A power-series description of strong subgroups of finite algebra groups *(preliminary report)*

Darci L. Kracht
darci@math.kent.edu

Advisor: Stephen M. Gagola, Jr.
Kent State University

2011 Zassenhaus Group Theory Conference
Towson University
May 28, 2011

Algebra groups

Definition

Let F be a field of characteristic p and order q. Let J be a finite-dimensional, nilpotent, associative F-algebra. Define $G = 1 + J$ (formally). Then G is a finite p-group. Groups of this form are called F-algebra groups. *We will assume this notation throughout.*

Example

Unipotent upper-triangular matrices over F

Theorem (Isaacs (1995))

All irreducible characters of algebra groups have q-power degree.

Algebra subgroups and strong subgroups

- Subgroups: $1 + X$ where $X \subseteq J$ is closed under the operation $(x, y) \mapsto x + y + xy$
- X need not be an algebra.

Definitions

- If L is a subalgebra of J, then $1 + L$ is an algebra subgroup of $G = 1 + J$.
- If $H \leq G$ such that $|H \cap K|$ is a q-power for all algebra subgroups K of G, then H is a strong subgroup of G.

Fact

Algebra subgroups are strong.

Strong subgroup example

Example

The subgroup

$$H = \left\{ \begin{pmatrix} 1 & \alpha & \binom{\alpha}{2} \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{pmatrix} \middle| \alpha \in F \right\}$$

is a strong subgroup (but not an algebra subgroup) of the algebra group of unipotent 3×3 upper-triangular matrices over F.

(Here $\binom{\alpha}{2} = \frac{\alpha(\alpha-1)}{2}$ is the generalized binomial coefficient.)

Why study strong subgroups?

Strong subgroups play an important role in the following:

- I.M. Isaacs, *Characters of groups associated with finite algebras*, J. Algebra (1995)
  (Strong subgroups are central to the proof of main theorem mentioned above.)
- A. Previtali, *On a conjecture concerning character degrees of some p-groups*, Arch. Math. (1995)
  (Similar results for character degrees of certain p-subgroups of the symplectic, orthogonal, and unitary groups (for p odd). The key to Previtali's proof is that certain sections of the groups are strong.)
- C.A.M. André, *Irreducible characters of groups associated with finite algebras with involution*, J. Algebra (2010)
  (Gets stronger and more general versions of the above results using the fact that certain fixed-point subgroups are strong.)

Strong subgroups as point stabilizers

Theorem (Isaacs 1995)

Under certain conditions, character stabilizers are strong.

- *If* $J^p = 0$, $N \trianglelefteq G$ *is an ideal subgroup, and* $\theta \in \mathrm{Irr}(N)$, *then the stabilizer in* G *of* θ *is strong.*
- *If* $N \trianglelefteq G$ *is an ideal subgroup, and* λ *is a linear character of* N, *then the stabilizer in* G *of* λ *is strong.*

Theorem (2010)

Let H *be an algebra subgroup of* $G = 1 + J$.

- *If* $J^{p+1} = 0$, *then* $N_G(H)$ *is strong.*
- *If* $J^{p+1} \neq 0$, *then examples exist for which* $|N_G(H)| = p \cdot q^a$, *and so* $N_G(H)$ *need not be strong.*

Example when $J^p = 0$: F-exponent subgroups

- If $J^p = 0$, define $\exp : J \to 1 + J$ and $\log : 1 + J \to J$ by the usual power series.

Definitions

- For $x \in J$ and $\alpha \in F$, define $(1+x)^\alpha = \exp(\alpha \log(1+x))$.
- We define an F-exponent subgroup to be a subgroup of the following form:
  - $(1+x)^F = \{(1+x)^\alpha | \alpha \in F\}$

  or equivalently
  - $\exp(F\hat{x}) = \{\exp(\alpha\hat{x}) | \alpha \in F\}$

Fact

F-exponent subgroups are strong.

Intersections of strong subgroups need not be strong.

Next, we construct an example to show that the collection of strong subgroups is not closed under intersection.

Suppose

- $q > p > 2$
- $x \in J$ with $x^p = 0$ but $x^2 \neq 0$
- $\epsilon : F \to F$ is a nonzero additive map with $\epsilon(1) = 0$
- $H = \left\{ (1+x)^\alpha \left(1+x^2\right)^{\epsilon(\alpha)} \middle| \alpha \in F \right\}$

Then

- H is an abelian subgroup of G of order q
- $1 + x \in H$ since $\epsilon(1) = 0$
- H is strong, as we will show.

$H = \left\{ (1+x)^{\alpha} \left(1+x^2\right)^{\epsilon(\alpha)} \middle| \alpha \in F \right\}$

To show H is strong (i.e., $|H \cap K|$ is a q-power $\forall$ algebra subgroups K):

- Let $A \subseteq J$, subalgebra, such that $H \cap (1+A) \neq 1$.
- Then $\exists\, \alpha_0 \in F$, $\alpha_0 \neq 0$ such that $(1+x)^{\alpha_0} \left(1+x^2\right)^{\epsilon(\alpha_0)} \in 1+A$.
- I.e., $(1+\alpha_0 x+\cdots)\left(1+\epsilon(\alpha_0)x^2+\cdots\right) \in 1+A$
- So $\alpha_0 x+\cdots \in A$
- This generates the algebra $xF[x]$, so $xF[x] \subseteq A$.
- So $H \cap (1+A) = H$.
- Therefore, $\forall$ algebra subgroups K of G, $H \cap K = 1$ or $H \cap K = H$
- Conclude: H is a strong subgroup of G.

$H = \left\{ (1+x)^{\alpha} \left(1+x^2\right)^{\epsilon(\alpha)} \middle| \, \alpha \in F \right\}$

Now consider the F-exponent group $(1+x)^F = \{(1+x)^{\alpha} | \, \alpha \in F\}$:

- A strong subgroup of G of order q.
- Distinct from H since ϵ is not the zero map.

Then we have

- $1 + x \in (1+x)^F \cap H$.
- So $1 < \left|(1+x)^F \cap H\right| < q$.
- Thus, $(1+x)^F \cap H$ is not strong.
- Conclude: the intersection of strong subgroups need not be strong.

Ideal frames

Definition

Let J be a nilpotent F-algebra with $\dim_F(J) = n$. An ideal frame of J is a basis $\{v_1, \ldots, v_n\}$ of J satisfying

$$v_i J, J v_i \subseteq \operatorname{Span}\{v_{i+1}, \ldots, v_n\}$$

for all $i = 1, \ldots, n$.

Such bases always exist. For example,

- Refine the chain $J \supset J^2 \supset \cdots \supset J^{m-1} \supset J^m = 0$ to a maximal flag $J = V_1 \supset V_2 \supset \cdots \supset V_{n-1} \supset V_n \supset 0$.
- Choose $v_i \in V_i \setminus V_{i+1}$ for all $i = 1, \ldots, n$.
- Then $\{v_1, \ldots, v_n\}$ is an ideal frame of J.

Stringent power series

Definition

We call a power series $J \to 1 + J$ stringent if it is of the form

$$x \mapsto 1 + x + \alpha_2 x^2 + \alpha_3 x^3 + \cdots$$

where $\alpha_2, \alpha_3, \ldots \in F$.

Example

$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ is stringent.

Let $S : J \to 1 + J$ be a stringent power series.

- Then $S(\alpha x)^{-1} = 1 - \alpha x + \cdots$.
- Define $S(Fx) = \{S(\alpha x) \,|\alpha \in F\}$.
- $S(Fx)$ is a subset, but not necessarily a subgroup, of G.

Expressing elements of G in terms of $f(Fx)$

Lemma

For a finite F-algebra group $G = 1 + J$, suppose

- $\{v_1, \ldots, v_n\}$ *is an ideal frame of J where $n = \dim_F(J)$*
- $S : J \to 1 + J$ *is a stringent power series*
- $V_i = \text{Span}\{v_i, \ldots, v_n\}$.

Then

- $1 + V_i = S(Fv_i)(1 + V_{i+1})$ *for all $i = 1, \ldots, n-1$;*
- *Every element of G has a unique representation of the form $S(\alpha_1 v_1)S(\alpha_2 v_2) \cdots S(\alpha_n v_n)$ where $\alpha_1, \ldots, \alpha_n \in F$.*
- *In particular, $G = \prod_{i=1}^{n} S(Fv_i)$.*

Description of strong subgroups

Main Theorem

For a finite F-algebra group $G = 1 + J$, suppose

- $\{v_1, \ldots, v_n\}$ *is an ideal frame of J where* $n = \dim_F(J)$
- $S : J \to 1 + J$ *is a stringent power series*
- $V_i = \mathrm{Span}\{v_i, \ldots, v_n\}$.

If H is a strong subgroup of G, then there exist

- *a partition* $I \dot{\cup} \hat{I} = \{1, \ldots, n\}$
- *functions* $\epsilon_{ij} : F \to F$ *for all $i \in I$ and $j \in \hat{I}$ with $j > i$*

such that, $\forall \alpha \in F$, $h_i(\alpha) = S(\alpha v_i) \prod_{\substack{j \in \hat{I} \\ j > i}} S\left(\epsilon_{ij}(\alpha) v_j\right)$ *is an element of H.*

Moreover, every $h \in H$ has a unique representation of the form $h = \prod_{i \in I} h_i(\alpha_i)$ *where* $\alpha_i \in F$.

Example

Suppose H is a strong subgroup of $G + 1 + J$ where $J^p = 0$, $\dim_F(J) = 6$, $S(\alpha x) = (1 + x)^\alpha$, and the partition given by the theorem turns out to be $I = \{1, 3, 5\}$ and $\hat{I} = \{2, 4, 6\}$.

Then there are functions $\epsilon_{1,2}, \epsilon_{1,4}, \epsilon_{1,6}, \epsilon_{3,4}, \epsilon_{3,6}, \epsilon_{5,6} : F \to F$ so that

$$h_1(\alpha_1) = (1 + v_1)^{\alpha_1}(1 + v_2)^{\epsilon_{1,2}(\alpha_1)}(1 + v_4)^{\epsilon_{1,4}(\alpha_1)}(1 + v_6)^{\epsilon_{1,6}(\alpha_1)}$$
$$h_3(\alpha_3) = (1 + v_3)^{\alpha_3}(1 + v_4)^{\epsilon_{3,4}(\alpha_3)}(1 + v_6)^{\epsilon_{3,6}(\alpha_3)}$$
$$h_5(\alpha_5) = (1 + v_5)^{\alpha_5}(1 + v_6)^{\epsilon_{5,6}(\alpha_5)}$$

are all elements of H.

Moreover, every $h \in H$ is of the form $h = h_1(\alpha_1)h_3(\alpha_3)h_5(\alpha_5)$ for unique $\alpha_1, \alpha_3, \alpha_5 \in F$.