That's Just the Way We Roll: Probability Distributions for Nonstandard Dice

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Ordinary 6-sided dice

Roll two fair ordinary six-sided dice:

$$\left\{\begin{array}{cccccc} (1,1), & (1,2), & (1,3), & (1,4), & (1,5), & (1,6), \\ (2,1), & (2,2), & (2,3), & (2,4), & (2,5), & (2,6), \\ (3,1), & (3,2), & (3,3), & (3,4), & (3,5), & (3,6), \\ (4,1), & (4,2), & (4,3), & (4,4), & (4,5), & (4,6), \\ (5,1), & (5,2), & (5,3), & (5,4), & (5,5), & (5,6), \\ (6,1), & (6,2), & (6,3), & (6,4), & (6,5), & (6,6) \end{array}\right\}$$

Sum:	2	3	4	5	6	7	8	9	10	11	12
Freq:	1	2	3	4	5	6	5	4	3	2	1
Prob:	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$
Prob:	.03	.06	.08	.11	.14	.17	.14	.11	.08	.06	.03

Sicherman dice: empirical

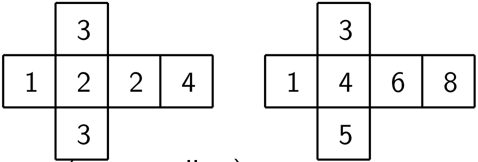
Roll the pair of Sicherman dice 25 times with your partner and record the frequencies of the sums.

Sum:	2	3	4	5	6	7	8	9	10	11	12
Freq:											
Prob:											
Compare:	.03	.06	.08	.11	.14	.17	.14	.11	.08	.06	.03

Compare this with the probability distribution for the ordinary dice.

Sicherman dice: theoretical

Sicherman Dice



(green, yellow)

$$\left\{\begin{array}{llllll}
(1,1), & (1,3), & (1,4), & (1,5), & (1,6), & (1,8), \\
(2,1), & (2,3), & (2,4), & (2,5), & (2,6), & (2,8), \\
(2,1), & (2,3), & (2,4), & (2,5), & (2,6), & (2,8), \\
(3,1), & (3,3), & (3,4), & (3,5), & (3,6), & (3,8), \\
(3,1), & (3,3), & (3,4), & (3,5), & (3,6), & (3,8), \\
(4,1), & (4,3), & (4,4), & (4,5), & (4,6), & (4,8)
\end{array}\right\}$$

Sicherman dice: theoretical

$$\left\{\begin{array}{cccccc}
(1,1), & (1,3), & (1,4), & (1,5), & (1,6), & (1,8), \\
(2,1), & (2,3), & (2,4), & (2,5), & (2,6), & (2,8), \\
(2,1), & (2,3), & (2,4), & (2,5), & (2,6), & (2,8), \\
(3,1), & (3,3), & (3,4), & (3,5), & (3,6), & (3,8), \\
(3,1), & (3,3), & (3,4), & (3,5), & (3,6), & (3,8), \\
(4,1), & (4,3), & (4,4), & (4,5), & (4,6), & (4,8)
\end{array}\right\}$$

Sum: Freq:	1	2	3	4	5	6	5	4	3	2	1
Prob:	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$
Prob:	.03	.06	.08	.11	.14	.17	.14	.11	.08	.06	.03

Question

How else can we label a pair of 6-sided dice with positive integers to get this same probability distribution?

Generating functions

- Generating function for the throw of one (ordinary) 6-sided die:

$$p(x) = x^1 + x^2 + x^3 + x^4 + x^5 + x^6$$

exponent: number of spots

coefficient: number of faces

$$p(x) = x(x+1)(x^2+x+1)(x^2-x+1)$$

- Generating function for the throw of a pair of (ordinary) 6-sided dice:

$$\begin{aligned} p(x)^2 &= x^2 + 2x^3 + 3x^4 + 4x^5 + 5x^6 + 6x^7 + 5x^8 + 4x^9 + 3x^{10} + 2x^{11} + x^{12} \\ &= x^2(x+1)^2(x^2+x+1)^2(x^2-x+1)^2 \end{aligned}$$

Generating functions

Find all polynomials $q(x), r(x)$ with

- $q(x)r(x) = p(x)^2 = x^2(x+1)^2(x^2+x+1)^2(x^2-x+1)^2$
- positive integer faces: polynomials
- at least one spot on each face:
- 6 faces on each die:

factor:	x	$x+1$	x^2+x+1	x^2-x+1
@ $x=0$	0	1	1	1
@ $x=1$	1	2	3	1

$$q(x) = x(x+1)(x^2+x+1)??$$
$$r(x) = x(x+1)(x^2+x+1)??$$

Conclusion

Only two partitions of the factors work:

- Ordinary dice:

$$q(x) = x(x+1)(x^2+x+1)(x^2-x+1)$$
$$r(x) = x(x+1)(x^2+x+1)(x^2-x+1)$$

- Sicherman dice:

$$q(x) = x(x+1)(x^2+x+1)$$
$$r(x) = x(x+1)(x^2+x+1)(x^2-x+1)^2$$

That is,

$$q(x) = x + 2x^2 + 2x^3 + x^4$$
$$r(x) = x + x^3 + x^4 + x^5 + x^6 + x^8$$

References and Acknowledgments

- M. Gardner, Mathematical Games, *Scientific American*, **238**/2 (1978), 19-32.
- Y. Nishiyama, Sicherman Dice: Equivalent Sums with a Pair of Dice, *International Journal of Pure and Applied Mathematics*, **81** (2012), 101–110.
- Sicherman Dice, Wikipedia, `http://en.wikipedia.org/wiki/Sicherman_dice`.

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Congratulations to the new inductees!