

MINIMALITY AND NON-DEGENERACY OF DEGREE-ONE GINZBURG-LANDAU VORTEX AS A HARDY'S TYPE INEQUALITY

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ABSTRACT. We consider the unique radially symmetric degree-one solution $w(x) = U(r)e^{i\theta}$ of the Ginzburg-Landau equation in \mathbb{R}^2

$$\Delta w + (1 - |w|^2)w = 0, \quad |w(x)| \rightarrow 1 \text{ as } |x| \rightarrow +\infty.$$

We provide an elementary proof of its locally minimizing character and non-degeneracy up to the obvious invariance of the equation, in the natural Hilbert space for its second variation bilinear form. The proof reduces to an optimal vector-valued form of Hardy's inequality. As a consequence, we establish a Fredholm alternative in this space for the associated linearized operator.

1. INTRODUCTION

We consider the complex-valued Ginzburg-Landau equation in the plane

$$\Delta w + (1 - |w|^2)w = 0, \quad \text{in } \mathbb{R}^2. \quad (1.1)$$

The standard one-vortex solution of degree one in the plane is the solution $w(x)$ of (1.1) of the form

$$w(x) = U(r)e^{i\theta}.$$

Where (r, θ) designate usual polar coordinates $x_1 = r \cos \theta$, $x_2 = r \sin \theta$, and $U(r)$ is the unique solution of the problem

$$\begin{cases} U'' + \frac{U'}{r} - \frac{U}{r^2} + (1 - U^2)U = 0, & \text{in } (0, \infty), \\ U(0) = 0, \quad U(+\infty) = 1. \end{cases} \quad (1.2)$$

It is known that $U'(0) > 0$ and that

$$U(r) \sim 1 - \frac{1}{r^2}, \quad U'(r) \sim \frac{2}{r^3}, \quad \text{as } r \rightarrow +\infty,$$

see for instance [3]. An important feature of this solution is its *locally minimizing character*. The energy functional associated to equation (1.1) is given by

$$E(v) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla v|^2 + \frac{1}{4} \int_{\mathbb{R}^2} (1 - |v|^2)^2.$$

While $E(w) = +\infty$, it turns out that for any ϕ smooth and compactly supported, $E(w) - E(w + \phi) \leq 0$, which implies $B(\phi, \phi) \geq 0$ for all such a

ϕ , where B is the bilinear form given by the (formal) second variation of E around w ,

$$B(\phi, \phi) = \int_{\mathbb{R}^2} |\nabla \phi|^2 - \int_{\mathbb{R}^2} (1 - U^2) |\phi|^2 + 2 \int_{\mathbb{R}^2} |\operatorname{Re}(\bar{w}\phi)|^2.$$

We observe that $B(\phi, \phi) = \langle L(\phi), \phi \rangle$ where here and in what follows

$$\langle u, v \rangle = \operatorname{Re} \int_{\mathbb{R}^2} u \bar{v},$$

and L is the linearization of (1.1) around w ,

$$L(\phi) = \Delta \phi + (1 - U^2)\phi - 2 \operatorname{Re}(\bar{w}\phi)w, \quad \text{in } \mathbb{R}^2. \quad (1.3)$$

Direct substitution shows that

$$L\left(\frac{\partial w}{\partial x_1}\right) = L\left(\frac{\partial w}{\partial x_2}\right) = L(iw) = 0,$$

which gives account of the invariance of equation (1.1) under space translations of the solution and under multiplication by complex scalars of absolute value one, which introduces degeneracy of this minimizer.

The locally minimizing character of w , $B(\psi, \psi) \geq 0$, follows by combining known results in the literature, as pointed out to us by P. Mironescu: A local minimizer v with $v(0) = 0$ can be found by considering global minimizers of the Ginzburg-Landau energy in a ball with large radius and boundary condition $e^{i\theta}$. The analysis in [1] shows that (taking as the origin one of its zeros) these functions converge, up to subsequences, locally over compacts of \mathbb{R}^2 to a solution of (1.1) with $v(0) = 0$, clearly a local minimizer, which besides satisfies $\int_{\mathbb{R}^2} (1 - |u|^2)^2 < +\infty$. From [2] and [11], it follows that v has a degree at infinity which is equal to 1 or -1 . From [8], v is necessarily radial, so that it must be equal to w or to its conjugate. Stability of radial solutions in a ball was previously studied in [6, 7].

A different proof of this result was obtained by Ovchinnikov and Sigal, [9] who analyze the spectrum of the operator L in L^2 and find that it has 0 as its lower limit. More specifically, see also [4, 5] for related results and method, $B(\phi, \phi)$ is positive if ϕ lies in an L^2 -orthogonal to the space

$$\mathcal{Z} = \operatorname{span} \left\{ \frac{\partial w}{\partial x_1}, \frac{\partial w}{\partial x_2}, iw \right\}.$$

Spectral theory for Schrödinger operators and a form of Perron-Frobenius method applied for spectral analysis of each element of a block decomposition of the operator L are the ingredients used in [9, 4, 5].

An observation we should make, however, is that in principle L^2 is not an ideal environment space for L since \mathcal{Z} is not contained in L^2 . In particular a decaying solution of $L(\phi) = h$ for, say, h compactly supported, should not decay in general at a faster rate than that of ∇w . Thus, it is not obvious how to produce a satisfactory solvability theory for this problem from the L^2 -information.

The purpose of this note is to present an elementary, direct proof of the strict minimizing character of w for perturbations ϕ in the “natural” Hilbert space H for the bilinear form B , of all locally H^1 functions for which

$$\|\phi\|_H^2 = \int_{\mathbb{R}^2} [|\nabla\phi|^2 + (1 - U^2)|\phi|^2 + |\operatorname{Re}(\bar{w}\phi)|^2] < +\infty .$$

Theorem 1. *We have that*

$$B(\phi, \phi) \geq 0 \quad \text{for all } \phi \in H. \quad (1.4)$$

Besides, if $\phi \in H$ is such that $B(\phi, \phi) = 0$, then

$$\phi = c_1 \frac{\partial w}{\partial x_1} + c_2 \frac{\partial w}{\partial x_2},$$

for certain real constants c_1, c_2 .

The proof of this result uses the standard observation that B can be decomposed in additive way among different Fourier modes in θ . Let us decompose ϕ into the form

$$\phi = \phi^0 + \sum_{j=1}^{\infty} \phi_j^1 + \sum_{j=1}^{\infty} \phi_j^2, \quad (1.5)$$

where

$$\phi^0 = e^{i\theta} [\phi_1^0(r) + i\phi_2^0(r)], \quad (1.6)$$

$$\phi_j^1 = e^{ij\theta} [\phi_{j1}^1(r) \sin j\theta + i\phi_{j2}^1(r) \cos j\theta], \quad (1.7)$$

and

$$\phi_j^2 = e^{ij\theta} [\phi_{j1}^2(r) \cos j\theta + i\phi_{j2}^2(r) \sin j\theta]. \quad (1.8)$$

Then we get that

$$B(\phi, \phi) = B(\phi^0, \phi^0) + \sum_{j=1}^{\infty} B(\phi_j^1, \phi_j^1) + \sum_{j=1}^{\infty} B(\phi_j^2, \phi_j^2). \quad (1.9)$$

This decomposition is naturally associated to the elements of \mathcal{Z} , since

$$iw = e^{i\theta} [0 + iU(r)],$$

$$\frac{\partial w}{\partial x_1} = e^{i\theta} [U'(r) \cos \theta - i \frac{U(r)}{r} \sin \theta],$$

$$\frac{\partial w}{\partial x_2} = e^{i\theta} [U'(r) \sin \theta + i \frac{U(r)}{r} \cos \theta].$$

We need to establish non-negativity of each of the individual terms in (1.9). The most delicate step is to establish that $B(\phi_1^\ell, \phi_1^\ell) \geq 0$, $\ell = 1, 2$, and that equality holds only if respectively $\phi_1^1 = \frac{\partial w}{\partial x_2}$, $\phi_1^2 = \frac{\partial w}{\partial x_1}$. We present a novel

proof of this fact, which amounts to a Hardy's type inequality for vector-valued functions, (Proposition 1 below) which is interesting in its own right and has an elementary proof.

As a corollary, we will show that the following Fredholm alternative for the operator L holds.

Theorem 2. *Consider the equation*

$$L(\phi) = h, \quad \text{in } \mathbb{R}^2, \quad (1.10)$$

where we assume that for certain $\sigma > 0$ $\int_{\mathbb{R}^2} |h|^2(1 + r^{2+\sigma}) < +\infty$. If additionally we assume

$$\left\langle h, \frac{\partial w}{\partial x_1} \right\rangle = \left\langle h, \frac{\partial w}{\partial x_2} \right\rangle = \langle h, iw \rangle = 0,$$

then (1.10) has a solution $\phi_0 \in H$ which satisfies

$$\|\phi_0\|_H^2 \leq C \int_{\mathbb{R}^2} |h|^2(1 + r^{2+\sigma}).$$

Moreover, all solutions $\phi \in H$ have the form

$$\phi = \phi_0 + c_1 \frac{\partial w}{\partial x_1} + c_2 \frac{\partial w}{\partial x_2},$$

with $c_1, c_2 \in \mathbb{R}$.

A solvability theory for the linearization is of crucial importance in the use of perturbative methods for the construction of vortex solutions of problems where the re-scaled vortex w provides a canonical profile. This is a subject broadly developed in the book [10], where an entirely different approach is used with respect to this issue.

We devote the rest of this paper to the proof of Theorems 1 and 2.

2. NON-DEGENERACY OF w

Let us consider first a smooth function ϕ , compactly supported, whose support does not contain the origin. It is convenient to define ψ by the relation

$$\phi = iw\psi,$$

and introduce the bilinear form

$$\mathbb{B}(\psi, \psi) = B(iw\psi, iw\psi).$$

Then we have, writing $\psi = \psi_1 + i\psi_2$,

$$\mathbb{B}(\psi, \psi) = \int_{\mathbb{R}^2} U^2 |\nabla \psi|^2 - 2 \operatorname{Re} \int_{\mathbb{R}^2} \frac{iU^2}{r^2} \frac{\partial \psi}{\partial \theta} \bar{\psi} + 2 \int_{\mathbb{R}^2} U^4 |\psi_2|^2.$$

On the other hand, defining $\phi^0 = iw\psi^0$ and $\phi_j^\ell = iw\psi_j^\ell$, for $j \in \mathbb{N}$ and $\ell = 1, 2$, and using (1.9), we find

$$\mathbb{B}(\psi, \psi) = \mathbb{B}(\psi^0, \psi^0) + \sum_{j=1}^{\infty} \mathbb{B}(\psi_j^1, \psi_j^1) + \mathbb{B}(\psi_j^2, \psi_j^2).$$

Let us set,

$$\begin{aligned} \psi^0 &= \psi_1^0(r) + i\psi_2^0(r), \\ \psi_j^1 &= \psi_{j1}^1(r) \cos j\theta + i\psi_{j2}^1(r) \sin j\theta, \end{aligned}$$

and

$$\psi_j^2 = \psi_{j1}^2(r) \sin j\theta + i\psi_{j2}^2(r) \cos j\theta.$$

Let us consider functions $\varphi : [0, \infty) \rightarrow \mathbb{R}^2$, and the bilinear forms

$$\mathcal{B}_j^\ell(\varphi, \varphi) = \int_0^\infty rU^2 |\varphi'|^2 + \int_0^\infty rU^2 B_j^\ell \varphi \cdot \varphi. \quad (2.1)$$

where B_j^ℓ is the matrix

$$B_j^\ell = \frac{1}{r^2} \begin{pmatrix} j^2 & (-1)^\ell 2j \\ (-1)^\ell 2j & j^2 + 2U^2 r^2 \end{pmatrix}. \quad (2.2)$$

With these definitions it is direct to check that the following fact holds:

$$\mathbb{B}(\psi_j^\ell, \psi_j^\ell) = \pi \mathcal{B}_j^\ell(\varphi_j^\ell, \varphi_j^\ell), \quad \ell = 1, 2, \quad (2.3)$$

where

$$\varphi_j^\ell(r) = (\psi_{j1}^\ell(r), \psi_{j2}^\ell(r)), \quad \ell = 1, 2. \quad (2.4)$$

At the core of the proof of Theorem 1 is the positivity of the bilinear forms \mathcal{B}_1^1 , which can be written as

$$\mathcal{B}_1^1(\varphi, \varphi) = \int_0^\infty rU^2 [|\varphi'|^2 + \frac{1}{r^2} |\varphi|^2 - \frac{4}{r^2} \varphi_1 \varphi_2 + 2U^2 |\varphi_2|^2] \geq 0. \quad (2.5)$$

This inequality is a vector-valued form of Hardy's inequality. In fact, Hardy's inequality for radially symmetric functions in \mathbb{R}^N , $N \geq 3$ asserts that

$$\int_0^\infty |u'|^2 r^{N-1} dr - \left(\frac{N-2}{2}\right)^2 \int_0^\infty \frac{|u|^2}{r^2} r^{N-1} dr \geq 0.$$

Let us observe that $\mathcal{B}_1^1(\varphi, \varphi) \geq 0$ for $\varphi = (v, -v)$ means

$$\int_0^\infty |v'|^2 U^2 r dr - \int_0^\infty \frac{|v|^2}{r^2} U^2 r dr + \int_0^\infty v^2 U^4 r dr \geq 0.$$

Near $r = 0$, $U^2(r)r \sim r^3$. Replacing v by $u(\frac{r}{\delta})$ with u compactly supported, and taking limit as $\delta \rightarrow 0$ we arrive to

$$\int_0^\infty |u'|^2 r^3 dr - \int_0^\infty \frac{|u|^2}{r^2} r^3 dr \geq 0,$$

which is precisely the optimal Hardy's inequality in dimension $N = 4$. In precise terms the result we obtain is the following.

Proposition 1. *For any \mathbb{R}^2 -valued smooth function φ , with compact support away from the origin we have,*

$$\mathcal{B}_1^1(\varphi, \varphi) = \int_0^\infty U^2 r |\varphi' - A_1(r)\varphi|^2 dr,$$

where $A_1(r)$ is a 2×2 matrix of smooth functions in $(0, \infty)$ with the property that the only solutions of the system $\varphi' = A_1(r)\varphi$ such that $\int_0^1 |\varphi|^2 U^2 r dr < +\infty$ are given by constant multiples of $\varphi_0(r) = (\frac{1}{r}, \frac{U'}{U})$.

Proof of Proposition 1. We will write $\mathcal{B}_1^1(\varphi, \varphi)$ in the form

$$\mathcal{B}_1^1(\varphi, \varphi) = \int_0^\infty U^2 r |\varphi' - A_1(r)\varphi|^2 dr, \quad (2.6)$$

where $A_1(r)$ is a two by two symmetric matrix of functions which we shall determine next. First we expand

$$\int_0^\infty U^2 r |\varphi' - A_1(r)\varphi|^2 = \int_0^\infty r U^2 |\varphi'|^2 - 2 \int_0^\infty r U^2 A_1 \varphi' \cdot \varphi + \int_0^\infty r U^2 A_1^2 \varphi \cdot \varphi.$$

Now,

$$2 \int_0^\infty r U^2 A_1 \varphi' \cdot \varphi = \int_0^\infty \frac{d}{dr} (r U^2 A_1 \varphi \cdot \varphi) - \int_0^\infty (r U^2 A_1)' \varphi \cdot \varphi.$$

Since φ is compactly supported in $(0, \infty)$ we get

$$\int_0^\infty U^2 r |\varphi' - A_1 \varphi|^2 = \int_0^\infty r U^2 |\varphi'|^2 + \int_0^\infty (r U^2 A_1)' \varphi \cdot \varphi + \int_0^\infty r U^2 A_1^2 \varphi \cdot \varphi.$$

Thus the requirement (2.6) is equivalent to

$$(r U^2 A_1)' + r U^2 A_1^2 = B_1^1 \quad (2.7)$$

where B_1^1 is the matrix defined in (2.2) with $j = 1$. Let us write

$$A_1 = \begin{pmatrix} a & c \\ c & b \end{pmatrix}.$$

Let us write $\varphi_0 = (\rho_1, \rho_2)$. Since $\mathcal{B}_1^1(\varphi_0, \varphi_0) = 0$, the matrix A_1 should satisfy

$$\varphi_0' = A_1 \varphi_0.$$

This yields the relations

$$a = \frac{\rho_1' - c \rho_2}{\rho_1}, \quad b = \frac{\rho_2' - c \rho_1}{\rho_2}.$$

Now, substituting these relations into (2.7), direct inspection leads to the fact that (2.7) holds if and only if

$$(U^2 r c)' = U^2 r c^2 \left(\frac{\rho_2}{\rho_1} + \frac{\rho_1}{\rho_2} \right) - U^2 r c \left(\frac{\rho_1'}{\rho_1} + \frac{\rho_2'}{\rho_2} \right) - \frac{2U^2}{r},$$

where, we recall, $\rho_1 = \frac{1}{r}$, $\rho_2 = \frac{U'}{U}$. Expanding the above equation we find

$$c' = -c\left(\frac{U''}{U} + \frac{U'}{U}\right) + c^2\left(\frac{\rho_2}{\rho_1} + \frac{\rho_1}{\rho_2}\right) - \frac{2}{r^2}. \quad (2.8)$$

This is a Ricatti equation. Let us observe that if u satisfies the equation

$$(p(r)u')' + q(r)u + p(r)z(r)u' = 0,$$

then $c = -pu'/u$ satisfies

$$c' = -\frac{(pu')'}{u} + p\frac{u'^2}{u^2} = q(r) + \frac{c^2}{p} - cz.$$

Let us set

$$p = \frac{\rho_1\rho_2}{\rho_2^2 + \rho_1^2}, \quad q = -\frac{2}{r^2},$$

$$z = \left(\frac{U'}{U} + \frac{U''}{U'}\right).$$

Then $c = -pu'/u$ satisfies equation (2.8) on $(0, \infty)$ if u is a positive solution of

$$\left(\frac{\rho_1\rho_2UU'}{\rho_2^2 + \rho_1^2}u'\right)' - \frac{2UU'}{r^2}u = 0, \quad r \in (0, \infty). \quad (2.9)$$

Observe that

$$\frac{\rho_1\rho_2UU'}{\rho_2^2 + \rho_1^2} = \frac{U'^2r}{1 + \frac{rU'}{U}}.$$

It is easy to check that this equation (2.9) has a solution $u(r)$ which decays to zero and it is positive as $r \rightarrow +\infty$; in fact as $r \rightarrow +\infty$ this equation resembles

$$(r^{-5}u')' - 2r^{-5}u = 0$$

or

$$u'' - \frac{5}{r}u' - 2u = 0,$$

which is a Bessel's type equation with a decaying solution $u(r) \sim r^{5/2}e^{-\sqrt{2}r}$. We can find with the use of barriers, a solution $u(r)$ of (2.9) with this property. Let us observe that then $u(r)$ is actually positive all over $r \in (0, \infty)$, since equation (2.9) satisfies maximum principle. As $r \rightarrow 0$ the equation gets similar to $(ru')' - \frac{4}{r}u = 0$ or

$$r^2u'' + ru' - 4u = 0.$$

Hence the behavior of u is like $u(r) \sim r^{-2}$ as $r \rightarrow 0$. In fact an application of Frobenius method for equation (2.9) provides this fact. Summarizing, we conclude that equation (2.8) has a globally defined, positive solution $c(r)$, $r \in (0, \infty)$, where $c = -pu'/u$. Since, from its definition $p(0^+) = \frac{1}{2}$, we get

$c(r) = \frac{1}{r} + o(r^{-1})$ as $r \rightarrow 0$. The matrix A_1 as desired has thus been built. Moreover, the following property for $A_1(r)$ is automatically checked:

$$A_1(r) = \begin{pmatrix} -\frac{2}{r} & \frac{1}{r} \\ \frac{1}{r} & -\frac{2}{r} \end{pmatrix} + o(r^{-1})$$

as $r \rightarrow 0$. The system $\varphi' = A_1(r)\varphi$ has as a solution $\varphi_0(r) = (\frac{1}{r}, \frac{U'}{U})$. Let $\varphi_1(r)$ be a second, linearly independent solution. Then Liouville's formula for the Wronskian gives us that

$$W(\varphi_0, \varphi_1) = C e^{-\int_r^{r_0} \text{tr}[A_1(s)] ds} \sim \frac{C}{r^4}$$

It follows that $|\varphi_1(r)| \geq \frac{C}{r^3}$ for all small $r > 0$. The conclusion is that the unique solutions of $\varphi' = A_1(r)\varphi$ for which $\int_0^1 |\varphi(r)|^2 U^2 r dr < +\infty$ are scalar multiples of φ_0 , and the proof of the proposition is complete. \square

Corollary 1. *For any \mathbb{R}^2 -valued smooth function φ , with compact support away from the origin we have,*

$$\mathcal{B}_1^2(\varphi, \varphi) = \int_0^\infty U^2 r |\varphi' - A_2(r)\varphi|^2 dr,$$

where $A_2(r)$ is a 2×2 matrix of smooth functions in $(0, \infty)$ with the property that the only solutions of the system $\varphi' = A_2(r)\varphi$ such that $\int_0^1 |\varphi|^2 U^2 r dr < +\infty$ are given by constant multiples of $\bar{\varphi}_0(r) = (\frac{1}{r}, -\frac{U'}{U})$.

Proof. It is enough to apply the Proposition 1 with $\bar{\varphi} = (\varphi_1, -\varphi_2)$ and consider

$$A_2 = \begin{pmatrix} a & -c \\ -c & b \end{pmatrix},$$

where a, b and c were defined in the proof of the proposition. \square

Proof of Theorem 1.

We have to estimate from below the quantities $\mathcal{B}_j^\ell(\varphi_j^\ell, \varphi_j^\ell)$, $j \geq 1$, $\ell = 1, 2$ with \mathcal{B}_j the bilinear form given by (2.1) and φ_j^ℓ defined by (2.4). Let us assume that $j \geq 2$. Then

$$\mathcal{B}_j^\ell(\varphi, \varphi) = \int_0^\infty r U^2 |\varphi'|^2 + \int_0^\infty r U^2 B_j^\ell \varphi \cdot \varphi,$$

where B_j^ℓ is given by (2.2). We observe that

$$(B_j^\ell - B_1^\ell) \varphi \cdot \varphi \geq \frac{j-1}{r^2} \begin{pmatrix} j+1 & (-1)^{\ell 2} \\ (-1)^{\ell 2} & j+1 \end{pmatrix} \varphi \cdot \varphi \geq \frac{(j-1)^2}{r^2} |\varphi|^2.$$

Hence we find

$$\mathcal{B}_j^\ell(\varphi, \varphi) \geq (j-1)^2 \int_0^\infty \frac{U^2}{r^2} |\varphi|^2 r dr.$$

Gathering the above estimates we have then found the following inequalities

$$\sum_{j=1}^{\infty} \mathbb{B}(\psi_j^\ell, \psi_j^\ell) \geq \int_0^\infty |\varphi_1^{\ell'} - A_\ell(r)\varphi_1^\ell|^2 U^2 r dr + \sum_{j=2}^{\infty} (j-1)^2 \int_{\mathbb{R}^2} \frac{U^2}{r^2} |\psi_j^\ell|^2,$$

$\ell = 1, 2$. On the other hand, let us observe that

$$\mathbb{B}(\psi^0, \psi^0) = \int_{\mathbb{R}^2} U^2 |\nabla \psi^0|^2 + 2 \int_{\mathbb{R}^2} U^4 |\psi_2^0|^2. \quad (2.10)$$

These facts give the following inequality: Whenever ϕ is smooth and compactly supported away from $r = 0$, with $\phi = iw\psi$, we have

$$\begin{aligned} B(\phi, \phi) &\geq \int_{\mathbb{R}^2} U^2 |\nabla \psi^0|^2 + 2 \int_{\mathbb{R}^2} U^4 |\psi_2^0|^2 \\ &\quad + \sum_{\ell=1}^2 \int_0^\infty |\varphi_1^{\ell'} - A_\ell(r)\varphi_1^\ell|^2 U^2 r dr \\ &\quad + \sum_{\ell=1}^2 \sum_{j=2}^{\infty} (j-1)^2 \int_{\mathbb{R}^2} |\psi_j^\ell|^2 \frac{U^2}{r^2}. \end{aligned} \quad (2.11)$$

We claim that inequality (2.11) remains true for ϕ smooth and with compact support now containing the origin. Let $\eta(s)$ be a smooth cut-off with $\eta = 0$ for $s < 1$ and $\eta = 1$ for $s > 2$, and set $\eta_\sigma(r) = \eta(r/\delta)$, then (2.11) is valid for ϕ replaced by $\eta_\delta \phi$. Now,

$$\int_{\mathbb{R}^2} |\nabla(\eta_\delta \phi)|^2 = \int_{\mathbb{R}^2} \eta_\delta^2 |\nabla \phi|^2 + 2 \int_{\mathbb{R}^2} \psi \eta_\delta \nabla \eta_\delta \nabla \phi + \int_{\mathbb{R}^2} \psi^2 |\nabla \eta_\delta|^2.$$

Integrating by parts,

$$2 \int_{\mathbb{R}^2} \phi \eta_\delta \nabla \eta_\delta \nabla \phi = - \int_{\mathbb{R}^2} \phi^2 |\nabla \eta_\delta|^2 - \int_{\mathbb{R}^2} \phi^2 \Delta \eta_\delta.$$

Now,

$$\int_{\mathbb{R}^2} \phi^2 \Delta \eta_\delta = \int_{\mathbb{R}^2} \phi^2 (\delta x) \Delta \eta dx.$$

Thus

$$\lim_{\delta \rightarrow 0} \int_{\mathbb{R}^2} \psi^2 \Delta \eta_\delta = \phi^2(0) \int_{\mathbb{R}^2} \Delta \eta = 0.$$

Combining the above computations we find

$$\lim_{\delta \rightarrow 0} \int_{\mathbb{R}^2} |\nabla(\eta_\delta \phi)|^2 = \int_{\mathbb{R}^2} |\nabla \phi|^2.$$

Using this, a density argument, and Fatou's lemma, inequality (2.11) is readily obtained, not only for ϕ compactly supported but actually for any $\phi \in H$. In particular $B(\phi, \phi) \geq 0$ for any $\phi \in H$.

Finally, let us assume that $\phi \in H$ is such that $B(\phi, \phi) = 0$. Inequality (2.11) then clearly implies $\psi^0 = 0$, $\psi_\ell^j = 0$ for all $j \geq 2$, $\ell = 1, 2$. Besides, we also have for $r > 0$ the validity of the differential equations

$$\varphi_1^{\ell'}(r) = A_\ell(r)\varphi_1^\ell(r).$$

On the other hand $\phi \in H$ implies $\int_0^1 |\varphi_1^\ell|^2 U^2 r dr < +\infty$. We then get $\varphi_1^\ell = c_\ell (\frac{1}{r}, (-1)^\ell \frac{U'}{U})$. This translates exactly into the desired form for ϕ , thus concluding the proof. \square

3. FREDHOLM ALTERNATIVE: PROOF OF THEOREM 2

Finding a solution of $L(\phi) = h$ in H corresponds to finding a critical point in H of the functional

$$J(\phi) = \frac{1}{2} B(\phi, \phi) - \langle \phi, h \rangle.$$

Assume that h has the form described in Theorem 2. We decompose h in the following way

$$h = h^0 + \sum_{j=1}^{\infty} h_j^1 + \sum_{j=1}^{\infty} h_j^2,$$

where now

$$\begin{aligned} h^0 &= e^{i\theta} [h_0(r) + ih_1(r)] , \\ h_j^1 &= e^{i\theta} [h_{j1}^1(r) \sin j\theta + ih_{j2}^1(r) \cos j\theta] , \\ h_j^2 &= e^{i\theta} [h_{j1}^2(r) \cos j\theta + ih_{j2}^2(r) \sin j\theta] . \end{aligned}$$

Let us decompose ϕ as in (1.5). Then the equation $L(\phi) = h$ is equivalent to solving each of the individual equations

$$L(\phi^0) = h^0, \tag{3.1}$$

and

$$L(\phi_j^\ell) = h_j^\ell, \quad j \in \mathbb{N}, \ell = 1, 2, \tag{3.2}$$

where ϕ^0, ϕ_j^ℓ have the form in (1.6)-(1.8). We begin by solving problem (3.2) for $\ell = 1, j = 1$. Let H_* be the space of functions $\tilde{\phi}(r) = (\tilde{\phi}_1(r), \tilde{\phi}_2(r))$, $r \in (0, \infty)$ such that $\phi = e^{i\theta} [\tilde{\phi}_1(r) \sin \theta + i\tilde{\phi}_2(r) \cos \theta] \in H$, endowed with norm $\|\tilde{\phi}\|_{H_*} = \|\phi\|_H$. Explicitly, we choose the (equivalent) norm,

$$\|\tilde{\phi}\|_{H_*}^2 = \int_0^\infty [|\tilde{\phi}'|^2 + \frac{1}{r^2}(\tilde{\phi}_1 - \tilde{\phi}_2)^2 + (1 - U^2)|\tilde{\phi}|^2 + U^2|\tilde{\phi}_2|^2] r dr.$$

We also set

$$\tilde{h}(r) = (\tilde{h}_1(r), \tilde{h}_2(r)) = (h_{11}^1(r), h_{12}^1(r)).$$

Solving (3.2) for $\ell = 1, j = 1$ in H corresponds exactly to finding a critical point of the functional J_1 in H_* defined as

$$J_1(\tilde{\phi}) = \frac{1}{2} \mathbf{B}(\tilde{\phi}, \tilde{\phi}) - \int_0^\infty \tilde{h}(r) \cdot \tilde{\phi}(r) r dr ,$$

where

$$\mathbf{B}(\tilde{\phi}, \tilde{\phi}) = \mathcal{B}_1^1(\varphi_1^1, \varphi_1^1)$$

and where \mathcal{B}_1^1 is given by (2.5) and $\varphi_1^1 = U^{-1}(\tilde{\phi}_2, -\tilde{\phi}_1)$. Theorem 1 referred in terms of the bilinear form \mathbf{B} reads like this: For any $\tilde{\phi} \in H_*$ we have the inequality

$$\mathbf{B}(\tilde{\phi}, \tilde{\phi}) \geq 0.$$

Equality holds if and only if $\tilde{\phi} = C(-U', \frac{U}{r})$, for some $C \in \mathbb{R}$. Set $Z_0(r) = (-U', \frac{U}{r})$ and observe that from the assumptions on h we get

$$\int_0^\infty \tilde{h}(r) \cdot Z_0(r) r dr = 0.$$

We will find a minimizer of $J_1(\tilde{\phi})$ in a suitable subspace of H_* . To do so we need to establish the following:

Lemma 1. *There exists a constant $C > 0$ such that for any $\tilde{\phi} \in H_*$ with*

$$\int_0^\infty (1 - U^2) \tilde{\phi} \cdot Z_0 r dr = 0,$$

one has

$$C \|\tilde{\phi}\|_{H_*}^2 \leq \mathbf{B}(\tilde{\phi}, \tilde{\phi}).$$

Proof of Lemma 1. We start by observing the following

$$\mathbf{B}(\tilde{\phi}, \tilde{\phi}) = \int_0^\infty [|\tilde{\phi}'|^2 + \frac{2}{r^2}(\tilde{\phi}_1 - \tilde{\phi}_2)^2 - (1 - U^2)|\tilde{\phi}|^2 + 2U^2|\tilde{\phi}_2|^2] r dr.$$

Since $1 - U^2(r) \sim \frac{1}{r^2}$ for large r , we see that, given $\delta > 0$ there exists $R > 0$ such that for all $r > R$ we have

$$\begin{aligned} \frac{2 - \delta}{r^2}(\tilde{\phi}_1 - \tilde{\phi}_2)^2 - (1 - \delta)(1 - U^2)|\tilde{\phi}|^2 + (2 - \delta)U^2|\tilde{\phi}_2|^2 &\geq \\ \frac{1 - 2\delta}{r^2}|\tilde{\phi}|^2 + (2 - 2\delta)|\tilde{\phi}_2|^2 - \frac{2}{r^2}|\tilde{\phi}_1||\tilde{\phi}_2| &\geq \frac{1}{2r^2}|\tilde{\phi}|^2. \end{aligned}$$

It then follows that for certain positive numbers C_1, C_2 we have

$$\mathbf{B}(\tilde{\phi}, \tilde{\phi}) \geq C_1 \|\tilde{\phi}\|_{H_*}^2 - C_2 \int_0^R |\tilde{\phi}|^2 r dr. \quad (3.3)$$

Now, in order to establish the lemma, we assume the opposite, namely existence of a sequence $\tilde{\phi}_n$ with $\|\tilde{\phi}_n\|_{H_*} = 1$ such that $\int_0^\infty (1 - U^2) \tilde{\phi}_n \cdot Z_0 r dr = 0$ and $\mathbf{B}(\tilde{\phi}_n, \tilde{\phi}_n) \rightarrow 0$. Let $\hat{\phi}$ be a weak limit of $\tilde{\phi}_n$ in the sense of $\|\cdot\|_{H_*}$. We claim that $\hat{\phi} \neq 0$. Indeed, $\tilde{\phi}_n \rightarrow \hat{\phi}$ locally strongly in L^2 -sense. Hence if $\hat{\phi} = 0$ we would have $\int_0^R |\tilde{\phi}_n|^2 r dr \rightarrow 0$, and estimate (3.3) would yield $\|\tilde{\phi}_n\|_{H_*} \rightarrow 0$, which is impossible. Strong L^2 convergence over compacts and weak lower semi-continuity of L^2 norms give $\mathbf{B}(\hat{\phi}, \hat{\phi}) = 0$. But, then we must have that $\hat{\phi} = CZ_0$. Weak convergence in $\|\cdot\|_{H_*}$ norm finally gives $\int_0^\infty (1 - U^2) \hat{\phi} \cdot Z_0 r dr = 0$, so $C = 0$, a contradiction that proves the lemma. \square

We consider the problem of minimizing the functional J_1 in the closed subspace of H_* ,

$$H_0 = \{\tilde{\phi} \in H_* / \int_0^\infty (1 - U^2) \tilde{\phi} \cdot Z_0 r dr = 0\}.$$

Let us observe that by assumption on h ,

$$\int_0^\infty |\tilde{h}|^2(1+r^{2+\sigma})rdr \leq \int_{\mathbb{R}^2} |h|^2(1+r^{2+\sigma})$$

and additionally that $\int_0^\infty \tilde{h} \cdot Z_0 r dr = 0$. From Lemma 1, it easily follows that the functional J_1 is continuous, coercive and strictly convex in H_0 . Hence there is a unique minimizer $\tilde{\phi}$ for this functional. Obviously, $\tilde{\phi}$ satisfies

$$\mathbf{B}(\tilde{\phi}, \eta) + \int_0^\infty \eta \cdot h r dr = 0,$$

for all $\eta \in H_0$. Now, any $\eta \in H_*$ can be decomposed as $\eta = \eta_1 + C_\eta Z_0$ with $\eta_1 \in H_0$, therefore the above equation is actually satisfied for all $\eta \in H_*$. This by definition means that $\tilde{\phi}$ is a critical point of J_1 in the whole H_* . Moreover, by this construction we easily see that

$$\int_0^\infty [|\tilde{\phi}'|^2 + (1-U^2)|\tilde{\phi}|^2] r dr \leq C \int_0^\infty |\tilde{h}|^2(1+r^2) r dr.$$

It is straightforward to check that the inherited solution ϕ_1^1 of (3.2) $\ell = 1$, $j = 1$ indeed satisfies

$$\|\phi_1^1\|_H^2 \leq C \int_{\mathbb{R}^2} |h_1^1|^2(1+r^{2+\sigma}).$$

In exactly symmetric way we find a solution ϕ_1^2 of $L(\phi_1^2) = h_1^2$ with analogous estimate. We will solve the remaining equations (3.2) for $j \geq 2$, all at once: Let us set

$$h^\perp = \sum_{j \geq 2} h_j^1 + h_j^2, \quad \phi^\perp = \sum_{j \geq 2} \phi_j^1 + \phi_j^2.$$

We then consider the equation

$$L(\phi^\perp) = h^\perp,$$

in the closed subspace H^\perp of all functions $\phi \in H$ that can be written in the form ϕ^\perp . A minimizer of the functional

$$J(\phi^\perp) = \frac{1}{2}B(\phi^\perp, \phi^\perp) - \langle h^\perp, \phi^\perp \rangle,$$

in H^\perp automatically gives a solution. Existence of such a minimizer is in this case a direct matter since we have from (2.11) the inequality

$$B(\phi^\perp, \phi^\perp) \geq c \int_{\mathbb{R}^2} \frac{1}{r^2} |\phi^\perp|^2,$$

with $c > 0$, from where it is straightforward to deduce

$$B(\phi^\perp, \phi^\perp) \geq c \|\phi^\perp\|_H^2,$$

with $c > 0$. Finally let us consider the mode zero case, (3.1). Then in terms of $\psi = -i\mathbf{w}^{-1}\phi^0 = \psi_1(r) + i\psi_2(r)$ we get two uncoupled equations:

$$\psi_1'' + \left(\frac{1}{r} + \frac{2U'}{U} \right) \psi_1' = g_1, \quad (3.4)$$

$$\psi_2'' + \left(\frac{1}{r} + \frac{2U'}{U}\right)\psi_2' - 2U^2\psi_2 = g_2, \quad \text{for } 0 < r < \infty, \quad (3.5)$$

where $-iw^{-1}h^0 = g_1(r) + ig_2(r)$. By the assumption made on h we have the properties,

$$\int_0^\infty rU^2g_1 = 0, \quad \int_0^\infty rU^2g_1^2(1+r^{2+\sigma})dr \leq C \int_{\mathbb{R}^2} |h|^2(1+r^{2+\sigma}).$$

The following explicit formula is then directly checked to represent a solution of (3.4):

$$\psi_1(r) = - \int_r^\infty \frac{ds}{sU^2(s)} \int_0^s g_1(t)tU^2(t)dt.$$

Moreover, ψ_1 satisfies

$$\int_0^\infty [|\psi_1'(r)|^2 + (1-U^2)\psi_1^2]rU^2dr \leq C \int_{\mathbb{R}^2} |h|^2(1+r^{2+\sigma}).$$

On the other hand, equation (3.5) has a solution which simply minimizes the functional

$$\frac{1}{2} \int_0^\infty [|\psi_1'(r)|^2 + 2U^2|\psi_2|^2]rU^2dr + \int_0^\infty g_2\psi_2rU^2dr,$$

in its natural H^1 -weighted space. With these definitions we inherit for ϕ^0 the estimate

$$\|\phi^0\|_H^2 \leq C \int_{\mathbb{R}^2} |h|^2(1+r^{2+\sigma}).$$

Adding up the above constructed solutions we have found a solution ϕ_0 of $L\phi = h$ with the required property. The fact that all solutions in H can be written as the sum of ϕ_0 and a linear combination of the partial derivatives of w follows immediately from Theorem 1. This concludes the proof. \square

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