjoyed the material covered in this course, especially the second half of the semester. (Geometric stuff)"