Some Subgroups of Finite Algebra Groups: Normalizers of Algebra Subgroups *(preliminary report)*

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Algebra groups

Definition

Let F be a field of characteristic p and order q. Let J be a finite-dimensional, nilpotent, associative F-algebra. Define $G = 1 + J$ (formally). Then G is a finite p-group. Groups of this form are called F-algebra groups. *We will assume this notation throughout.*

Example

Unipotent upper-triangular matrices over F

Theorem (Isaacs (1995))

All irreducible characters of algebra groups have q-power degree.

Algebra subgroups and strong subgroups

- Subgroups: $1 + X$ where $X \subseteq J$ is closed under the operation $(x, y) \mapsto x + y + xy$
- X need not be an algebra.

Definitions

- If L is a subalgebra of J, then $1 + L$ is an algebra subgroup of $G = 1 + J$.
- If $H \leq G$ such that $|H \cap K|$ is a q-power for all algebra subgroups K of G, then H is a strong subgroup of G.

Fact

Algebra subgroups are strong.

Strong subgroups as point stabilizers

Theorem (Isaacs (1995))

Under certain conditions, character stabilizers are strong.

- *If* $J^p = 0$, $N \trianglelefteq G$ *is an ideal subgroup, and* $\theta \in \mathrm{Irr}(N)$, *then the stabilizer in* G *of* θ *is strong.*
- *If* $N \trianglelefteq G$ *is an ideal subgroup, and* λ *is a linear character of* N, *then the stabilizer in* G *of* λ *is strong.*

Question

Are normalizers of algebra subgroups strong?

When $J^p = 0$: The Exponential Map

- If $J^p = 0$, define $\exp : J \to 1 + J$ and $\log : 1 + J \to J$ by the usual power series.

Definitions

- For $x \in J$ and $\alpha \in F$, define $(1+x)^\alpha = \exp(\alpha \log(1+x))$.
- We define an F-exponent subgroup to be a subgroup of the following form:
  - $(1+x)^F = \{(1+x)^\alpha | \alpha \in F\}$

  or equivalently
  - $\exp(F\hat{x}) = \{\exp(\alpha\hat{x}) | \alpha \in F\}$

Fact

F-exponent subgroups are strong.

Exponentially closed subgroups

Definition

The subgroup H is said to be exponentially closed if $\exp(Fx) \subseteq H$ whenever $\exp(x) \in H$.

- Also called partitioned subgroups

Fact

Exponentially closed subgroups are strong.

Normalizers of algebra subgroups when $J^p = 0$

Theorem

Suppose $J^p = 0$. If H is an algebra subgroup of $G = 1 + J$, then $N_G(H)$ is exponentially closed (hence strong).

Sketch of proof.

- Let $H = \exp(L)$ be an algebra subgroup of G.
- To show: if $\exp(x) \in N_G(H)$, then $\exp(\alpha x) \in N_G(H)$ for all $\alpha \in F$.
- Key: $N_G(H) = \exp N_J(L)$ where $N_J(L) = \{x : [L, x] \subseteq L\}$.
  - $y^{(\exp x)^{-1}} = y^{\exp(-x)} = \exp(\operatorname{ad} x)(y)$.
  - $\exp(x) \in N_G(H) \iff \exp(\operatorname{ad} x)$ stabilizes L $\iff \operatorname{ad} x$ stabilizes L

□

A generalization of the exponential map.

Goal

To find an analog of exp *that works if* $x^p \neq 0$.

- From the study of Witt rings and p-adic analysis:

Definition

Fix a prime p and a nilpotent algebra X over a field of characteristic 0. The Artin-Hasse exponential function, $\operatorname{hexp} : X \to 1 + X$, is defined by

$$\begin{aligned} \operatorname{hexp}(x) &= \exp\left(x + \frac{x^p}{p} + \frac{x^{p^2}}{p^2} + \frac{x^{p^3}}{p^3} + \cdots\right) \\ &= \exp(x) \exp\left(\frac{x^p}{p}\right) \exp\left(\frac{x^{p^2}}{p^2}\right) \exp\left(\frac{x^{p^3}}{p^3}\right) \cdots \end{aligned}$$

Another formula for hexp

Miracle

The coefficients in hexp(x) *are p-integral.*

Sketch of proof.

- $\text{hexp}(x) = \sum \frac{\left|\cup \text{Syl}_p(S_n)\right|}{n!} x^n$
- Frobenius: The highest power of p that divides $n! = |S_n|$ also divides $\left|\cup \text{Syl}_p(S_n)\right|$.

□

- hexp makes sense in characteristic p

hexp lacks some nice properties of exp

- Suppose $xy = yx$. Then $\exp(x)\exp(y) = \exp(x+y)$.
- Does $\operatorname{hexp}(x)\operatorname{hexp}(y) = \operatorname{hexp}(x+y)$?
- Not usually:
  $\left(\exp\left(x + \frac{x^p}{p} + \frac{x^{p^2}}{p^2} + \cdots\right)\right)\left(\exp\left(y + \frac{y^p}{p} + \frac{y^{p^2}}{p^2} + \cdots\right)\right) \neq \left(\exp\left((x+y) + \frac{(x+y)^p}{p} + \frac{(x+y)^{p^2}}{p^2} + \cdots\right)\right)$
- We have

$$\begin{aligned}\operatorname{hexp}(x)\operatorname{hexp}(y) &= \operatorname{hexp}(s_1)\operatorname{hexp}(s_p)\operatorname{hexp}(s_{p^2})\cdots \\ \text{where}\quad s_1 &= x+y \\ s_p &= \frac{x^p + y^p - (x+y)^p}{p}\end{aligned}$$

  and the remaining polynomials s_{p^n} can be shown to have p-integral coefficients.

Hexponent and hexponentially closed subgroups

Definitions

- A hexponent subgroup is defined to be a subgroup of the following form: $\operatorname{hexp}(Fx)\operatorname{hexp}(Fx^p)\operatorname{hexp}(Fx^{p^2})\cdots$.
- A subgroup H is said to be hexponentially closed if $\operatorname{hexp}(\gamma x) \in H$ for all $\gamma \in F$ whenever $\operatorname{hexp}(x) \in H$.

Facts

- *Hexponent and hexponentially closed subgroups are strong.*
- *Hexponent subgroups are not necessarily hexponentially closed.*
- *If $J^{2p-1} = 0$, then $\operatorname{hexp}(Fx)\operatorname{hexp}(Fx^p)$ is hexponentially closed.*

Hexponentials and normalizers.

Question

Can we use the hexponential map to show normalizers of algebra subgroups are strong?

Answer

Only if $J^{p+1} = 0$.

Theorem

Let H *be an algebra subgroup of* $G = 1 + J$.

- *If* $J^{p+1} = 0$, *then* $N_G(H)$ *is hexponentially closed (hence strong).*
- *If* $J^{p+1} \neq 0$, *then examples exist for which* $|N_G(H)| = p \cdot q^a$, *and so* $N_G(H)$ *need not be strong.*

Sketch of proof

- We find a function had analogous to ad so that $\mathrm{hexp}(x) \in N_G(H)$ $\iff$ $\mathrm{hexp}(\mathrm{had}\, x)$ stabilizes L $\iff$ $\mathrm{had}\, x$ stabilizes L
- $\mathrm{had}\, x = \mathrm{ad}\, x + \delta_x$ where δ_x is not linear
- If $J^{2p-1} = 0$, $\delta_x = \dfrac{L_x^p - R_x^p - (L_x - R_x)^p}{p}$, where L_x, R_x are left and right multiplication by x, respectively.
- $\delta_x(y) \in J^{p+1}$
- $J^{p+1} = 0 \implies$
  - $\mathrm{had}\, x = \mathrm{ad}\, x$
  - $\mathrm{had}\, x$ stabilizes L $\iff$ $\mathrm{had}(\alpha x)$ stabilizes L for all $\alpha \in F$
  - $N_G(H)$ is hexponentially closed (hence strong)

Sketch of proof, cont'd

- Suppose $J^{p+1} \neq 0$.
- $\mathrm{hexp}(\alpha x) \in N_G(H) \iff \mathrm{had}(\alpha x) = \alpha \,\mathrm{ad}\, x + \alpha^p \delta_x$ stabilizes L
- Now $\mathrm{had}\, x = \mathrm{ad}\, x + \delta_x$ stabilizes $L \iff$
  $\alpha^p \,\mathrm{had}\, x = \alpha^p(\mathrm{ad}\, x + \delta_x) = \alpha^p \,\mathrm{ad}\, x + \alpha^p \delta_x$ stabilizes L for all α.
- So $\mathrm{had}(\alpha x)$ stabilizes $L \iff (\alpha^p - \alpha)\,\mathrm{ad}\, x$ stabilizes L
- Examples exist (for all p) for which this happens $\iff \alpha \in GF(p)$ and so $|N_G(H)| = p \cdot q^a$.
- So if $J^{p+1} \neq 0$ and $|F| = q > p$, examples exist for which normalizers of algebra subgroups are not strong.

Other uses for hexp?

- Algebra subgroups are strong, but is the converse true? Are strong subgroups at least isomorphic to algebra subgroups?
- Not necessarily.
- Suppose $x^{p+1} = 0$, but $x^p \neq 0$; $H = \mathrm{hexp}(Fx)\,\mathrm{hexp}(Fx^p)$.
- H is a strong subgroup of exponent p^2.
- Assume there is an isomorphism $H \to 1 + A$, some F-algebra group.
- $\dim_F(A) = 2$
- There is some $u \in A$ with $o(1 + u) = p^2$, so $u, u^2, \ldots, u^p$ are non-zero, linearly independent, a contradiction, if p is odd.
- Similar example exists for $p = 2$.