
There are 3 possibilities for \( \sum \frac{b_n}{z^n} \):

i) It diverges \( \forall z \in \mathbb{C} \setminus \{0\} \)

ii) It converges \( \forall z \in \mathbb{C} \setminus \{0\} \)

iii) \( \exists R: 0 < R < \infty \) and the series converges for \(|z| > R\) and it diverges for \(|z| < R\). \( R \) is called the radius of convergence and Hadamard's and Abel's Thms tell something about it. In case i) let \( R = 0 \).

Then, \( A \) laurent series : \( \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n = \sum_{n=1}^{\infty} \frac{c_{(-n)}}{(z-z_0)^n} + \sum_{n=0}^{\infty} c_n (z-z_0)^n \)

where \( z_0 \in \mathbb{C} \) or \( S = S_1 + S_2 \).

Defn. \( S \) converges at \( z \in \mathbb{C} \) \( \iff \) \( S_1 \) and \( S_2 \) converge at \( z \).

\( S_1: \exists R_1: S_1 \) converges for \(|z| > R_1\)

\( S_2: \exists R_2: S_2 \) converges for \(|z| < R_2\).

If \( R_1 < R_2 \), then \( S = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n \) converges on this annulus : \( \{ z: R_1 < |z-z_0| < R_2 \} \). Also, it converges uniformly on any \( \{ z: R_1 \leq |z-z_0| \leq R_2 \} \) where \( R_1 < R_1 < R_2 < R_2 \). So by the Weierstrausa thm, the series is analytic in the annulus.

Also, any analytic fn. can be expanded into a laurent series:

let \( 0 \leq R_1 < R_2 \leq \infty \) and \( z_0 = 0 \) and \( f(z) \) is analytic in \( A \). \( \exists r_1, r_2 : R_1 < r_1 < |z| < r_2 < R_2 \). let \( C_1 = \{ z: |z| = r_1 \} \)

\( C_2 = \{ z: |z| = r_2 \} \)

Consider the cycle \( -C_1 + C_2 \sim O(\text{mod } A) \) since \( n(-C_1+C_2,a) = 0 \) \( \forall a \in A \)

=> the Cauchy - Integral Thm applies.
The Cauchy Formula: \[ n(C_1 + C_2, z) \oint_{C_1 + C_2} \frac{f(z)}{z - \zeta} \, dz = 2\pi i \int_{C_1} \frac{f(z)}{z - \zeta} \, dz + \int_{C_2} \frac{f(z)}{z - \zeta} \, dz \Rightarrow \forall \zeta \in A, \]

\[ f(z) = -\frac{1}{2\pi i} \oint_{C_1} \frac{f(z)}{z - \zeta} \, dz + \frac{1}{2\pi i} \oint_{C_2} \frac{f(z)}{z - \zeta} \, dz = f_1(z) + f_2(z) \quad \text{where} \]

\[ f_1(z) \quad \text{is analytic in} \quad \{ z : |z| > R_1 \}, \quad \text{and} \quad f_2(z) \quad \text{is analytic in} \quad \{ z : |z| < R_2 \}, \]

even though we have \( R_1 \) and \( R_2 \), we can make them arbitrarily close to \( R_1 \) and \( R_2 \). Since \( f_1(z) \) can be written as a Taylor series in \( \frac{1}{z} \) and \( f_2(z) \) can be written as a Taylor series. Hence, these give the Laurent series for \( f(z) \), or Laurent expansion.

There are other ways to represent funs.: products, partial fractions, etc.

**Partial Fractions**

Let \( \Omega \subset \mathbb{C} \) be a domain, and \( f \) be meromorphic in \( \Omega \), \( \mathbb{C} \setminus \{ b_v \} \) with each \( b_v \) a pole of \( f \).

So, \( f(z) = \sum \frac{A_v}{(z - b_v)} + \ldots + \frac{A_{-1}}{(z - b_v)} + \sum_{k=0}^{n} \frac{P_v}{(z - b_v)^k} \) where \( P_v(z) \) is analytic in \( \Omega^\ast \).

Also, the poles, \( b_v \) don't cluster; i.e., for \( 0 < |z - b_v| < r \), sufficiently small, \( f \) is analytic.

\[ f(z) = \sum_{v} P_v \left( \frac{1}{z - b_v} \right) + \sum_{n=0}^{\infty} a_n (z - b_v)^n. \]

If \( f \) is a finite number of singular parts in \( \frac{1}{z - b_v} \), \( b_v \), \( \Rightarrow f(z) = \sum_{v} P_v \left( \frac{1}{z - b_v} \right) = g(z) \) where \( g(z) \) is analytic in \( \Omega \) \( \Rightarrow f(z) = \sum_{v} P_v \left( \frac{1}{z - b_v} \right) + g(z) \). Hence, any meromorphic fun. can be written as a sum of an analytic fun. and a finite (easy to deal with) or maybe infinite # of series.
\[
\begin{align*}
\text{Eq. } f(z) &= \frac{\Pi^2}{\sin^2 \pi z} \quad \text{normalized } \quad \Omega = \mathbb{C}, \text{ } f(z) \text{ is meromorphic w/ poles at } n \in \mathbb{Z} \\
\text{and } \sin^2 \pi z &= \sin^2 \pi (z-n) \text{ so it is enough to find what happens at } 0.
\end{align*}
\]

So by \( z = 0 \) is a pole of order 2 since \( \frac{1}{\sin^2} \) has order 1 at \( z = 0 \).

\[
\begin{align*}
\frac{\Pi^2}{\sin^2 \pi z} &= \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + \varphi(z) \quad \text{with } \quad \varphi \left( \frac{1}{z} \right) = \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z}, \\
\text{let } Z = \pi \frac{1}{z} \quad \text{let } \lim_{Z \to 0} \\
\frac{\varphi(Z)}{Z^2} &= \frac{\frac{2}{\sin^2 Z}}{\lim_{Z \to 0}} = \left( \lim_{Z \to 0} \frac{2}{\sin^2 Z} \right)^2 = 1
\end{align*}
\]

\[
\begin{align*}
\frac{2}{\sin^2 Z} &= \left( \lim_{Z \to 0} \frac{2}{\sin^2 Z} \right)^2 = \left( \lim_{Z \to 0} \frac{2}{Z^2 - \frac{\pi^2}{6} + \frac{\pi^4}{30} + \ldots} \right)^2 = 0
\end{align*}
\]

\[
\begin{align*}
\frac{\Pi^2}{\sin^2 \pi z} &= \left\{ \begin{array}{ll}
\frac{1}{z^2} + \varphi(z) & \text{near } 0 \\
\frac{1}{(z-n)^2} + \varphi(z) & \text{near } n
\end{array} \right.
\end{align*}
\]

consider \( \frac{1}{z^2} + \sum_{n \neq 0} \frac{1}{(z-n)^2} \). It converges \( \forall z \notin \mathbb{Z} \).

So \( \frac{\Pi^2}{\sin^2 \pi z} \) is meromorphic in \( \mathbb{C} \)

\[
\begin{align*}
\frac{\Pi^2}{\sin^2 \pi z} &= \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} + g(z) \quad \text{where } g(z) \text{ is an entire fcfn.}
\end{align*}
\]

So's find \( g(z) \). Claim: \( g(z) \) is periodic with period = 1.

If \( |y| > 1 \), then \( \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} \) converges uniformly. \( \lim_{y \to \infty} \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} = 0 \).

\( \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} \) is bounded for \( |y| \leq 1 \) and \( \lim_{y \to \infty} \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} = 0 \).
Continuing the example: Consider \( \frac{\pi^2}{\sin^2 \pi z} \) with \( z = x + iy \). Let \( z = \pi z = \pi x + i\pi y \).

\[
\sin^2 z = \frac{1}{2i} (e^{iz} - e^{-iz}) \quad \text{and} \quad \frac{1}{\sin^2 z} = -\frac{1}{2i} (e^{-iz} - e^{iz})
\]

\[\left| \sin^2 z \right| = \frac{1}{4} (e^{i(z - \frac{\pi}{2})} - e^{i(z + \frac{\pi}{2})} - e^{-(z - \frac{\pi}{2})} + e^{-(z + \frac{\pi}{2})}) \]

\[\frac{\pi^2}{\sin^2 \pi z} \quad \text{is bounded for} \quad |y| \geq 1 \quad \text{and} \quad \lim_{|y| \to \infty} \frac{\pi^2}{\sin^2 \pi x} = 0\]

\[\Rightarrow g(z) \quad \text{is bounded} \quad \Rightarrow \text{Liouville's Thm} \quad g(z) = \text{Constant} = 0\]

\[\Rightarrow \frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} \quad \text{We were lucky here because the formal sum} \quad \sum_{n} \frac{1}{(z-n)^2} \quad \text{converges} \quad \forall z \in \mathbb{C}, \ z \neq \mathbb{Z}.
\]

4-30-09 We reviewed (orally) what we did last time which was the beginning of approximation theory. From last class \( \frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} \)

Next e.g. \( f(z) = \cot(z) = \frac{\cos z}{\sin z} \); we'll normalize later. Notice \( f \) is analytic in \( \mathbb{C} \setminus \{ \mathbb{Z} \} \) \( \forall n \in \mathbb{Z} \), and \( f'(z) = -\frac{1}{\sin^2 z} \). Then for \( g(z) = -\pi \cot \pi z \), \( g'(z) = \frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} \quad \text{formally} \quad = \left( -\sum_{n} \frac{1}{z-n} \right)^0 \)

The singularities of \( g(z) = \pi \cot \pi z \) are \( \frac{1}{z-n} + \frac{1}{n} \) near \( z=n \).

Consider \( \frac{1}{z} + \sum_{n \neq 0} \left( \frac{1}{z-n} + \frac{1}{n} \right) = \frac{1}{z} + \sum_{n \neq 0} \frac{z}{(z-n)n} \quad \text{well defined, since it uniformly converges as} \quad n \to \infty. \)
So, \( \pi \cot \pi z = \frac{1}{z} + \sum_{n \neq 0} \left( \frac{1}{z-n} + \frac{1}{n} \right) + g(z) \) \( \Rightarrow \) which is entire in \( \mathbb{C} \).

The main problem after doing something like this is finding what \( g(z) \) is. Consider the derivative is one way:

\[
(\pi \cot \pi z)' = \left( \frac{1}{z} + \sum_{n \neq 0} \left( \frac{1}{z-n} + \frac{1}{n} \right) \right)' + g'(z) = 0 \text{ in } \mathbb{C} \implies g'(z) = 0 \text{ in } \mathbb{C} \implies g(z) \text{ is a constant.}
\]

Andriyusiky skipped this part (talk over it).

\[\pi \cot \pi z = \frac{1}{z} + \sum_{n \neq 0} \left( \frac{1}{z-n} + \frac{1}{n} \right) + c \quad \text{Same with } n \rightarrow -n\]

Let \( z = -z \)

\[-\pi \cot \pi z = \frac{1}{-z} + \sum_{n \neq 0} \left( \frac{1}{z-n} - \frac{1}{n} \right) + c \quad \text{Add these } 0 = 2c \Rightarrow c = 0.\]

\[\pi \cot \pi z = \frac{1}{z} + \sum_{n \neq 0} \left( \frac{1}{z-n} + \frac{1}{n} \right)\]

A general construction:

**Theorem (Mittag-Leffler)** Let \( \{ b_n \} \) be a sequence \( \subset \mathbb{C} \) such that \( \lim_{n \to \infty} b_n = \infty \), and let \( p_n(z) \) be polynomials without constant terms. Look at the examples and a singularity and consider the reciprocals. \( \ldots \text{?} \) Then, \( f \) functions that are meromorphic in the entire \( \mathbb{C} \) plane with poles at \( b_n \) and the corresponding singular parts \( p_n \left( \frac{1}{z-b_n} \right) \). Moreover, the most general meromorphic function of this kind can be written in the form

\[ f(z) = \sum_{n} [p_n \left( \frac{1}{z-b_n} \right) - p_n(z)] + g(z) \]

where \( p_n(z) \) are polynomials and \( g(z) \) is an entire fun.
pf: WLOG assume $b_\nu \neq 0$ (or consider $f(z) - P, \left(\frac{1}{z}\right)$ if $b_\nu = 0$).

Consider $h_\nu(z) = P_\nu \left(\frac{1}{z-b_\nu}\right)$ which is analytic around zero. Then in the disk $C_\nu = \{z : |z| = \frac{|b_\nu|}{2}\}$, there is a maximum,

$$\max_{z \in C_\nu} |h_\nu(z)| = M_\nu. \text{ For } |z| \leq \frac{|b_\nu|}{4}$$

$$p_\nu(z) = h_\nu(0) + \cdots + \frac{h_\nu^{(n_\nu)}(0)}{n_\nu!} z^{n_\nu}$$

$$|h_\nu(z) - p_\nu(z)| = \left| \frac{1}{2\pi i} \int_{C_\nu} \frac{h_\nu(\zeta)}{\zeta^{n_\nu+1}} (z-\zeta) \, d\zeta \right| |z|^{n_\nu+1}$$

So, start with $h_\nu(z)$ and look at its Taylor polynomial and

Thus, $$|h_\nu(z) - p_\nu(z)| \leq \frac{1}{2\pi} \frac{M_\nu}{(1/|b_\nu|)^n_\nu+1} \cdot \frac{|z|^{n_\nu+1}}{2} \leq \frac{1}{2\pi} \frac{M_\nu}{(1/|b_\nu|)^n_\nu+1} \cdot \frac{|z|^{n_\nu+1}}{2} \leq M_\nu \left(\frac{4|z|}{|b_\nu|}\right)^{n_\nu+1}.$$ 

Consider a series $\sum_{\nu} M_\nu \left(\frac{4}{|b_\nu|}\right)^{n_\nu+1} \cdot Z^{n_\nu+1}$. It converges in all of $C$ if (we can only work with $n_\nu$) \ldots we Hadamard's formula to get $R = \infty$.

Use $h(z) = \sum_{\nu} \left[ P_\nu \left(\frac{1}{z-b_\nu}\right) - p_\nu(z) \right]$ is well-defined in $C \setminus \{b_\nu\}$

Hadamard's formula: \lim_{\nu \to \infty} M_\nu^{\frac{1}{n_\nu+1}} = e^{-\frac{1}{2}(\text{omitted b/c})} 0 < \frac{1}{|b_\nu|} \leq 0$ so the upper limit is zero, and many coefficients are zero, depending on the $n_\nu$'s.

Take logarithm here: \lim_{\nu \to \infty} \left(\frac{\log M_\nu}{n_\nu+1} - \log \frac{1}{|b_\nu|}\right) = -\infty$; so now
choose \( n > \log M \) \( \implies \) the first part of the proof is shown.

State it here:

\[ f(z) \text{ is an arbitrary meromorphic fn. with poles at } b_j \]

and singular parts \( P_n \left( \frac{1}{z-b_j} \right) \implies \text{Consider } f(z) - h(z) = g(z) \)

and \( g(z) \) is an entire fn.

It appears this thm is due to Weierstrass, mostly. But Weierstrass used "canonic" products to represent fns.

\textbf{Infinite products (not just an arbitrary product)}

\textbf{Defn: } \( p_1, p_2, \ldots = \prod_{n=1}^{\infty} p_n \) where \( p_n \in \mathbb{C} \) is called an infinite product.

With sums we had some algebra to deal with the sums, such as \( \sum C x_n = C \sum x_n \). We will need to look at an algebra for infinite product, and to find where they converge.

Notice if \( \exists i \in \mathbb{Z}: p_i = 0 \), \( \prod_{n=1}^{\infty} p_n = 0 \). A partial product \( p_n = \prod_{j=1}^{n} p_j \).

\textbf{Defn: } \( \prod_{n=1}^{\infty} p_n \) is said to converge if at most a finite number of its factors vanish and the partial products \( p_n^* \) formed by the nonvanishing factors satisfy: \( \lim_{n \to \infty} p_n^* \neq 0 \).

\textbf{Remarks:}

1. \( p_n = \frac{p_n^*}{p_{n-1}^*} \implies n \to \infty \implies 1 \implies \prod (1 + a_n) \text{ converges} \implies \lim_{n \to \infty} a_n = 0 \).

2. To deal with the product, we use logarithm, especially the principal branch \( \log (1 + a_n) \). So \( -\pi \leq 1 + a_n \leq \pi \).

So \( \log p_n^* (1 + a_n) = \sum \log (1 + a_n) = \sum \log (1 + a_n) \).
\[ \text{Thm} \quad \sum_{n} \log(1+an) \quad \text{and} \quad \prod_{n}(1+an), \quad \text{where} \quad 1+an \neq 0, \quad \text{converge simultaneously.} \]

\[ \text{Pf} \quad \text{Problems could arise because of the special nature of the logarithm \textsc{fn}.} \]

Assume \( \sum_{n}(1+an) \) converges \( \Rightarrow \) For \( S_n = \sum_{j=1}^{n} \log(1+aj) \), \( \exists \lim_{n \to \infty} S_n = S. \)

Now \( P_n = e^{S_n} \xrightarrow{n \to \infty} e^S \neq 0 \quad : \quad \prod_{n=1}^{\infty} P_n = \lim_{n \to \infty} P_n \text{ exists and} = e^S. \)

\[ \text{Partial product} \]

\[ \text{Factor} \]

Starting with \( \exists \lim_{n \to \infty} \prod_{j=1}^{n} p_j = P \) we have to be careful about which branch of the multi-valued logarithm \textsc{fn} we will be on. Consider \( \log P, \log(1+an). \)

\[ \arg(P-\pi) \leq \arg P_n < \arg(P+\pi) \]

\[ \sum_{j=1}^{n} \log(1+an) = \log P_n + \frac{1}{2}n \pi i \]

\[ \text{an integer} \]

\[ \text{Final for Homework} \]

\[ \sum_{n} \log(1+an) \text{ converges} \]
Infinite products

\[ P = \prod_{n=1}^{\infty} (1 + a_n) \quad \text{and} \quad \sum_{n=1}^{\infty} \log(1 + a_n) \quad \text{converge simultaneously} \]

\[ = \sum_{n=1}^{\infty} \log(1 + a_n) = S \]

\[ \log P = S + (2\pi i) n \]

Defn \[ \prod_{n=1}^{\infty} (1 + a_n) \quad \text{converges absolutely} \quad \text{iff} \quad \sum_{n=1}^{\infty} |\log(1 + a_n)| < \infty \]

Defn \[ \prod_{n=1}^{\infty} (1 + a_n(z)) \quad \text{converges uniformly on} \quad E \subset \mathbb{C} \quad \text{iff} \quad \sum_{n=1}^{\infty} \log(1 + a_n(z)) \quad \text{converges uniformly on} \quad E \subset \mathbb{C} \]

Remark let \( a_n \) be entire fns. (a sequence of ...)

Then \( S(z) = \sum_{n=1}^{\infty} \log(1 + a_n(z)) \) and \( P(z) = \prod_{n=1}^{\infty} (1 + a_n(z)) \)

both are also entire

\[ S(z) = \sum_{n=1}^{\infty} \log(1 + a_n(z)) \quad \text{converges uniformly on compact subsets of} \quad \mathbb{C} \]

\[ \Rightarrow \quad S(z) \quad \text{is an entire fn} \quad \Rightarrow \quad P(z) \quad \text{converges} \quad \Rightarrow \quad P(z) \quad \text{is also entire} \]

Remark Consider the entire fn \( f(z) : f(z) \neq 0 \ \forall z \in \mathbb{C} \)

\[ \Rightarrow f(z) = e^{g(z)} \quad \text{where} \quad g(z) \quad \text{is an entire fn} \]

Indeed, Fix a pt. \( z \quad \in \quad \mathbb{C} \)

\[ F(z) = \int_{\gamma}^{z} \frac{f'(z)}{f(z)} \quad \text{d}z \]

\[ h(z) = f(z) e^{-F(z)} \quad \text{is entire} \]

\[ h'(z) = e^{-F(z)} (f'(z) - f(z) \frac{f'(z)}{f(z)}) = 0 \quad \Rightarrow \quad h(z) = h(0) e^{-F(0)} \]

\[ \Rightarrow f(z) = e^{F(z)} \cdot f(0) = e^{F(z)} \cdot \text{e}^{-F(0)} = \text{e}^{g(x)} = f(0) + \text{e}^{g(x)} g(x) \]
Remark: Consider \( f(z) \) as an entire function with a finite number of zeros on \( \mathbb{C} \). The zeros of \( f \) (with multiplicities) \( 0, 0, \ldots, 0, a_1, a_2, \ldots, a_N \) list according to multiplicities \( m_0, m_1, \ldots, m_N \). So \( f(z) = z^{m_0} e^{g(z)} \prod_{n=1}^{N} \left( 1 - \frac{z}{a_n} \right) \) is a finite "canonical" product.

Consider: \( 0, 0, \ldots, 0, a_1, a_2, \ldots \) an infinite number of zeros, with \( \lim_{n \to \infty} a_n = \infty \) with \( 0 < |a_1| < |a_2| \leq \ldots \) and each equality can only be repeated a finite number of times because

\[
\sum_{n} \left| \frac{1}{a_n} \right| < \infty \implies \exists R > 0 : \forall n \geq n_0(R) \left| a_n \right| > 2R
\]

We want to get \( f(z) = z^n \times \prod_{n=1}^{\infty} \left( 1 - \frac{z}{a_n} \right) \). The \( e^{g(z)} \) can be divided out to combine with \( f(z) \) getting some new \( f(z) \).

\[
f(z) = z^n \prod_{n=1}^{\infty} \left( 1 - \frac{z}{a_n} \right) \text{ converges (uniformly)} \quad \text{for } |z| \leq R \text{ iff}
\]

\[
\sum_{n=n_0}^{\infty} \log \left( 1 - \frac{z}{a_n} \right) \text{ converges (uniformly)}.
\]

\[
\lim_{z \to 0} \frac{\log(1+z)}{z} = 1 \quad \Rightarrow \forall \epsilon > 0 \exists \delta > 0 : \log(1+z) \in \left[ (1-\epsilon)z, (1+\epsilon)z \right]
\]

Use Weierstrass M-test to show \( \sum_{n=n_0}^{\infty} \log \left( 1 - \frac{z}{a_n} \right) \) converges uniformly in \( \{|z| \leq R\} \).
\[ \left| \log \left(1 - \frac{z}{a_n} \right) \right| \leq C \frac{R}{|a_n|} = M \frac{1}{|a_n|} \]

\[ \sum_1^n \frac{1}{|a_n|} < \infty \implies \sum_1^n \log \left(1 - \frac{z}{a_n} \right) \text{ converges uniformly by Weierstrass M-Test.} \]

\[ \implies f(z) = z^n \prod_{n=1}^\infty \left(1 - \frac{z}{a_n} \right) \text{ is well defined.} \]

Not all functions satisfy this restrictive property. 

E.g., \[ f(z) = \sin \pi z : \sum_1^n \frac{1}{|a_n|} = \infty \]

\[ = \sum_1^n \frac{1}{ln} \]

Also, the \( \Gamma \) function has zeros at all the negative integers, so again \[ \sum_1^n \frac{1}{|a_n|} = \sum_1^n \frac{1}{n} = \infty. \text{ Note that in both these cases } \]

\[ \sum_1^n \frac{1}{|a_n|^2} \text{ converges, so } f(z) = z^n \prod_{n=1}^\infty \left(1 - \frac{z}{a_n} \right) \text{ does not solve our problem in general. So we would like to} \]

have \[ f(z) = z^n \prod_{n=1}^\infty \left(1 - \frac{z}{a_n} \right) e^{p_n(z)} \]

where \( p_n(z) \) is a correcting polynomial. \[ f(z) = z^n \prod_{n=1}^\infty \left(1 - \frac{z}{a_n} \right) e^{p_n(z)} \text{ converges (uniformly on} \]

compact subsets of \( C \) \[ \text{iff } \sum_1^n R_n(z) = \sum_1^n \left[ \log \left(1 - \frac{z}{a_n} \right) + p_n(z) \right] \]

where \( \text{Im}(R_n(z)) \in (-\pi, \pi] \). So \( p_n(z) \) will be chosen so that \[ \text{Im}(R_n(z)) \text{ is in the interval } (-\pi, \pi]. \text{ Consider } R > 0 \]

\[ |a_n| > 2R \]

\[ f_n > n_o \]

\[ \text{So } |z| < R \implies \]

\[ \left| \frac{z}{a_n} \right| < \frac{1}{2} \]
Since \(|z_{an}| < \frac{1}{2}\), \(\log(1 - \frac{z}{a_n}) = -\frac{z}{a_n} - \frac{1}{2} \left(\frac{z}{a_n}\right)^2 - \frac{1}{3} \left(\frac{z}{a_n}\right)^3 - \ldots\).

So let \(p_n(z) = \frac{z}{a_n} + \frac{1}{2} \left(\frac{z}{a_n}\right)^2 + \frac{1}{3} \left(\frac{z}{a_n}\right)^3 + \ldots + \frac{1}{m_n} \left(\frac{z}{a_n}\right)^{m_n}\) so that \(m_n\) is large enough so that \(\log(1 - \frac{z}{a_n}) + p_n(z)\) is small enough to be in the interval \((-\pi, \pi]\).

Now \(|r_n(z)| \leq \frac{1}{m_{n+1}} \left(\frac{R}{|a_n|}\right)^{m_{n+1}} + \frac{1}{m_{n+2}} \left(\frac{R}{|a_n|}\right)^{m_{n+2}} + \ldots\)

\[\leq \frac{1}{m_{n+1}} \left(\frac{R}{|a_n|}\right)^{m_{n+1}} \left(1 + \frac{R}{|a_n|} + \left(\frac{R}{|a_n|}\right)^2 + \ldots\right)\]

\[= \frac{1}{m_{n+1}} \left(\frac{R}{|a_n|}\right)^{m_{n+1}} \left(\frac{1}{1 - \frac{R}{|a_n|}}\right) \leq \frac{1}{m_{n+1}} \left(\frac{R}{|a_n|}\right)^{m_{n+1}} \left(\frac{1}{1 - \frac{R}{2}}\right)\]

\[\leq \frac{1}{m_{n+1}} \left(\frac{1}{2}\right)^{m_n} \rightarrow 0\]

So \(\lim_{n \to \infty} r_n = 0\). \(\Rightarrow\) Also, \(f(z) = z^n \prod_{k=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{p_n(z)}\) is well-defined.

To find \(m_n\), try to choose the smallest \(m_n\) so that \(\text{Im}(r_n(z)) e^{-\pi m_n}\).

So, for any given choice of \(a_n\) will there be an \(m_n\)?

Note: \(m_n - n \sum_{n=1}^{\infty} \left(\frac{R}{|a_n|}\right)^{n+1}\) converges for any \(R\) since \(\frac{R}{|a_n|} < \frac{1}{2}\).

This \(m_n\) is too large for most practical purposes. We would like to have \(m_n\) that did not even depend on \(n\), and is as small as possible.
Thm (Weierstrass) There exists an entire fn. with arbitrarily prescribed zeros at provided that \( \lim_{n \to \infty} a_n = \infty \).

Every entire fn. with this condition and no other zeros can be written in the form

\[
f(z) = z^n \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right)^{\frac{1}{a_n}} e^{\frac{1}{a_n}(\frac{z}{a_n})^n}
\]

where \( g \) is an entire fn.

The proof is already sketched out in the previous work.

Remark a. If \( f(z) \) and \( g(z) \) are entire fns then

\[
h(z) = \frac{f(z)}{g(z)} \text{ is a meromorphic fn.}
\]

\( \Rightarrow \) only has poles,

b) Every meromorphic fn. \( h(z) \) can be written in the form: \( h(z) = \frac{f(z)}{g(z)} \), \( f, g \) are entire fns.

Let \( 0, 0, \ldots, 0, a, a, \ldots \) be the poles of \( h(z) \). Then, by Weierstrass Thm, \( \exists \) \( g(z) \) : \( g \) is entire and the poles of \( h(z) \) are the zeros (to the same order) of \( g \).

\( \Rightarrow h(z) = \frac{f(z)}{g(z)} \) is an entire fn. \( h(z)g(z) = f(z) \Rightarrow h(z) = \frac{f(z)}{g(z)} \) if \( f, g \) are entire.

Remark: Choice of \( m_n \):

E.g.: \( \sin \pi z \) zeros, \( n \in \mathbb{Z} \), \( \sum_{n=0}^{\infty} \frac{1}{a_n} = \infty \), but \( \sum_{n=0}^{\infty} \frac{1}{|a_n|^2} < \infty \)

In general,

Now, \( \exists h \geq 0 \), \( h \in \mathbb{Z} : \sum_{n=0}^{\infty} \frac{1}{|a_n|^h} = \infty \), but \( \sum_{n=0}^{\infty} \frac{1}{|a_n|^h+1} < \infty \).

Let \( m_n = h \Rightarrow \sum_{n=0}^{\infty} \frac{1}{m_{n+1}} (\frac{R}{|a_n|})^{m_{n+1}} \) only depends on this part.
Thus, \( f(z) = e^{g(z)} \left[ \prod_{n=1}^{\infty} \left( 1 - \frac{z}{a_n} \right) e^{\frac{z}{a_n} + \ldots + \frac{1}{h} \left( \frac{z}{a_n} \right)^{h+1}} \right] \)

called the canonical product, with \( h \) called the genus of the canonical product, showing
\[
\sum |a_n|^{-h} = \infty \quad \text{but} \quad \sum |a_n|^{-h_1} < \infty
\]

Now, \( \sin \pi z = z e^{g(z)} \prod_{n \neq 0} \left( 1 - \frac{z}{n} \right) e^{\frac{z}{n}} \), where \( g(z) \) can be found, but we are out of time, so \( g(z) = \log \pi \)

\[ \therefore \sin \pi z = \pi z \prod_{n \neq 0} \left( 1 - \frac{z}{n} \right) e^{\frac{z}{n}} \]

Letting \( n \) be negative, the positive kills many of the terms, so

\[ \sin \pi z = \pi z \prod_{n=1}^{\infty} \left( 1 - \frac{z^2}{n^2} \right) \]