An Application of the Artin-Hasse Exponential to Finite Algebra Groups

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Algebra groups

Definition

Let F be a field of characteristic p and order q. Let J be a finite-dimensional, nilpotent, associative F-algebra. Define $G = 1 + J$ (formally). Then G is a finite p-group. Groups of this form are called F-algebra groups. *We will assume this notation throughout.*

Example

Unipotent upper-triangular matrices over F

Theorem (Isaacs 1995)

All irreducible characters of algebra groups have q-power degree.

Algebra subgroups and strong subgroups

- Subgroups: $1 + X$ where $X \subseteq J$ is closed under the operation $(x, y) \mapsto x + y + xy$
  - X need not be an algebra.
- If L is a subalgebra of J, then $1 + L$ is an algebra subgroup of $G = 1 + J$.
- If $H \leq G$ such that $|H \cap K|$ is a q-power for all algebra subgroups K of G, then H is a strong subgroup of G.
  - In particular, H strong $\implies |H|$ is a q-power.
- Algebra subgroups are strong.

Strong subgroups as point stabilizers

Theorem (Isaacs 1995)

Under certain conditions, character stabilizers are strong.

- *If $J^p = 0$, $N \trianglelefteq G$ is an ideal subgroup, and $\theta \in \mathrm{Irr}(N)$, then the stabilizer in G of θ is strong.*
- *If $N \trianglelefteq G$ is an ideal subgroup, and λ is a linear character of N, then the stabilizer in G of λ is strong.*

Question

Are normalizers of algebra subgroups strong?

When $J^p = 0$: The Exponential Map

- If $J^p = 0$, define $\exp : J \to 1 + J$ by the usual power series. For $x \in J$,

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

- Connects multiplication in group to addition and Lie structure in underlying algebra.
- For $x \in J$, $\exp(Fx) = \{\exp(\alpha x) | \alpha \in F\}$ is called an F-exponent subgroup.
- F-exponent subgroups are strong.
- The subgroup H is said to be exponentially closed if $\exp(Fx) \subseteq H$ whenever $\exp(x) \in H$. (Also called partitioned.)
- Exponentially closed subgroups are strong.

Lie algebra structure and exp

- Put a Lie algebra structure on J by defining the Lie bracket $[\cdot,\cdot]\colon J \times J \to J$ by $[x, y] = xy - yx$ for $x, y \in J$.
- For $x \in J$, define $\operatorname{ad} x : J \to J$ by $\operatorname{ad} x(y) = [x, y]$ for $y \in J$.

Identity (anon)

If $J^p = 0$, then $y^{(\exp x)^{-1}} = \exp(\operatorname{ad} x)(y)$ for all $x, y \in J$.

- Let $H = \exp(L)$, where L is a subalgebra of J.
- $N_G(H) = \exp N_J(L)$ where $N_J(L) = \{x : [L, x] \subseteq L\}$.
- $\exp(x) \in N_G(H) \iff \exp(\operatorname{ad} x)$ stabilizes L $\iff \operatorname{ad} x$ stabilizes L

What if $J^p \neq 0$?

- Stick with $x \in J$ for which $x^p = 0$.
- Reduction to the universal case (characteristic 0)
  - Halasi (2004)
- Find another bijection $J \to 1 + J$
  - truncated exponential
  - Previtali (1995): $x + \sqrt{1 + x^2}$ (for odd p only)

Goal

To find an analog/generalization of exp *that works if* $x^p \neq 0$.

The Artin-Hasse exponential

Definition

Fix a prime p and a nilpotent algebra X over a field of characteristic 0 or p. The Artin-Hasse exponential series, $E_p : X \to 1 + X$, is defined by

$$E_p(x) = \exp\left(x + \frac{x^p}{p} + \frac{x^{p^2}}{p^2} + \frac{x^{p^3}}{p^3} + \cdots\right)$$

In characteristic 0:

$$E_p(x) = \exp(x)\exp\left(\frac{x^p}{p}\right)\exp\left(\frac{x^{p^2}}{p^2}\right)\exp\left(\frac{x^{p^3}}{p^3}\right)\cdots$$

Example

In characteristic 0:

$$
\begin{aligned}
E_3(x) &= \exp\left(x + \frac{x^3}{3} + \frac{x^9}{9} + \frac{x^{27}}{27} + \cdots\right) \\
&= \exp(x)\exp\left(\frac{x^3}{3}\right)\exp\left(\frac{x^9}{9}\right)\exp\left(\frac{x^{27}}{27}\right)\cdots \\
&= \left(1 + X + \frac{X^2}{2!} + \cdots\right)\left(1 + \left(\frac{X^3}{3}\right) + \frac{\left(\frac{X^3}{3}\right)^2}{2!} + \cdots\right)\cdots \\
&= 1 + X + \frac{X^2}{2} + \frac{X^3}{2} + \frac{3X^4}{8} + \frac{7X^5}{40} + \cdots .
\end{aligned}
$$

It looks like we can reduce modulo 3:

$$
E_3(X) = 1 + X + 2X^2 + 2X^3 + 0X^4 + X^5 + \cdots .
$$

Another formula for E_p

Theorem (Artin, Hasse 1928)

The coefficients in $E_p(x)$ are p-integral.

Sketch of combinatorial proof.

- $$E_p(x) = \sum \frac{\left|\cup \mathrm{Syl}_p(S_n)\right|}{n!} x^n$$
- Frobenius: The highest power of p that divides $n! = |S_n|$ also divides $\left|\cup \mathrm{Syl}_p(S_n)\right|$.

□

E_p lacks some nice properties of exp

- If $XY = YX$, then $\exp(X)\exp(Y) = \exp(X+Y)$.
- Above example in characteristic 3:

$$\begin{aligned} E_3(X)E_3(Y) &= \left(1 + X + 2X^2 + 2X^3 + \cdots\right)\cdot\left(1 + Y + 2Y^2 + 2Y^3 + \cdots\right) \\ &= 1 + (X+Y) + 2(X+Y)^2 + 2(X^3 + X^2Y + XY^2 + Y^3) \\ &\qquad + (2x^3Y + X^2Y^2 + 2XY^3) + \cdots \end{aligned}$$

On the other hand,

$$\begin{aligned} E_3(X+Y) &= 1 + (X+Y) + 2\,(X+Y)^2 + 2\,(X+Y)^3 + 0\,(X+Y)^4 + \cdots \\ &= 1 + (X+Y) + 2\,(X+Y)^2 + 2\left(X^3 + Y^3\right) + 0\,(X+Y)^4 + \cdots \end{aligned}$$

- So $E_3(X)E_3(Y) \neq E_3(X+Y)$.

An Artin-Hasse exponential law

$$E_p(x)E_p(y) = E_p(X+Y)E_p(\Theta_1) = E_p(X+Y)E_p(S_1)E_p(\Theta_2)$$

$$\text{where} \quad S_1(X,Y) = \frac{X^p + Y^p - (X+Y)^p}{p}$$

Theorem (Blache 2005)

If X and Y are commuting indeterminates, then

$$E_p(X)E_p(Y) = \prod_{k \geq 0} E_p(S_k(X,Y)),$$

where S_k is a homogeneous polynomial of degree p^k having integer coefficients.

- $S_k(X,Y) = s_k(X,0,\ldots,0;Y,0,\ldots,0)$ where s_k is the k-th Witt addition polynomial.

An Artin-Hasse Freshman Dream

Proposition

In characteristic p, $E_p(X)^{p^a} = E_p\left(X^{p^a}\right)$, *for all integers* $a \geq 0$.

$$\begin{aligned}
\text{In char 0:}\quad E_p(X)^p &= \exp\left(X + \frac{X^p}{p} + \frac{X^{p^2}}{p^2} + \frac{X^{p^3}}{p^3} + \cdots\right)^p \\
&= \exp\left(pX + X^p + \frac{X^{p^2}}{p} + \frac{X^{p^3}}{p^2} + \cdots\right) \\
&= \exp(pX)\exp\left(X^p + \frac{(X^p)^p}{p} + \frac{(X^p)^{p^2}}{p^2} + \cdots\right) \\
&= \exp(pX) E_p\left(X^p\right).
\end{aligned}$$

$$\text{So in char } p:\quad E_p(X)^p = E_p\left(X^p\right)$$

Similarly for p^a. □

Subgroups defined by the Artin-Hasse exponential

Definitions

- $E_p(Fx) = \{E_p(\alpha x) | \alpha \in F\}$
  - Usually not closed under multiplication
- AH-exponent subgroup: $\mathcal{E}_p^F(x) = E_p(Fx)E_p(Fx^p)E_p(Fx^{p^2})\cdots$.
- A subgroup H is said to be AH-closed if $E_p(\gamma x) \in H$ for all $\gamma \in F$ whenever $E_p(x) \in H$.

Facts

- $\mathcal{E}_p^F(x) =< E_p(Fx) >$
- $\mathcal{E}_p^F(x)$ *and AH-closed subgroups are strong.*
- $\mathcal{E}_p^F(x)$ *is not necessarily AH-closed.*
- *If* $J^{2p-1} = 0$, *then* $\mathcal{E}_p^F(x) = E_p(Fx)E_p(Fx^p)$ *is AH-closed.*

Application: normalizers of algebra subgroups

Main Theorem

Let J be a finite-dimensional nilpotent associative algebra over a finite field of positive characteristic p and order q. Suppose $G + 1 + J$ and H is an algebra subgroup of G.

- *If $J^{p+1} = 0$, then $N_G(H)$ is AH-closed (hence strong).*
- *If $J^{p+1} \neq 0$, then examples exist for which $|N_G(H)| = p \cdot q^a$, and so $N_G(H)$ need not be strong.*

Sketch of proof

- Let $H = 1 + L$, an algebra subgroup.
- Recall: for $x^p = 0$, $\exp(x) \in N_G(H) \iff \operatorname{ad} x$ stabilizes L
- We want had analogous to ad so that $E_p(x) \in N_G(H) \iff E_p(\operatorname{had} x)$ stabilizes $L \iff \operatorname{had} x$ stabilizes L
- $\operatorname{had} x = \operatorname{ad} x + \theta_x$ where θ_x is not linear
- If $J^{2p-1} = 0$, $\theta_x = \dfrac{L_x^p - R_x^p - (L_x - R_x)^p}{p}$, where L_x, R_x are left and right multiplication by x, respectively.
- $\theta_x(y) \in J^{p+1}$
- $J^{p+1} = 0 \implies$
  - $\operatorname{had} x = \operatorname{ad} x$
  - $\operatorname{had} x$ stabilizes $L \iff \operatorname{had}(\alpha x)$ stabilizes L for all $\alpha \in F$
  - $N_G(H)$ is AH-closed (hence strong)

Sketch of proof, cont'd

- Suppose $J^{p+1} \neq 0$.
  - $E_p(\alpha x) \in N_G(H) \iff \text{had}(\alpha x) = \alpha \, \text{ad}\, x + \alpha^p \theta_x$ stabilizes L
  - Now $\text{had}\, x = \text{ad}\, x + \theta_x$ stabilizes $L \iff$ $\alpha^p \, \text{had}\, x = \alpha^p(\text{ad}\, x + \theta_x) = \alpha^p \, \text{ad}\, x + \alpha^p \theta_x$ stabilizes L for all α.
  - So $\text{had}(\alpha x)$ stabilizes $L \iff (\alpha^p - \alpha)\, \text{ad}\, x$ stabilizes L
  - Examples exist (for all p) for which this happens $\iff \alpha \in GF(p)$ and so $N_G(H)$ is not AH-closed if $|F| = q > p$
  - In fact, in these examples $|N_G(H)| = p \cdot q^a$.
  - So if $J^{p+1} \neq 0$ and $|F| = q > p$, examples exist for which normalizers of algebra subgroups are not strong. □