Consider $f_1(z) = \begin{cases} \frac{f(z)}{z}, & |z| < 1, \ z \neq 0 \\ f'(0) = \lim_{z \to 0} \frac{f(z)}{z}, & z = 0 \end{cases}$

$f_1$ is analytic in $D$.

Let $0 < r < 1$ and $\overline{D}_r = \{z \mid |z| < r\}$. Now $f_1$ is analytic on $\overline{D}_r$.

By remark 2, on previous page, from maximum principle, the maximum for $f_1(z)$ on $\overline{D}_r$ occurs on the boundary, $\partial \overline{D}_r$; so for $r \in \overline{D}_r$, $|f_1(z)| \leq \frac{1}{r}$. As $r \to 1$, $|\frac{f(z)}{z}| \leq 1 \implies |f(z)| \leq |z| \implies f'(0) \leq 1$

Now, suppose equality holds at one point. Then by the maximum principle, remark 1, $f_1(z)$ is constant: $f_1(z) = c$. But $|f_1(z)| = |\frac{f(z)}{z}|$ and $f(z) = z$ at 1 pt., so $|f_1(z)| = 1$.

$\implies |c| = 1$ and $f(z) = cz$. This will lead to many consequences, including a non-Euclidean geometry.

General situation: $f(z)$ analytic in $|z| < R$, $z_0$, $|z_0| < R$.

$|f(z)| \leq M$. $f(z_0) = w_0$, $|w_0| < M$

We will use linear fractional transformations to be able to apply Schwarz Lemma.

\[ T(x) = \frac{Tz + 1}{Tz - 1} \]
T: \( \{ z \mid |z| < R \} \rightarrow \{ z \mid |z| < 1 \} \) with \( T_{z_0} = 0 \)

\[
T(z) = \frac{R(z-z_0)}{R^2 - \overline{z}_0z} \quad (z_0 \rightarrow 0 \text{ and } \overline{z}_0 \rightarrow \infty)
\]

\( S: \{ w \mid |w| < M^2 \} \rightarrow \{ z \mid |z| = 1 \} \quad S_{w_0} = 0 \)

\[
S(w) = \frac{M(w-w_0)}{M^2 - \overline{w}_0w}
\]

Consider \( g(z) = Sf(T^{-1}z) \) \quad Take \( z \) in \( |z| < 1 \). Then take \( T^{-1} \) to get back to \( |z| < R \); Apply \( f \) to find \( f(z) = w \in \{ w \mid |w| < M^2 \} \)

Then apply \( S \) to get inside the unit ball again.

Apply Schwarz lemma in \( g(z) : z = Tz \quad |g(z)| = |z| \)

\[
|Sf(z)| \leq |T(z)|
\]

\[
\frac{|M(f(z)-w_0)|}{M^2 - \overline{w}_0f(z)} \leq \left| \frac{R(z-z_0)}{R^2 - \overline{z}_0z} \right|.
\]

This is called the general form of Schwarz Inequality.

It is one of the possible applications of the maximum principle.

Cauchy's Thm: we have proved it for the situation: \( f \) analytic in a disk \( D \) with \( \gamma \) a closed curve, \( \gamma \subset D \).

\[
\int_{\gamma} f(z) \, dz = 0.
\]

This is a local situation. To make this global we will use an approach of Artin (1930's \( \rightarrow \) 1940's)
If the domain is "meagre", we need to consider other things.

What is the "domain" in general where \( \int f(z) \, dz = 0 \)?

We will generalize the notion of a closed curve to arbitrary cycles. Consider the arcs \( \gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_n \), each piecewise differentiable. The formal sum is \( \gamma_1 + \gamma_2 + \ldots + \gamma_n \), called a chain, often considered the union of the arcs. Let \( \Omega \) be a domain so all \( \gamma_i \) arcs in \( \Omega \), with \( f \) continuous in \( \Omega \). \( \int_\gamma f(z) \, dz = \int_{\gamma_1} f(z) \, dz + \int_{\gamma_2} f(z) \, dz + \ldots + \int_{\gamma_n} f(z) \, dz. \)

We can add chains. We can multiply chains by a positive integer. -1 to change the orientation.

**Remark** If \( \gamma_1 \) and \( \gamma_2 \) are chains, then we will have linearity; i.e., \( \int_\gamma f(z) \, dz = \int_{\gamma_1} f(z) \, dz + \int_{\gamma_2} f(z) \, dz \). But we will need cycles. **Defn** a cycle is a chain which can be represented as a sum of a finite number of closed curves. \( \int_\gamma f(z) \, dz \) Sum is a cycle.

**Remark** Results about integration over closed curves can be reformulated for cycles.

**Ex** Let \( \Omega \) be a domain with \( \gamma \) a cycle in \( \Omega \). If \( p \) and \( q \) are continuous, then with \( \int p \, dx + q \, dy = dU \), an exact differential, then \( \int_\gamma p \, dx + q \, dy = 0 \); i.e., \( \frac{\partial U}{\partial x} = p \), \( \frac{\partial U}{\partial y} = q \).
Let \( \gamma \) be a cycle, \( a \in C \setminus \gamma \).

Then the index of \( \gamma \) w.r.t. \( \gamma \): \( n(\gamma, a) = \frac{1}{2\pi i} \oint_{\gamma} \frac{dz}{z-a} \) which is just the sum of indices: \( n(\gamma, a) = \sum_{j=1}^{m} n(\gamma_j, a) \).

So, we are trying to find the most general domains where Cauchy's Thm. holds. It turns out these are just what are called simply connected domains.

**Defn:** A domain \( \Omega \subset C \) is called simply connected if \( \overline{C \setminus \Omega} \) is connected. A topologist would argue that this is not a good way to do this, so this is nonstandard defn.

**Remark:** Note that the complement is not connected:

1. \( \Omega \) is a domain without holes.
2. defn. is nonstandard for topology: A torus in 3-D is not simply connected because there is no transformation that will shrink it to a pt.
3. \( \overline{C \setminus \Omega} \) is closed.

To be not connected means \( \exists A, B \subset \overline{C \setminus \Omega} \) where \( A, B \) are closed and \( A \cap B = \emptyset \), \( A \cup B = \overline{C \setminus \Omega} \).

**Examples:**

a) \( D = \{ z \mid \text{Im} z > 0 \} \) is simply connected.

b) \( D = \{ z \mid |z| < 1 \} \) simply connected.
c) \( S = \{ z \mid 0 < \text{Im}(z) < 1 \} \) is simply connected since \( \overline{C \setminus S} \) is connected (through \( \infty \)).

d) \( \Omega = \{ z \mid |z| > 1 \} \) is not simply connected since \( \overline{C \setminus \Omega} = D \cup \{ \infty \} \) which is not connected.

**Theorem:** A domain \( \Omega \) is simply connected if for all cycles \( \gamma \) in \( \Omega \) and all \( a \) in \( \overline{C \setminus \Omega} \) one has \( n(\gamma, a) = 0 \).

**Proof:** Let \( \Omega \) be simply connected and \( a \in \overline{C \setminus \Omega} \). Note: \( \gamma \) is a cycle (formal sum) in \( \Omega \). \( \overline{C \setminus \Omega} \) is connected by def. of simply connected. So \( \overline{C \setminus \Omega} \) belongs to one of the components of \( \overline{C \setminus \gamma} \). \( \infty \in \overline{C \setminus \Omega} \Rightarrow \overline{C \setminus \Omega} \) belongs to the unbounded component of \( \overline{C \setminus \gamma} \), so for \( b \) near \( \infty \), \( n(b, \gamma) = 0 \).

(\( \Rightarrow \)) Suppose \( n(\gamma, a) = 0 \). If \( \Omega \) is not simply connected \( \overline{C \setminus \Omega} \) is not connected \( \Rightarrow \overline{C \setminus \Omega} = A \cup B \) with \( A, B \neq \emptyset \) and \( A \cap B = \emptyset \) and \( A \cup B \) both closed. So \( \overline{C \setminus \Omega} \) has \( A, B \) like

Let \( d = \text{dist}(A, B) = \inf \{ |z - z'| \mid z \in A, z' \in B \} > 0 \)

Now cover \( C \) with a grid of \( \frac{\delta}{2} \).

Let \( Q_j \) be a square in this grid.

Consider \( \gamma = \sum_j Q_j \) \( Q_j \cap A \neq \emptyset \)

\( n(\gamma, \gamma) = \sum_j n(Q_j, a) = 1 \)

After cancellations, \( \gamma \cap A = \emptyset \). \( \gamma \in \Omega \), \( a \in C \setminus \Omega \) and \( n(\gamma, a) = 1 \).
\( \Omega \subset \mathbb{C} \), is simply connected iff \( \overline{\mathbb{C} \setminus \Omega} \) is connected.

Remark: If \( \Omega \) is not simply connected then \( \exists \gamma \subset \Omega, \ a \in \mathbb{C} \setminus \Omega \) such that \( n(\gamma, a) \neq 0 \). Consider \( f(z) = \frac{1}{z-a}, \ f \) analytic in \( \Omega \).

\[
\oint_{\gamma} f(z) \, dz = \oint_{\gamma} \frac{dz}{z-a} = 2\pi i \ n(\gamma, a) \neq 0 \implies \text{Cauchy's Thm. does not hold.}
\]

This means that Cauchy's Thm, which holds in the unit disk, also holds in any simply connected domain.

\textbf{Thm (Cauchy's general)}: Let \( \Omega \) be simply connected and \( \gamma \subset \Omega \). Then \( \forall f \), analytic \( \text{on \ } \Omega \), \( \oint_{\gamma} f(z) \, dz = 0 \).

The proof is mostly from real analysis, so it is often considered local and global. A more general form of this statement is: \textbf{Thm}: The differential \( px + qy \, dy \) whose coefficients are defined and continuous in a simply connected domain \( \Omega \subset \mathbb{C} \) is exact in \( \Omega \) iff \( \int p(x,y) \, dx + q(x,y) \, dy = 0 \) \( \forall \) rectangle \( R \subset \Omega \).

Remark: \( p = p(x,y) \) and \( q(x,y) = q \), so \( p(x,y) \, dx + q(x,y) \, dy = dU(x,y) \).

The domain could be very complicated, but this should hold for any small rectangles and large rectangles could be subdivided. Hence \( px + qy \, dy \) is often called a locally exact differential.

3. What does \( f \) being analytic in simply connected domain \( \Omega \) mean? Recall Goursat's Thm: \( \int_{\partial R} f(z) \, dz = 0 \) for \( R \subset \Omega \) \( \implies \oint_{\gamma} f(z) \, dz = dF(z) \atop \gamma \subset \Omega \).
pf: If \( \int p\,dx + q\,dy = \int U(x,y) \) is an exact diff, in \( \Omega \Rightarrow (**) \)

by Goursat's Thm.

Now, suppose \( \int p\,dx + q\,dy = 0 \) \( \forall \mathcal{R} \), a rect, in \( \Omega \). Fix \( z_0 \in \Omega \). Then \( \forall z \in \Omega \), \( \exists \) a polygonal path, \( \sigma \), with sides parallel to the axes, joining \( z_0 \) to \( z \).

Consider \( U(x,y) = \int p\,dx + q\,dy \) (if it is well defined). We need to show that

\[
\int_{\sigma} p\,dx + q\,dy = \int_{\sigma'} p\,dx + q\,dy \iff \int_p p\,dx + q\,dy = 0
\]

for a polygon, \( \gamma = \sigma - \sigma' \subset \Omega \). Consider the simpler picture:

From each vertex on each polygonal path, \( \sigma \) and \( -\sigma' \), draw horizontal and vertical lines to make a grid:

For the trivial case a) \( \gamma \) is on either a single horizontal or vertical line.

or otherwise b) there are finite rectangles, \( R_i \), and unbounded rectangles, \( R_j' \).

Fix \( a_i \in R_i \setminus \partial R_i \); i.e. \( a_i \) is an interior pt. in \( R_i \). Also, fix \( a_j' \in R_j' \setminus \partial R_j' \).

Now let \( \gamma = \sum_i n(\gamma, a_i) \mathcal{R}_i \). \( \gamma \) will be precisely \( \gamma \).

\[
n(\gamma, a_k) = \sum_i n(\gamma, a_i) n(\mathcal{R}_i, a_k) = n(\gamma, a_k)
\]

\[
\begin{cases}
  1, & i = k \\
  0, & i \neq k
\end{cases}
\]

So \( n(\gamma, a_j') = 0 \) and \( n(\gamma, a_j') = 0 \).

So, \( n(\gamma, a) = n(\gamma_0, a) \forall a = a_i \) and \( \forall a = a_j' \).

So, we claim \( \gamma - \gamma_0 = 0 \) (or \( \gamma - \gamma_0 = \emptyset \)) (good)
Indeed, there are 3 cases to consider:

a) \( T_{ik} \) is the common interval for \( R_i \) and \( R_k \).

If \( \gamma - \gamma_0 \) includes \( C T_{ik} \), consider \( \gamma - \gamma_0 - C D R_i \), a cycle since \( \gamma, \gamma_0 \), and \( C D R_i \) are all cycles. \( \gamma - \gamma_0 - C D R_i \) does not include \( T_{ik} \), which means now \( a_i \) and \( a_k \) are in the same component of the new cycle, so

\[
\eta(\gamma - \gamma_0 - C D R_i, a_i) = \eta(\gamma - \gamma_0 - C D R_i, a_k) = -C = 0
\]

So \( C = 0 \Rightarrow T_{ik} \) is not in \( \gamma - \gamma_0 \).

b) \( T_{ik} \) is the common interval for \( R_i \) and \( R_k' \).

\( \Rightarrow T_{ik} \) is not in \( \gamma - \gamma_0 \).

c) \( T_{ik} \) is the common interval for \( R_i' \) and \( R_k' \) ⇒ impossible because \( C T_{ik} \subset \gamma - \gamma_0 \) ⇒ \( \gamma - \gamma_0 = \emptyset \). ⇒ \( \gamma = \gamma_0 = \sum_{i} \eta(\gamma, a_i) dR_i \).

Many of the \( \eta(\gamma_i, a_i) = 0 \). If \( \eta(\gamma, a_i) \) does not vanish, \( \Rightarrow R_i \subset \Omega \) because \( \Omega \) is the unbounded component of \( \gamma \) ⇒ \( \eta(\gamma, a_i) = 0 \) ⇒ \( \eta(\gamma, a_i) = 0 \) since \( \eta(\gamma, a_i) \neq 0 \).

So,

\[
\int_{\gamma} p dx + q dy = \sum_{i} \eta(\gamma, a_i) \int_{R_i} p dx + q dy = 0.
\]

Remark: We only used simple connectivity here.

1. Given: a domain \( \Delta \) and a polygon \( \gamma \subset \Delta \), \( \eta(\gamma, a) = 0 \) \( \forall a \in C \setminus \Delta \).

\( p dx + q dy \) is a locally exact differential \( \Rightarrow \int_{\gamma} p dx + q dy = 0 \) \( \forall R \), rect., \( \subset \Delta \).

Thus, \( \int_{\gamma} p dx + q dy = 0 \).

2. If \( f(z) \) is analytic and does not vanish in a simply connected domain, a single value analytic branch of \( \log(f(z)) \) can be defined in \( \Delta \), i.e.,

\[
\exists h(z), \text{ analytic in } \Omega \text{ and } f(z) = e^{h(z)}, h(z) = \log f(z).
\]

(continued)
Indeed, let \( F(z) = \int_{\gamma} \frac{f'(z)}{f(z)} \, dz \); this is well-defined since \( f \) is analytic and Cauchy's Thm. states this does not depend on choice of \( \gamma \).

Hence, \( g(z) = f(z) e^{-F(z)} \) is analytic in \( \Omega \).

\[
\begin{align*}
g'(z) &= e^{-F(z)} \left[ f'(z) - f(z) F'(z) \right] \\
&= e^{-F(z)} \left[ f'(z) - f(z) \frac{f'(z)}{f(z)} \right] = 0
\end{align*}
\]

\[\Rightarrow \quad g(z) \text{ is constant; i.e., } g(z) = g(z_0) \quad \forall \ z \in \Omega.\]

\[
f(z) e^{-F(z)} = f(z_0) = g(z_0)
\]

\[
\log f(z_0) = \log |f(z_0)| + i \arg f(z_0)
\]

\[
\text{fix one of the values}
\]

\[
f(z) = e^{F(z)} f(z_0) = e^{F(z) + \log f(z_0)}
\]

\[
h(z) = F(z) + \log f(z_0).
\]

Example: \( \log z \) for \( f(z) = z \). Fix \( \log z_0 \), then \( \log z \) is analytic and single valued in \( \Omega \).

This is used for \( f(z)^a = e^{a \log f(z)} \).

Now we are ready for the most general form of the Cauchy integral Thm (after one more remark).

Run \( \Omega \subset \mathbb{C} \) be a domain with \( \gamma \subset \Omega \) being a cycle.

\( \forall \ a \in \mathbb{C} \setminus \Omega, \ n(\gamma, a) = 0 \). We say \( \gamma \) is homologous to \( 0 \) in \( \Omega \). This is written as \( \gamma \sim 0 \pmod{\Omega} \) and read "\( \gamma \) is homologous to \( 0 \) in \( \Omega \)."
The General Cauchy Integral Thm: If \( f(z) \) is analytic in \( \Omega \),
then \( \oint_{\gamma} f(z) \, dz = 0 \) for every cycle \( \gamma \subset \Omega \),
where \( \gamma \sim 0 \pmod{\Omega} \).

4/9/09 went over #3 on test
Multiply-connected domains

If \( \Omega \subset \mathbb{C}, \gamma, \) a cycle, \( \gamma \subset \Omega \)
\( \gamma \sim 0 \pmod{\Omega} \) iff \( n(\gamma, a) = 0, \ a \in \mathbb{C} \setminus \Omega \)

Then \( \int_{\gamma} pdx + qdy \) a locally exact differential \( \iff \int_{\gamma} pdx + qdy = 0 \)
for every rect \( R \subset \Omega \).

Thm: If \( pdx + qdy \) is locally exact in \( \Omega \), then \( \int_{\gamma} pdx + qdy = 0 \)
for every cycle \( \gamma \subset \Omega \), i.e. \( \gamma \sim 0 \pmod{\Omega} \).

Pf: For simplicity, let \( \gamma \) be a curve, instead of a cycle.

\( \gamma : z = z(t), \ t \in [a, b], \ z(t) \) is continuous \( \Rightarrow z(t) \) is uniformly continuous.

Let dist \( (\gamma, \mathbb{C} \setminus \Omega) = \delta \). If \( \Omega \neq \mathbb{C}, \ \delta > 0 \). Consider \( t \in [a, b] \).

\[ |z(t) - z(t_j)| < \frac{\delta}{2} \text{ for } t_j \leq t \leq t_{j+1}. \]

Let \( \gamma_j = z_{t_j} z(t) + z(t) \). So, \( \gamma = \gamma_1 + \gamma_2 + \ldots + \gamma_n \), where

\( \gamma_j : z = z(t), \ t \in [j-1, j] \)

Consider a disk of radius \( \delta \). The disk is in \( \Omega \). Also, \( \int_{\gamma_j} \frac{dz}{z-a} = \int_{\gamma_{j-1}} \frac{dz}{z-a} \)
for all \( \gamma_j \).
\[ \sigma = \sigma_1 + \sigma_2 + \ldots + \sigma_n. \]

\[ \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z-a} = \frac{1}{2\pi i} \int_{\sigma} \frac{dz}{z-a} \Rightarrow n(\gamma, a) = n(\sigma, a) = 0 \]

\[ \int_{\gamma_i} p\,dx + q\,dy = \int_{\sigma_j} p\,dx + q\,dy \Rightarrow \int_{\gamma_i} p\,dx + q\,dy = \int_{\sigma_j} p\,dx + q\,dy \text{ this is zero, since } \sigma \text{ is a polygonal path homologous to zero. For a cycle, finite number of closed curves, we just find a } \gamma_i \text{ for each } \gamma \text{ and then combine them.} \]

\[ n(\gamma_i + \gamma_j, a) = 0 \]

**General Cauchy's Thm:** If \( f \) is analytic in a domain \( \Omega \subset \mathbb{C} \), then \[ \int_{\gamma} f(z)\,dz = 0 \] for every cycle \( \gamma \subset \Omega : \gamma \sim 0 \text{ (mod } \partial \Omega) \).

**General Cauchy Integral Formula:** If \( f \) is analytic in a domain \( \Omega \subset \mathbb{C} \), then
\[ n(\gamma, a)f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)\,dz}{z-a} \text{ for } a \in \Omega \setminus \partial \Omega \]
and a cycle \( \gamma \subset \Omega : \gamma \sim 0 \text{ (mod } \partial \Omega) \).

Let \( \Omega_1, \Omega_2, \ldots, \Omega_n \) be components of \( \mathbb{C} \setminus \Omega \). Let \( \Omega_n \) be the unbounded component; i.e., \( \infty \in \Omega_n \).

**Defn** If \( n > 1 \), \( \Omega \) is called multiply-connected.

Let \( \gamma \subset \Omega \) be a cycle.
Remarks

1. For $a, b \in A_j$, $n(\gamma, a) = n(\gamma, b)$

2. For any $j \in \mathbb{Z}$, $\gamma_j \cap \gamma_k = \emptyset$ for all $k \neq j$

$$n(\gamma_j, a) = \begin{cases} 1, & a \in A_j \\ 0, & a \in (\mathbb{C} \setminus \Omega) \setminus A_j \end{cases}$$

This is true for all closed components, i.e., $V_j \neq \emptyset$ since $\infty \in A_n$.

Let $f$ be analytic in $\Omega$. Fix $a_j \in A_j$. Let $n(\gamma_j, a_j) = c_j$. Note $c_n = 0$.

Let $\gamma = \gamma_1 + c_2 \gamma_2 + \ldots + c_{n-1} \gamma_{n-1}$, with $\gamma_1, \ldots, \gamma_{n-1}$ disjoint polygonal paths and $c_1, \ldots, c_{n-1} \in \mathbb{Z}^+$. Then

$$n(\gamma, a) = c_j \quad \text{for} \quad a \in A_j, \quad V_j \in \{1, \ldots, n\}$$

Thus, $\gamma - \sigma \sim 0 \pmod{\Omega}$ G.C. Thm. $\Rightarrow \int_{\gamma} f(z) \, dz = 0$

$$\Rightarrow \int_{\gamma} f(z) \, dz = \sum_{j=1}^{n-1} c_j \int_{\gamma_j} f(z) \, dz$$

Each $\int_{\gamma_j} f(z) \, dz$ is called a period of $f$ in $\Omega$. Let $P_j = \int_{\gamma_j} f(z) \, dz$.

Then

$$\int_{\gamma} f(z) \, dz = \sum_{j=1}^{n-1} n(\gamma, a_j) P_j = \sum_{j=1}^{n-1} c_j P_j$$

The calculus of residues

Given $\Omega \subset \mathbb{C}$, a domain with $f$ analytic in $\Omega'$ (see below)

Let $z_1, z_2, \ldots, z_j, \ldots, z_n \in \Omega$.

Let $\delta_j > 0$: $\{z : 0 < |z - z_j| \leq \delta_j\} \subset \Omega \setminus \bigcup_{j=1}^{n-1} \{z_j\} = \Omega'$

Let $C_j = \{z : |z - z_j| = \delta_j\}$. Let $P_j(f) = \int_{C_j} f(z) \, dz$.

Consider $R_j(f) = \frac{P_j(f)}{2\pi i}$. Let $g(z) = f(z) - \frac{R_j(f)}{z - z_j}$.

$g$ is also analytic on $\Omega'$.
\[
\int g(z) \, dz = \int f(z) \, dz - \int \frac{R_j}{z-z_j} \, dz
\]
\[= \text{Res}_{z=a} f(z) - R_j \cdot 2\pi i
\]
\[= \text{Res}_{z=a} f(z) - \text{Res}_{z=a} f(z) = 0
\]
\[\therefore \text{Res}_{z=a} f(z) = 0
\]

The residue, \( R = \text{Res}_{z=a} f(z) \) of \( f \) at an isolated singularity \( a \) is the unique complex number \( R \) which makes \( f(z) - \frac{R}{z-a} \) to be the derivative of a single-valued analytic function in \( \left\{ z : 0 < |z-a| < \delta \right\} \)

\[
\text{Res}_{z=a} f(z) = \frac{1}{2\pi i} \int_{\gamma} f(z) \, dz. \text{ So this}
\]

\( R \) is the coefficient of \( \frac{1}{z-a} \) that is the main part of why \( f(z) \) did not have an antiderivative. (411 Midterm 2)

End of Cauchy's Int. Thm. + More Residue Theory

\( \Omega \subseteq \mathbb{C} \), a domain \( \Omega \setminus \bigcup_{j=1}^{n} \{ a_j \} = \Omega' \), each with a disk around it; enclosed in a circle \( C_j \). Consider \( f \) is analytic on \( \Omega' \). \( f \) is on \( \left\{ z : 0 < |z-a_j| < \delta_j \right\} = D_j' \)

let \( \gamma \) be a cycle with \( \gamma \subseteq D_j' \). Let \( g(z) = f(z) - \frac{R_j}{z-a_j} \)

\[
\int_{C_j} g(z) \, dz = 0 \text{ or } \exists G(z): G'(z) = g(z)
\]
Let \( \gamma \in \Gamma' \) and \( \gamma \sim 0 \pmod{\Omega} \)
\[
\gamma \sim \sum_j n(\gamma, a_j) C_j \pmod{\Omega'}
\]

If a pt. is chosen outside \( \Omega \), both sides are equal.

If \( a_k \) is chosen as one of the isolated singularities (centers of circles)
\[
n(\gamma, a_k) = \sum_j n(\gamma, a_j) C_j a_k
\]

\[
\int_{\gamma} f(z) \, dz = \sum_j n(\gamma, a_j) \int_{C_j} f(z) \, dz = 2\pi i \text{ Res}_z f(z)
\]

\[
= \sum_j n(\gamma, a_j) 2\pi i \text{ Res}_z f(z)
\]

\[
= 2\pi i \sum_j n(\gamma, a_j) \text{ Res}_z f(z)
\]

The **residue Thm.** let \( f \) be analytic except for isolated singularities \( a_j \) in a domain \( \Omega \). Then \( \frac{1}{2\pi i} \int_{\gamma} f(z) \, dz = \sum_{j=1}^n n(\gamma, a_j) \text{ Res}_z f(z) \)

for any cycle \( \gamma \) such that \( \gamma \sim 0 \pmod{\Omega} \) and \( \gamma \) does not pass through any of the pts. \( a_j \).

Note that only a finite number of \( a_j \)'s satisfy the restriction that \( n(\gamma, a_j) \neq 0 \). Thus, there can be an infinite number of singularities.

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\( \text{Go to page } 93 \)
Defn: A func \( f \) is modular if
a) \( f \) is meromorphic in \( \mathcal{F} \)
\[ f(A \tau) = f(\tau) \quad \forall A \in \Gamma, \quad \tau \in \mathcal{F} \]
b) \( f \) has at worst a pole (of finite order, say \( m \)) at \( i\infty \);

i.e., the Fourier expansion of \( f \) has the form
\[ \sum_{n=-\infty}^{\infty} a(n)e^{2\pi in}\tau \]

Thm: If \( f \) is a nonzero modular func, then the number of zeros equals the number of poles in the closure of the standard fundamental region, \( \mathcal{F} \), with suitable conventions on \( \partial \mathcal{F} \). If a zero pole is on \( \partial \mathcal{F} \) then the only one counted is the leftmost one in the closure \( \mathcal{F} \).

The order of a zero/pole at \( \rho \) is to be divided by 3, and the order of a zero/pole at \( i\infty \) is to be divided by 2.

\( \chi = \rho \), measured in the variable \( \chi = e^{2\pi i \tau} \).

\[ N = P \quad \text{in closure} (\mathcal{F}) \]

Thm: \( N-P = \frac{1}{2\pi i} \int_0^1 \frac{f'(\tau)}{f(\tau)} d\tau \)

Applications: Any modular func is just a rational func. of \( \tau \).

Corollary: If \( f \) is modular and nonconstant, then \( f \) assumes every value equally often in \( \text{cl}(\mathcal{F}) \). Pf. For \( f(\tau) = c \), apply the Thm to \( f-C \), also \( f-C \) modular.
Corollary to the Corollary: If \( f \) is bounded and modular in \( \mathbb{H} \), then \( f \) is constant.

**Pf:** \( f \) bounded \( \Rightarrow \) \( \Delta \) omits some values \( \Rightarrow f \) can't be non-constant \( \Rightarrow f \) is constant.

Aside: Thm. The func \( J \) takes every value exactly once in \( \text{cl}(\mathbb{R}_p) \).

In particular, \( J(p) = 0 \), \( J(i) = 1 \), and \( J(i \infty) = i \infty \).

**Pf:** \( g_2(p) = g_3(i) = 0 \). From \( p^3 = 1, p^2 + p + 1 = 0 \). The Eisenstein series

\[
\frac{1}{60} g_2(p) = \sum_{m,n} \frac{1}{(m+n^2)^4} = \sum_{m,n} \frac{1}{(mp^3 + np)^4} = \frac{1}{p^4} \sum_{m,n} \frac{1}{(mp^2 + n)^4}
\]

\[
= \frac{1}{p^4} \sum_{m,n} \frac{1}{(n-m-p-m)^4} = \frac{1}{p} \sum_{m,n} \frac{1}{(N+M)^4}
\]

\[
= \frac{1}{p} \left( \frac{1}{60} g_2(p) \right)
\]

\( g_2(p) = \frac{1}{p} g_2(p) \Rightarrow g_2(p) = 0 \)

Similarly, get \( g_3(i) = 0 \). So, \( J(p) = \frac{(g_2(p))^3}{\Delta(p)} = 0 \), and

\[
J(i) = \frac{(g_2(i))^3}{g_2^2(i) - \frac{27}{g_3^2(i)}} = 1
\]

Also, \( J \) has a 1\textsuperscript{st} order pole at \( i \infty \), a triple zero at \( p \), and \( J(i) = 1 \) has a double zero at \( i \). (from the Thm on \( N = p \))

**Thm.** Every rational func. in \( J \) is modular, and conversely.
Pf of convergence. Suppose $f$ is modular and has zeros at $z_1, \ldots, z_n$ and poles at $p_1, \ldots, p_m$ (same number if each $b/c$ of $N = p$). Need for modular

Let $g(\tau) = \prod_{k=1}^{n} \frac{J(\tau) - J(z_k)}{J(\tau) - J(p_k)}$, where we insert a factor of $1$ whenever $f$ has a zero/pole at $\infty$. Then $g$ has the same zero/poles as $f$ in $\text{cl}(R)$, each with proper multiplicity, so $f/g$ has no zero/poles, and is, thus, constant.

(1) $\text{Im}(\tau) = M$ with $M > 0$ and large enough so that the truncated region (call it $R$) contains all the $z/p$'s. [There are infinitely many $p$'s since there would be an accumulation point $i \infty$ otherwise, contradicting condition (c) of the defn. of $f$ being modular. Likewise, there are only finitely many zeros, since, otherwise, $f \equiv 0$.]

Then $N - P = \frac{1}{2\pi i} \int_{\partial R} \frac{f'(\tau)}{f(\tau)} \, d\tau = \frac{1}{2\pi i} \left( \int_{(1)} + \int_{(3)} \right)$

Have $\int_{(1)} + \int_{(4)} = 0$ since $f(\tau + i) = f(\tau)$ by transl. $T$

$\int_{(2)} + \int_{(3)} = 0$ since $f(-1/\tau) = f(\tau)$ by transformation $S$

Under $U = S(\tau) = -\frac{1}{\tau}$; i.e. $T = S^{-1}U$

Then $f(S(u)) = f(S(u)) = f(u) \Rightarrow f(S(u)) \cdot S'(u) = f'(u) \Rightarrow f'(S(u)) \cdot S'(u) = f'(u)$

So $\frac{f'(\tau)}{f(\tau)} \, d\tau = \frac{f'(u)}{f(u)} \, du$. 

\begin{align*}
\end{align*}
\[ N - P = \frac{1}{2\pi i} \oint_{\gamma} \frac{f'(\tau)}{f(\tau)} \, d\tau. \]

Now, go to the \( x \)-plane \((x = e^{2\pi i \tau})\) on (5), where \( \tau = \imath a + i \frac{1}{2} \). Then \( x = e^{-2\pi i a} e^{2\pi i \imath} \), so \( x \) varies around a circle \((\text{call it } \mathcal{C})\) with radius \( e^{-2\pi a} \) about \( x = 0 \), in the "neg" \( \text{\textit{opposite}} \) direction. Since the pts. above (5) map inside \( \mathcal{C} \), \( f \) has no \( \frac{1}{2}/p \)'s inside \( \mathcal{C} \) except possibly at \( x = 0 \) (corresponding to \( \tau = i \infty \)).

The Fourier expansion of \( f \) is
\[ f(\tau) = a^-_m x^{-m} + a^+_m x^m + \ldots = F(x), \text{ say, } f'(\tau) = F'(x) dx/d\tau, \]
so \( \frac{f'(\tau)}{f(\tau)} \, d\tau = \frac{F'(x)}{F(x)} \, dx \). So \( N - P = \frac{1}{2\pi i} \oint \frac{F'(x)}{F(x)} \, dx \)
\[ = - (N_F - P_F) = P_F - N_F. \]

Suppose \( \exists \) \( p \) with order \( m \) at \( x = 0 \). Then \( m > 0 \) and \( N_F = 0 \), \( P_F = m \), so \( N = P + m \). \((\text{\textit{remember opposites}})\)

Likewise, if \( \exists \) \( z \) of order \( n \) at \( x = 0 \), then \( m = -n \) and \( P_F = 0 \) and \( N_F = n \), so \( N + n = P \).
Also \( e^x \cdot e^x = e^{2x} \) by Taylor series and Cauchy expansion. 

What you will need to use binomial then also.

\[
1 = 1 + 0x + 0x^2 + \ldots \\
p(0) = 1
\]

\[
0 = p(n) - p(n-1) - p(n-2) + p(n-3) + p(n-7) - \ldots
\]

\[
p(n) = p(n-1) - p(n-2) - p(n-5) - \ldots
\]

Truncate to \( p(3) \)

\[
p(5) = p(4) + p(3) - p(0)
\]

\[
= (p(3) + p(2)) + p(3) - p(0)
\]

\[
= (p(2) + p(1)) + p(2) + (p(2) + p(1)) - p(0)
\]

\[
= \sum_{i=0}^{2} p(i) + 2p(1) + 2p(1) + p(1) - p(0)
\]

\[
= 2 + 1 + 2 + 2 + 1 - 1 = 7
\]
Go back to the middle of page 91
To find residues consider types of singularities:

1. **Removable singularity**, \(a\)

Consider \(C\) with \(\int_c f(z) \, dz = 0\). Since \(f'\) can be defined on the disk inside \(C\) that is analytic,

\[
\sum_{z=a}^{c} \text{Res}_{z=a} f(z) = B_1 + \lim_{z \to a} (z-a) \varphi(z)
\]

To find \(B_1\), consider

\[
\lim_{z \to a} f(z)(z-a) = B_1 + \lim_{z \to a} (z-a) \varphi(z)
\]

If \(h > 1\), multiply by \((z-a)^h\) and take \(h-1\) derivatives.

\[
\text{Res}_{z=a} f(z) = \frac{1}{(h-1)!} \lim_{z \to a} \frac{d^{(h-1)}}{dz^{(h-1)}} \left[ f(z)(z-a)^h \right]
\]

2. **Pole**, \(a\), of order \(h \geq 1\).

From Taylor's formula, near \(a\), for \(0 < |z-a| < \delta\),

\[
f(z) = \frac{B_h}{(z-a)^h} + \frac{B_{h-1}}{(z-a)^{h-1}} + \ldots + \frac{B_1}{z-a} + \varphi(z)
\]

where \(\varphi(z)\) is analytic in some nbhd. \(a\).

3. **Essential singularity**, \(a\), then we will use Laurent series and we will do this later.

**To summarize:** tying together Cauchy Integral Thm, Cauchy Integral Formula, and Residue Theorem. Let's do this for a simple curve:

**Defn:** Let \(\Omega \subset \mathbb{C}\) be a domain. A cycle \(\gamma \subset \partial \Omega\) is said to bound \(\Omega\) iff \(\nu(\gamma, a) = \int_{\gamma} f \, dz = 0\) if \(a \in \Omega\), \(\nu(\gamma, a) = 0\) if \(a \notin \Omega\).

**Eq. a.** \(\Omega = \{ z : |z-a| < \delta \}\)

\[
\gamma = a + \epsilon e^{i\theta}, \quad \theta \in [0, 2\pi]\]

\(\nu = \epsilon\).

**Eq. b.** Consider a multiply-connected domain:

winding numbers will be \(2\), so doesn't bound \(c\) if zero.

**Remark:** Let \(\gamma\) bound \(\Omega\) and let \(f\) be analytic in \(\Omega \cup \gamma\). Then

\[
\int_{\gamma} f(z) \, dz = 0\]

Since \(\nu(\gamma, a) = 1\) \(\forall a \in \Omega\), \(f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z-a} \, dz\).
Suppose \( \Omega \) bounds \( \Omega \) and \( f \) is analytic in \( \Omega \cup \Omega' \cup \{a_j\} \) where \( a_j \) is an isolated singularity \( \forall j \in \mathbb{Z}^+ \) (could be finite or infinite). Then
\[
\frac{1}{2\pi i} \oint \frac{f'(z)}{f(z)} \, dz = \sum_{j} \text{Res}_{z=a_j} f(z).
\]
These last 2 forms are what is usually used.

Applications and the Argument Principle

If \( f(z) \) is meromorphic in \( \Omega \), with zeros \( \{a_j\} \) and poles \( \{c_k\} \) which are both counted according to their multiplicities, then the integral of the logarithmic derivative:
\[
\frac{1}{2\pi i} \oint \frac{f'(z)}{f(z)} \, dz = \sum_{j} n(\gamma, a_j) - \sum_{k} n(\gamma, c_k)
\]
for every cycle \( \gamma \) such that \( \gamma \sim 0 \text{ (mod } \partial \Omega) \) and \( \gamma \) does not pass through any zeros or poles.

Rem: (a)

- Circles could be used as arcs to pick up exactly the pts. desired.

(b) What is analyzed or argued with this:

If \( f(z) \) is continuous, \( f(\gamma) \) is also a curve:
\[
\gamma \rightarrow f(\gamma)
\]
with \( \Gamma = f(\gamma) \)

Pf: (Using the Residue Thm)

Let \( F(z) = \frac{f'(z)}{f(z)} \). To find \( \oint F(z) \, dz \),

we need to find the residues at all singularities.

Suppose \( z = a_j \); \( a_j \) is a zero of \( f \) of order \( h \geq 1 \). Then \( f(z) = (z-a_j)^h f_h(z) \)

where \( f_h \) is analytic at \( a_j \), \( f_h(a_j) \neq 0 \), \( f'(z) = h (z-a_j)^{h-1} f_h(z) + (z-a_j)^h f_h'(z) \).

\[
F(z) = \frac{f'(z)}{f(z)} = \frac{h}{z-a_j} + \frac{f_h'(z)}{f_h(z)} \quad \Rightarrow \quad \text{Res}_{z=a_j} F(z) = \frac{h}{z-a_j} \text{, the order of the zero at } a_j.
\]

Suppose \( b_k \) is a pole of order \( h \) : \( f(z) = (z-b_k)^{-h} f_h(z) \) where \( f_h \) is analytic at \( b_k \), \( f_h(b_k) \neq 0 \) (and \( f_h'(b_k) \) exists)

\[
f'(z) = -h (z-b_k)^{-h-1} f_h(z) + (z-b_k)^{-h} f_h'(z) \quad \Rightarrow \quad F(z) = -h (z-b_k)^{-h} + \frac{f_h'(z)}{f_h(z)}
\]

\( \therefore \text{Res}_{z=b_k} F(z) = -h \). Use the Residue Thm to substitute and finish the proof.