Applications of the Artin-Hasse Exponential Series and its Generalizations to Finite Algebra Groups

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Algebra groups

Definition

Let F be a field of characteristic p and order q. Let J be a finite-dimensional, nilpotent, associative F-algebra. Define $G = 1 + J$ (formally). Then G is a finite p-group. Groups of this form are called F-algebra groups. *We will assume this notation throughout.*

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Theorem (Isaacs (1995))

All irreducible characters of algebra groups have q-power degree.

Algebra subgroups and strong subgroups

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- If $H \leq G$ such that $|H \cap K|$ is a q-power for all algebra subgroups K of G, then H is a strong subgroup of G.

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Fact

Algebra subgroups are strong.

Strong subgroup example

Example

The subgroup

$$H = \left\{ \begin{pmatrix} 1 & \alpha & \binom{\alpha}{2} \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{pmatrix} \middle| \alpha \in F \right\}$$

is a strong subgroup (but not an algebra subgroup) of the algebra group of unipotent 3×3 upper-triangular matrices over F.

(Here $\binom{\alpha}{2} = \frac{\alpha(\alpha-1)}{2}$ is the generalized binomial coefficient.)

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- *If* $N \trianglelefteq G$ *is an ideal subgroup, and* λ *is a linear character of* N, *then the stabilizer in* G *of* λ *is strong.*

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Question

Are normalizers of algebra subgroups strong?

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Fix a prime p and a nilpotent algebra X over a field of characteristic 0.

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Definition

Fix a prime p and a nilpotent algebra X over a field of characteristic 0. The Artin-Hasse exponential series, $\mathrm{E_p} : X \to 1 + X$, is defined by

$$\begin{aligned}\mathrm{E_p}(x) &= \exp\left(x + \frac{x^p}{p} + \frac{x^{p^2}}{p^2} + \frac{x^{p^3}}{p^3} + \cdots\right) \\ &= \exp(x)\exp\left(\frac{x^p}{p}\right)\exp\left(\frac{x^{p^2}}{p^2}\right)\exp\left(\frac{x^{p^3}}{p^3}\right)\cdots\end{aligned}$$

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- Frobenius: The highest power of p that divides $n! = |S_n|$ also divides $\left|\cup \mathrm{Syl}_p(S_n)\right|$.

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- $\mathrm{E_p}$ makes sense in characteristic p

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- Not usually:

$$\left(\exp\left(x + \frac{x^p}{p} + \frac{x^{p^2}}{p^2} + \cdots\right)\right)\left(\exp\left(y + \frac{y^p}{p} + \frac{y^{p^2}}{p^2} + \cdots\right)\right) \neq$$

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- We have

$$\mathrm{E_p}(x)\,\mathrm{E_p}(y) = \mathrm{E_p}(S_0)\,\mathrm{E_p}(S_1)\,\mathrm{E_p}(S_2)\cdots$$
$$\text{where}\quad S_0 = x + y$$
$$S_1 = \frac{x^p + y^p - (x+y)^p}{p}$$

and the remaining polynomials S_n can be shown to have p-integral coefficients.

$\mathcal{E}_p^F$ and AH-closed subgroups

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- *$\mathcal{E}_p^F$ and AH-closed subgroups are strong.*
- *$\mathcal{E}_p^F$ is not necessarily AH-closed.*
- *If $J^{2p-1} = 0$, then $\mathcal{E}_p^F = \mathrm{E_p}(Fx)\,\mathrm{E_p}(Fx^p)$ is AH-closed.*

E_p and normalizers.

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- *If $J^{p+1} = 0$, then $N_G(H)$ is AH-closed (hence strong).*
- *If $J^{p+1} \neq 0$, then examples exist for which $|N_G(H)| = p \cdot q^a$, and so $N_G(H)$ need not be strong.*

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- We find a function had analogous to ad so that $\mathrm{E_p}(x) \in N_G(H)$ $\iff$ $\mathrm{E_p}(\mathrm{had}\, x)$ stabilizes L

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- $\vartheta(y) \in J^{p+1}$
- $J^{p+1} = 0 \implies$
  - $\mathrm{had}\, x = \mathrm{ad}\, x$
  - $\mathrm{had}\, x$ stabilizes $L \iff \mathrm{had}(\alpha x)$ stabilizes L for all $\alpha \in F$
  - $N_G(H)$ is AH-closed (hence strong)

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- Examples exist (for all p) for which this happens $\iff \alpha \in GF(p)$ and so $|N_G(H)| = p \cdot q^a$.
- So if $J^{p+1} \neq 0$ and $|F| = q > p$, examples exist for which normalizers of algebra subgroups are not strong.

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Suppose

- $q > p > 2$
- $x \in J$ with $x^p = 0$ but $x^2 \neq 0$
- $\epsilon : F \to F$ is a nonzero additive map with $\epsilon(1) = 0$
- $H = \left\{ (1+x)^{\alpha} \left(1+x^2\right)^{\epsilon(\alpha)} \middle| \alpha \in F \right\}$

Then

- H is an abelian subgroup of G of order q
- $1 + x \in H$ since $\epsilon(1) = 0$
- H is strong, as we will show.

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To show H is strong (i.e., $|H \cap K|$ is a q-power $\forall$ algebra subgroups K):

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- This generates the algebra $xF[x]$, so $xF[x] \subseteq A$.

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- Therefore, $\forall$ algebra subgroups K of G, $H \cap K = 1$ or $H \cap K = H$

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- Therefore, $\forall$ algebra subgroups K of G, $H \cap K = 1$ or $H \cap K = H$
- Conclude: H is a strong subgroup of G.

$H = \left\{ (1+x)^{\alpha} \left(1+x^2\right)^{\epsilon(\alpha)} \middle| \, \alpha \in F \right\}$

Now consider the F-exponent group $(1+x)^F = \{ (1+x)^{\alpha} | \, \alpha \in F \}$:

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- Conclude: the intersection of strong subgroups need not be strong.

Ideal frames

Definition

Let J be a nilpotent F-algebra with $\dim_F(J) = n$. An ideal frame of J is a basis $\{v_1, \ldots, v_n\}$ of J satisfying

$$v_i J, J v_i \subseteq \operatorname{Span}\{v_{i+1}, \ldots, v_n\}$$

for all $i = 1, \ldots, n$.

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- Refine the chain $J \supset J^2 \supset \cdots \supset J^{m-1} \supset J^m = 0$
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- Choose $v_i \in V_i \setminus V_{i+1}$ for all $i = 1, \ldots, n$.
- Then $\{v_1, \ldots, v_n\}$ is an ideal frame of J.

Stringent power series

Definition

We call a power series $J \to 1 + J$ stringent if it is of the form

$$x \mapsto 1 + x + \alpha_2 x^2 + \alpha_3 x^3 + \cdots$$

where $\alpha_2, \alpha_3, \ldots \in F$.

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- Define $\sigma(Fx) = \{\sigma(\alpha x) \,|\alpha \in F\}$.
- $\sigma(Fx)$ is a subset, but not necessarily a subgroup, of G.

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- *In particular,* $G = \prod_{i=1}^{n} \sigma(Fv_i)$.

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Moreover, every $h \in H$ has a unique representation of the form $h = \prod_{i \in I} h_i(\alpha_i)$ where $\alpha_i \in F$.

Example

Suppose H is a strong subgroup of $G + 1 + J$ where $J^p = 0$, $\dim_F(J) = 6$, $\sigma(\alpha x) = (1 + x)^\alpha$, and the partition given by the theorem turns out to be $I = \{1, 3, 5\}$ and $\hat{I} = \{2, 4, 6\}$.

Example

Suppose H is a strong subgroup of $G+1+J$ where $J^p=0$, $\dim_F(J)=6$, $\sigma(\alpha x)=(1+x)^\alpha$, and the partition given by the theorem turns out to be $I=\{1,3,5\}$ and $\hat{I}=\{2,4,6\}$.

Then there are functions $\epsilon_{1,2}, \epsilon_{1,4}, \epsilon_{1,6}, \epsilon_{3,4}, \epsilon_{3,6}, \epsilon_{5,6} : F \to F$ so that

$$h_1(\alpha_1) = (1+v_1)^{\alpha_1}(1+v_2)^{\epsilon_{1,2}(\alpha_1)}(1+v_4)^{\epsilon_{1,4}(\alpha_1)}(1+v_6)^{\epsilon_{1,6}(\alpha_1)}$$
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