2/9/09 look at a metric. 

For $z, \bar{z} \in \mathbb{C}$, $|z - \bar{z}|$ = distance between $z$ and $\bar{z}$.

Some basics of metrics and topology of $\mathbb{C}$.

**Defn.** The $\delta$-nbhd of a pt.

$$N_\delta(z_0) = \{z \mid |z - z_0| < \delta \} \subset \mathbb{C}$$

**Defn.** A nbhd of $z_0$ is any set $N$ s.t.

$$N_{\delta}(z_0) \subseteq N \subseteq \mathbb{C}$$

**Defn.** $S \subset \mathbb{C}$ is an open set iff $S$ is a neighborhood of each of its pts.

**Ex.** $N_{\delta}(z_0)$ is an open set b/c (Indeed)

$$\forall z \in N_{\delta}(z_0), \quad |z - z_0| < \delta$$

Consider $\delta' = \delta - |z - z_0| > 0$.

$$\forall z \in N_{\delta'}(z)$$

$$|z - z_0| = |z - z + z - z_0| \leq |z - z| + |z - z_0|$$

$$\leq \delta - |z - z_0| + |z - z_0| = \delta'$$

$$\therefore |z - z_0| < \delta' \Rightarrow z \in N_{\delta'}(z_0) \Rightarrow$$

$$N_{\delta'}(z) \subseteq N_{\delta}(z_0) \Rightarrow N_{\delta}(z_0)$$ is open.

**Remark.** $S \subset \mathbb{C}$ is a closed set iff $\mathbb{C} \setminus S$ is open.

$\emptyset$ and $\mathbb{C}$ by def. are both open and closed.

**Properties of Open and Closed Sets**

1. The intersection of a finite number of open sets is open.
2. The union of any collection of open sets is open.
3. The union of a finite number of closed sets is closed.
4. The intersection of any number of closed sets is closed.
pf: 1. Let \( U_1, \ldots, U_n \) be open sets, and \( U = \bigcap_{j=1}^{n} U_j \). If \( z \in U \), \( V \subseteq U \), then \( \exists \delta > 0 \) such that \( N_{\delta}(z) \subseteq U \). Let \( \delta = \min_{j=1}^{n} \delta_j > 0 \). Then \( N_{\delta_j}(z) \subseteq U \) and \( \bigcap_{j=1}^{n} U_j \subseteq U \). Thus, \( \bigcap_{j=1}^{n} U_j \) is open.

2. Let \( S_1, \ldots, S_n \) be closed \( \Rightarrow U_j = C \setminus S_j \) is open.

Using De Morgan's law, \( C \setminus (\bigcup_{j=1}^{n} S_j) = \bigcap_{j=1}^{n} (C \setminus S_j) \)

Let \( S = \bigcup_{j=1}^{n} S_j \)

\( C \setminus S \) is open so \( S \) is closed.

Algebra of sets: involves sum (union) and product (intersection) of sets.

Connectedness can be looked at as linearly connected, but our view will be more general. Our definitions and explanations may not be formally perfect topologically speaking.

Defn: An open set \( S \subseteq C \) if it cannot be represented as the union of 2 disjoint open sets, none of which is empty.

Rem. \( S \), open, is not connected iff \( \exists S_1, S_2 \) nonempty and open \( S \cap S_2 = \emptyset \), \( S_1 \cup S_2 = S \).

Ex: \( S = N_1(0) \cup N_1(t+i) \)

(by its defn.) is not connected.

(6) \( S = N_1(0) \) is connected

(Use from real analysis, \( (a, b) \) is connected, any interval of the real line is connected)

This is not trivial to prove.

Def: \( \frac{z_1, z_2, \ldots, z_n \in C, [z_1, z_2]}{z} = \left\{ z + t(z_2 - z_1) \mid 0 \leq t \leq 1 \right\} \). This is a standard parameterization for a segment in \( C \).
Def Let $z_1, \ldots, z_n \in \mathbb{C}$. The polygonal path joining $z_1, \ldots, z_n$ is 

$$[z_1, z_2] \cup [z_2, z_3] \cup \ldots \cup [z_{n-1}, z_n]$$

Note: Don't close the path.

Thm: A nonempty set $S \subseteq \mathbb{C}$ is connected if any two of its points can be joined by a polygonal path which lies in the set.

Necessity ($\Rightarrow$) Assume $S$ is connected. Consider, with fixed $z_0 \in S$, sets 

$$S_1 = \{ z \in S \mid z_0 \text{ can be joined to } z \text{ by a polygonal path in } S \}$$

and 

$$S_2 = \{ z \in S \mid z_0 \text{ cannot be joined to } z \}.$$

Let $S = S_1 \cup S_2$ and $S_1 \cap S_2 = \emptyset$. Both are open because $S$ is connected. $S_1$ open by $\forall z \in S \subseteq S \exists \delta > 0 \forall \delta \in N_\delta(z) \subseteq S$.

$\forall z \in N_\delta(z)$, so it can be connected to $z$, hence to $z_0$, so $3 \in S_1 \Rightarrow N_{\delta}(z) \subseteq S_1 \Rightarrow S_1$ is open.

Let $z \in S_2 \subseteq S \Rightarrow \exists \delta > 0 \Rightarrow N_{\delta}(z) \subseteq S$. $\forall z \in N_{\delta}(z)$, $z \in S_2$.

In $S_2$, $z_0 \in S_1 \Rightarrow S_1 \neq \emptyset \Rightarrow S_2 = \emptyset \Rightarrow S = S_1$.

($\Leftarrow$) Sufficiency: Assume $S$ is not connected. $S = S_1 \cup S_2$. $S_1 \cap S_2 = \emptyset$, $S_1, S_2$ open, $S_1 \neq \emptyset$, $S_2 \neq \emptyset$. Let $z_1 \in S_1$ and $z_2 \in S_2$.

In $S_1$, $z_1 \in S_1$ and $z_2 \in S_2$. Let $P$ be a polygonal path joining $z_1$ and $z_2$.

Without loss of generality, assume $P = [z_1, z_2] = \{ z(t) = z_1 + t(z_2 - z_1) \mid t \in [0, 1] \}$. 

$(z(t))_{t \in [0, 1]}$ is a polygonal path in $S_1$ but not in $S_2$.
Let $A_1 = \{ t \mid z(t) \in S_1 \} \setminus \{ 0 \}$ and $A_2 = \{ t \mid z(t) \in S_2 \} \setminus \{ 1 \}$.

$A_1 \cup A_2 = (0,1)$ and $A_1 \cap A_2 = \emptyset$, $A_1 \neq \emptyset$, $A_2 \neq \emptyset$.

$A_1$, $A_2$ open w.r.t. $z(t) \in S_1 \Rightarrow z(\tau) \in S_1$ where $\exists \delta > 0$ so $|t - \tau| < \delta$.

Thus $(0,1)$ is not connected $\iff$ Therefore, the assumption is wrong, so $S$ is connected.$\blacksquare$

Remark: The previous thm can be extended to state that the polygonal path can have sides parallel to the coordinate axis.

Def. A domain (region) = a nonempty, open, connected set.

Via the Heine - Borel property, let's consider compactness.

Def. A sequence, $z_1, z_2, \ldots$

is any function from $\mathbb{Z}^+ \to \mathbb{C}$.

$\exists n_0 \geq 0$ s.t. $|z - z_0| < \varepsilon \ \forall \ n > n_0$.

Remark: Let $z_n = x_n + iy_n$ and $z_0 = x_0 + iy_0$; then

$\lim_{n \to \infty} z_n = z_0 \iff \lim_{n \to \infty} x_n = x_0$ and $\lim_{n \to \infty} y_n = y_0$.

Def. Fundamental or Cauchy sequence $\{z_n\}$ if $\forall \varepsilon > 0 \exists n_0 > 0$ s.t. $|z_n - z_m| < \varepsilon \ \forall n, m > n_0$.

Remark: (a) $\{z_n\}$ is fundamental (Cauchy) $\iff \{x_n\}, \{y_n\}$ are both fundamental (Cauchy) seq.

(b) $\{z_n\}$ is convergent iff $\{z_n\}$ is fundamental (follows from and real analysis).

i.e., $\mathbb{C}$ is a complete space $\iff$ all sequences are fundamental.
Def. A set \( S \subseteq \mathbb{C} \) is complete if any fundamental \( \{Z_n\} \subseteq S \), i.e., \( \forall n, Z_n \in S \), sequence converges to a pt. \( z_0 \in S \).

Remark (as exercise) \( S \) is complete iff \( S \) is closed.

2/5/09 Review: \( S \) open \( \iff \) \( S \) nbhd. \( S \) closed \( \iff \overline{S} \) is open \( \iff \overline{S} \) is connected \( \iff \) polygonal path \( S \) complete \( \iff \) any seq. is Cauchy (fundamental)

Rem. \( S \) closed \( \iff \) \( S \) complete \( \Rightarrow \) Indeed (PF)

\( S \) closed \( \Rightarrow \) consider \( \{Z_n\} = \text{fundamental} \) \( Z_n \in S \)

\( \Rightarrow \exists \lim_{n \to \infty} Z_n = z \in \mathbb{C} \) \( \Rightarrow \) we claim: \( z \in S \). If not, i.e., \( z \in \overline{S} \), open

\( \Rightarrow \exists \delta > 0 : N_\delta(z) \subseteq \mathbb{C} \setminus S \Rightarrow \{Z_n\} \to z \) \( \Rightarrow \) since \( |Z_m - Z_n| < \delta \)

\( \forall \varepsilon > 0 \)

\( \Rightarrow z \in S \Rightarrow S \) is complete.

Suppose \( S \) is complete \( \Rightarrow \) let \( z \in \mathbb{C} \setminus S \) (to show \( \mathbb{C} \setminus S \) is open)

We claim \( \forall z \in \mathbb{C} \setminus S, \exists \delta > 0 : N_\delta(z) \subseteq \mathbb{C} \setminus S \). If not, then

\( \exists \{Z_n\} : Z_n \in S, \lim_{n \to \infty} Z_n = z \Rightarrow \{Z_n\} \) is fundamental \( \Rightarrow z \in S \)

Thus, \( \mathbb{C} \setminus S \) is open and \( S \) is closed.

Compactness using the Heine–Borel property.

Defn. Let \( S \subseteq \mathbb{C} \). An open covering of \( S \) = \{open sets, \( U_x \) \}

Defn. A subcovering = a subcollection of \( \{U_x\} \) with the same property.

Defn. A finite covering = an open covering which consists of a finite number of sets.
Def. A set $S \subseteq \mathbb{C}$ is compact if every open covering of $S$ contains a finite subcovering. Rem. Every compact set is complete $\iff$ closed. Pf. Let $S$ be a compact set. Consider $\{Z_n\}$, fundamental sequence $Z_n \rightarrow z \in \mathbb{C}$. We claim $z \in S$. If not, i.e., if $z \notin S$, then consider $\exists 3 \in S$. Thus $|z - 3| > 0$. Let $\delta = \frac{|z - 3|}{2}$. But $\exists n_0$ such that $Z_n \in N_{\delta}(z)$ for all $n > n_0$. Thus $N_{\delta}(z)$ intersects only a finite number of $Z_n$'s.

Consider $\{N_{\delta}(z)\}$ an open covering of $S \Rightarrow \exists$ a finite subcovering of $\{N_{\delta}(z)\}_{n \geq 3 \in S}$. So $S$ includes only a finite number of $Z_n$'s if not, $z \notin S$. Thus $z \in S$. $\Rightarrow$ $S$ is complete.

Rem. $S = \mathbb{C}$ since every sequence of complex numbers which converges, converges to a complex number. $\Rightarrow S$ is complete. However, $\mathbb{C}$ is not compact.

Pf. Consider $\{N_n(0)\}_{n=1}^{\infty}$ an open covering of $\mathbb{C}$, but this has no finite subcovering.

Defn. $S \subseteq \mathbb{C}$ is bounded iff $\exists M > 0$ s.t. $|z| < M$ for all $z \in S$.

Rem. Any compact set is bounded.

Pf. $S$ compact, consider $\{N_n(0)\}_{n=1}^{\infty}$. $\exists$ a finite subcovering of $S$: $\{N_{n_1}(0), N_{n_2}(0), \ldots, N_{n_m}(0)\}$. Let $M = \max \{n_1, n_2, \ldots, n_m\}$.
Rem. Let \( S \subseteq \mathbb{C} \) be bounded. Then \( \forall \varepsilon > 0 \) \( S \) can be covered by finitely many \( \varepsilon \)-neighs of its points.

Proof:
If \( R_j \cap S \neq \emptyset \). Fix any \( z_j \in R_j \cap S \)
with \( N_{\varepsilon_j}(z_j) = R_j \Rightarrow \{N_{\varepsilon_j}\} \) is a finite subcollection \( \Rightarrow S \) is compact. \( \varepsilon \frac{1}{2} \)

Thm: A set \( S \subseteq \mathbb{C} \) is compact if and only if it is closed and bounded.

Proof:
\( \Rightarrow \) \( S \) compact \( \Rightarrow S \) closed
\( \Rightarrow S \) bounded

\( \Leftarrow \)
Suppose \( S \) is closed and bounded. Suppose \( S \) is not compact.

Then \( \exists \) an open covering \( \{U_j\} \) of \( S \) without a finite subcovering.

Let \( \varepsilon_n = 2^{-n} \). 3 steps to get contradiction:

Step 1: Start with \( \varepsilon_1 = \frac{1}{2} \). According to the Remark at the top,
\( \exists z_1 \in S \): \( S \not\subset N_{\varepsilon_1}(z_1) \) cannot be covered by a finite covering of \( U_j \).

Step 2: \( \exists z_2 \in S \cap N_{\varepsilon_1}(z_1) \): \( S \cap N_{\varepsilon_2}(z_2) \) cannot be covered by a finite number of \( U_j \).

We construct \( \{z_n\} : z_n \in S, \Rightarrow |z_{n+1} - z_n| < \varepsilon_n = \frac{1}{2^n} \)

\( \forall n \in \{1, 2, 3, \ldots\}, |z_{n+p} - z_n| = |z_{n+p} - z_{n+p-1} + z_{n+p-1} - z_{n+p-2} + \ldots + z_{n+1} - z_n| \)

\( \leq |z_{n+p} - z_{n+p-1}| + |z_{n+p-1} - z_{n+p-2}| + \ldots + |z_{n+1} - z_n| \)

\( \leq \frac{1}{2^{n+p-1}} + \frac{1}{2^{n+p-2}} + \ldots + \frac{1}{2^n} < \frac{1}{2^n} \)

\( \Rightarrow \{z_n\} \) is fundamental since \( \forall \varepsilon > 0 \), \( \exists n_0 > 0 \) s.t.
\( |z_n - z_m| < \varepsilon \) \( \forall n, m > n_0 \). \( \exists N_{\varepsilon}(z_n) \cap S \) cannot be covered by a finite number of \( U_j \).

\( \exists \lim_{n \to \infty} z_n = z \in S \). Consider \( \{U\} \) an open covering.
\( \exists \delta > 0 : N_{\varepsilon}(z) \subset U_{\delta} \). Consider \( \varepsilon : \varepsilon > 0 \), \( |z - z_n| < \delta \). \( \forall z \in U_i \)

\( \exists \delta > 0 : N_{\varepsilon}(z) \subset U_i \). Consider \( \varepsilon : \varepsilon > 0 \), \( |z - z_n| < \delta \). \( \forall z \in U_i \)
\[ \forall z \in N_{\varepsilon_n}(z_0), \quad |z-z_0| \leq |z-z_n| + |z_n-z_0| \]
\[ \leq \varepsilon_n + \frac{\delta}{2} < \frac{\delta}{2} + \frac{\delta}{2} = \delta \]
\[ \Rightarrow U_i \text{ covers } N_{\varepsilon_n}(z_0) \text{ which contradicts that } N_{\varepsilon_n}(z_0) \cap S \text{ is not covered by a finite number of } U_i's. \]
\[ \therefore S \text{ is compact.} \]

Ares and closed curves are much simpler in complex analysis than in real analysis. **Def** An arc is an image of a closed interval under a continuousfcn. **Ex** \( Y: z = z(t) = x(t) + iy(t) \), \( t \in [a, b] \).

The "t" parametric representations gives us an orientation for free. However, this representation is not unique.

Change of variable Consider a non-decreasing continuous fcn \( t = \varphi(z) \), \( a' \leq z \leq b' \), where \( \varphi(a') = a \) and \( \varphi(b') = b \).

Now \( Y: z = z(t) = z(\varphi(z)) \), \( z \in [a', b'] \).

**Rem:** Let \( z'(t_0) \neq 0 \), i.e. \( z'(t_0) = x'(t_0) + iy'(t_0) \neq 0 \) for \( t_0 \in (a, b) \),

**Def:** \( Y \) is differentiable iff \( z'(t) \) is continuous.

**Def:** \( Y \) is regular iff \( z'(t) \) is continuous and \( z'(t) \neq 0 \).
Completion of our work on topology we will need for integration theory in $\mathbb{C}$ and some work on analytic sets.

An arc $\gamma : z = z(t)$, $a \leq t \leq b$, $z = x(t) + i y(t)$, cont. on $t \in [a, b]$, is an arc iff $\exists z'(t)$ differentiable arc continuously on $[a, b]$, $z'(t) = x'(t) + i y'(t)$ and $z'(t) \neq 0$.

(regular arc) $z(1) \circlearrowleft z(0)$

most general piecewise [reg.]

$\Rightarrow \gamma$ consists of a finite number diff. or reg. interval arcs.

A Jordan arc is an arc with no intersections, i.e., $z(t_1) = z(t_2)$ iff $t_1 = t_2$.

A closed curve is an arc with $z(a) = z(b)$, could be any type arc.

Ex $[z_1, z_2] 
\gamma_1 : z = z_1 + (z_2 - z_1)t \quad t \in [0, 1] 
\gamma_2 : z = z_2 + (z_1 - z_2)t \quad t \in [0, 1]$

Ex Circle w/ ctr. $z_0$, radius $r$
$\gamma : z = z(\theta) = z_0 + re^{i\theta}, 
\theta \in [0, 2\pi]$ 
a closed curve.

Defn. Given $\gamma : z = z(t) + \in [a, b]$
$\quad \gamma(1) \circlearrowleft \gamma(a) \quad \Leftrightarrow \quad \gamma(-1) \circlearrowleft \gamma(-a)$ 
$\gamma(-t) : z = z(-t), \ t \in [-b, -a]$
Properties of Analytic Fns:

**Thm.** Let \( f \) be analytic in domain \( D \subset \mathbb{C} \) (sometimes this may include \( \infty \)), and let \( f'(z) = 0 \) for \( z \in D \). Then \( f(z) \) is constant in \( D \). We will use this for uniqueness of fns (up to a constant) i.e., 2 fns with the same derivative differ by a constant.

**Pf.** Let \( f(z) = u(x,y) + iV(x,y) \) and \( z = x + iy \)

\[
\begin{align*}
f'(z) &= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}.
\end{align*}
\]

\( f'(z) = 0 \implies \frac{\partial u}{\partial x} = 0, \frac{\partial v}{\partial x} = 0 \) (C-R)\( \implies u \) is constant on any vert. line in \( D \)

\( \implies v \) is constant on any horizontal line in \( D \)

Fix \( z_0 \in D \); \( \forall z \in D \), \( u(z) = u(z_0) \implies f(z) = f(z_0) \).

**Rem.** \( f \) is constant in \( D \) if a) both real part is constant

or b) both imaginary part is constant.

or if c) its absolute value is constant.

or if d) its argument is constant.

Indeed, a) \( f = u + iv \) u constant \( \implies \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0 \) \( \implies f'=0 \) by (C-R)

\( \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 0 \) and \( \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 0 \implies f' = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial y} = 0 \implies f \) is constant.

b) \( v \) is constant and \( v = \text{Re}(-if) \) is constant \( \implies f \) is constant.

c) \( u^2 + v^2 \) is constant \( \implies \) case 1: \( u^2 + v^2 = 0 \implies u,v = 0 \implies f \equiv 0 \) in \( D \).

\( \begin{align*}
\begin{cases}
\frac{\partial u}{\partial x} - v \frac{\partial u}{\partial y} = 0 \\
v \frac{\partial u}{\partial x} + u \frac{\partial u}{\partial y} = 0 \\
u^2 + v^2 \neq 0
\end{cases}
\end{align*} \implies \text{u constant in } D.
\]
(d) $\arg(f(z)) \equiv \text{constant}$ \implies \exists a, b \in \mathbb{R} \text{ s.t. } au + bv = 0 \quad (35)

\[ au + bv = \text{Re}(a - ib)f = 0 \]

= $\frac{\text{Re}(a - ib)}{u + iv} = 0 \implies (a - ib)f \equiv \text{const.}$

\[ \implies f \text{ is constant in } D. \]

**Remark:** This holds for connected sets $D$. $D$ connected is important for this Thm:

**Ex:**

\[ U_1 \cup U_2 \]

$D = U_1 \cup U_2$, both open and nonempty.

\[ f \text{ is a conformal mapping } \iff f \text{ is analytic and } f'(z) \neq 0. \]

Let $D$ be a domain

\[ z(t) \rightarrow z_0, \quad t \rightarrow t_0, \quad f(z(t)) \rightarrow f(z_0) \]

\[ f(z(t)) = \begin{cases} 1, & z \in U_2 \\ 0, & z \in U_1 \end{cases} \]

\[ \Rightarrow f'(z) \equiv 0 \text{ on } D = U_1 \cup U_2, \quad \text{but } f(z) \text{ is not constant.} \]

\[ f : D \rightarrow D^* \text{ continuous, also a domain } \]

\[ W = f \circ z : W \text{ is continuous b/c it is the composition of continuous fns.} \]

Let $W = W(t) = f(z(t))$

Let $a \leq t_0 \leq b$ \quad $z_0 = z(t_0)$ \quad $W_0 = W(t_0) = f(z_0)$

\[ f \text{ analytic in } D \text{ and } z \text{ a regular arc } \implies W'(t_0) = f'(z_0) z'(t_0) \]

Indeed, \[ W'(t_0) = \lim_{t \to t_0} \frac{W(t) - W_0}{t - t_0} = \lim_{t \to t_0} \frac{W(t) - W_0}{t - t_0} \]

\[ = \lim_{z \to z_0} \frac{z - z_0}{t - t_0} = \frac{z - z_0}{t - t_0} \]

\[ \Rightarrow f'(z_0) \neq 0 \text{ defn of conformal mapping!} \]

$\Rightarrow \quad z'(t_0) \neq 0$ \quad (regular)

$W'(t_0) \neq 0 \Rightarrow \exists$ tangent vector to $z$ at $z_0$ and $\nabla f$ at $W_0$.

\[ \text{arg} W'(t_0) = \text{arg } f'(z_0) + \text{arg } z'(t_0) \]

depends on choice of curve.

\[ \text{arg}(W(t_0) - W_0) = \text{arg } f'(z_0) \]

does not depend on choice of curve.
Therefore, we can conclude that two curves which form an angle at \( z_0 \) are mapped upon curves forming the same angle. (Under conformality, angles are preserved.)

Another form of conformality

\[
\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0)
\]

Consider abs. value:

\[
\lim_{z \to z_0} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| = |f'(z_0)|. \text{ So, if } 
\]

\( z \) is "close to" \( z_0 \), \( |f(z) - f(z_0)| \) is close to \( |f'(z_0)||z - z_0| \). Consider a "very small" circle

i.e.

\[
\left| \frac{W'(t_0)}{Z'(t_0)} \right| = |f'(z_0)| \text{ if } r \text{ is small, } f(C) \text{ is a curve "close to" } C^* \]

along their curves! so this "scaling" in "small circles" only depends on the map \( f \) and not the curves. Any map that maps small circles to small circles in analytic.

Rand Both kinds of conformality imply separately the existence of a derivative (under the assumption that \( \frac{\partial f}{\partial x} \) and \( \frac{\partial f}{\partial y} \) are continuous in a nbhd of \( z_0 \)).

Let \( W(t) = f(Z(t)) \).

\[
W'(t_0) = \frac{\partial f}{\partial x}(z_0) x'(t_0) + \frac{\partial f}{\partial y}(z_0) y'(t_0) = \frac{1}{2} \left( \frac{\partial f}{\partial x}(z_0) - \frac{\partial f}{\partial y}(z_0) \right) z'(t_0) + \frac{1}{2} \left( \frac{\partial f}{\partial x}(z_0) + \frac{\partial f}{\partial y}(z_0) \right) \bar{z}'(t_0)
\]

\[
W(t_0) = \frac{\partial f}{\partial x}(z_0) z'(t_0) + \frac{\partial f}{\partial y}(z_0) \bar{z}'(t_0)
\]
Conformality

$\text{$f(z)$ is analytic on $D \subseteq \mathbb{C}$}$

\[ f(z) = \frac{w(\zeta)}{\zeta} \]

\[ f = f(z) \]

\[ w(\zeta) = f(z(\zeta)) \]

\[ \frac{w'(t_0)}{z'(t_0)} = f'(z(t_0)) \]

\[ \arg \text{ does not depend on $\gamma$} \]

Remark

Let $f$ be such that $\frac{df}{dx}, \frac{df}{dy}$ are continuous in a nbhd of $z_0$

\[ W'(t_0) = \frac{1}{2} \left( \frac{\partial^2 f}{\partial x^2}(z_0) - i \frac{\partial^2 f}{\partial y^2}(z_0) \right) z'(t_0) + \frac{1}{2} \left( \frac{\partial^2 f}{\partial x^2}(z_0) + i \frac{\partial^2 f}{\partial y^2}(z_0) \right) \bar{z}'(t_0) \]

\[ \frac{\partial f}{\partial \bar{z}}(z_0), \frac{\partial f}{\partial z}(z_0) \]

\[ \frac{W'(t_0)}{z'(t_0)} = \frac{\frac{\partial f}{\partial \bar{z}}(z_0)}{\frac{\partial f}{\partial z}(z_0)} + \frac{\frac{\partial f}{\partial \bar{z}}(z_0) \bar{z}'(t_0)}{\frac{\partial f}{\partial z}(z_0) z'(t_0)} = A + B \frac{z'(t_0)}{z'(t_0)} \]

Recall $\text{Arc $\gamma$ is regular } \Rightarrow z'(t) \neq 0$.

But \[ \left| \frac{z'(t_0)}{z'(t_0)} \right| = 1 \]

So \[ \left| \frac{z'(t_0)}{z'(t_0)} \right| = 1 \]

\[ \arg \frac{z'(t_0)}{z'(t_0)} = \arg \bar{z}'(t_0) - \arg z'(t_0) \]

\[ = -2 \arg z'(t_0) \]

So the circle degenerates to a pt. with radius $B = 0$.

So \[ \frac{\partial f}{\partial \bar{z}}(z_0) = 0 \] which is equivalent to $C - B$ eq.

\[ \Rightarrow \exists f'(z_0) \]

\[ \frac{W'(t_0)}{z'(t_0)} \text{ does not depend on $\gamma$ could happen if(i) $B = 0$ } \Rightarrow f'(z_0) \text{ analytic} \]

or ii) $A = 0$ \[ \Rightarrow \frac{\partial f}{\partial z}(z_0) = 0 \]

\[ \Rightarrow \exists f'(z_0) \text{ analytic} \]

(a) and (b) separately imply the analytic condition for complex fcn.
Linear Transformations

**Defn.** A linear (fractional) transformation is a function of the form \( W = S(z) = \frac{az + b}{cz + d} \) where \( ad - bc \neq 0 \), \( a, b, c, d \) fixed \( \in \mathbb{C} \).

**(a)** If \( c = 0 \), then \( S(\infty) = \infty \) is prescribed.

**(b)** If \( c \neq 0 \), then \( S(\infty) = \frac{a}{c} \) and \( S\left(-\frac{d}{c}\right) = \infty \).

**Remark:** The ratio \( \frac{az + b}{cz + d} \) ensures the ratio is not constant.

**S:** \( \mathbb{C} \rightarrow \mathbb{C} \) closure of \( \mathbb{C} \) using \( \frac{cz + d}{cz + d} \) is one-to-one, since \( z = f(w) \) also \( (cz + d)w = az + b \)

**Note:** inverse of \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) \( z = \frac{b - wd}{cw - a} = \frac{dw - b}{a - cw} \)

is \( \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \) \( \text{det} = ad - bc \) \( \Rightarrow S^{-1}(w) = \frac{wd - b}{-wc + a} \)

So we can use \( 2 \times 2 \) matrices to represent linear transformations.

Let \( z = \frac{z_1}{z_2} \) and \( W = \frac{w_1}{w_2} \) \( W = S(z) \) where \( S = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) \( (w_1) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \frac{z_1}{z_2} \)

Then \( \text{in } 2 \text{ tranformations, } W = S_1\left(S_2(z)\right) = S_1S_2z \).

Since the \( a, b, c, d \) are chosen up to some complex constant, set \( ad - bc = 1 \) so that \( S^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \).

**Remark:** A linear transform can be written as a composition of:

(a) a parallel translation: \( W = z + b, \ b \in \mathbb{C} \)

(b) a rotation \( W = k z, \ |k| = 1 \)

(c) a homothetic transform: \( W = k z, \ k > 0 \)

(d) an inversion \( W = \frac{1}{z} \)

So to check a property of linear transforms, it is enough to check the property for all 4 types.

**Indeed,**

(a) \( c = 0 \) \( \Rightarrow W = \frac{a}{d} z + \frac{b}{d} \) \( \Rightarrow z \rightarrow z_1 = \frac{a}{d} z \rightarrow z_2 = z_1 + \frac{b}{d} \)

(b) \( c \neq 0 \) \( W = \frac{az + b}{cz + d} = \frac{ac\zeta + bc}{c(z + d)} = \frac{a \left(\frac{z + d}{c}\right) - ad + bc}{c(z + d)} \)

\[ = \frac{a}{c} + \frac{bc - ad}{c(cz + d)} = \frac{bc - ad}{c^2} + \frac{a}{c} \]

**rotation and homothetic**

Thus \( z \rightarrow z_1 = z + \frac{d}{c} \rightarrow z_2 = \frac{1}{z_1} \rightarrow z_3 = \frac{bc - ad}{c^2} z_2 \rightarrow W = z_3 + \frac{a}{c} \)
There are some invariant for conformal mappings.

**The Cross Ratio** Defn. Given distinct points $z_1, z_2, z_3, z_4 \in \mathbb{C}$, the cross ratio $(z_1, z_2, z_3, z_4)$ is the image of $z, \in \mathbb{C}$ under the linear tran that carries $z_1, z_3, z_4$ to $1, 0, \infty$.

\[ T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

\[ Tz = (z_1, z_2, z_3, z_4) \]

**Remark:** $(z_1, z_2, z_3, z_4)$ is uniquely determined. Indeed let $S$ be another linear tran with the same properies as $T$. Consider $TS^{-1} = U$

\[ U(0) = T(S^{-1}(0)) = T(z_3) = 0 \]
\[ U(1) = T(S^{-1}(1)) = T(z_2) = 1 \]
\[ U(\infty) = T(S^{-1}(\infty)) = T(z_4) = \infty \]

\[ W = Uz = \frac{az + b}{cz + d} \quad U(\infty) = \infty \Rightarrow \frac{a}{c} \rightarrow \infty \Rightarrow c = 0 \]
\[ U(0) = 0 \Rightarrow \frac{b}{d} = 0 \quad U(1) = 1 \Rightarrow \frac{a}{d} = 1 \Rightarrow W = Uz = z \Rightarrow U = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

\[ TS^{-1} = I \Rightarrow S = T \Rightarrow T \text{ is uniquely determined. So } z_1, z_2, z_4 \text{ uniquely determine } T. \text{ Consider (a) } z, z_2, z_3, z_4 \in \mathbb{C} \text{ (w/o } \infty \text{) and } \]

\[ S = (z_1, z_2, z_3, z_4) \quad S_{z_3} = 0 \Rightarrow z - z_3 \text{ in numerator } \]

\[ S_{z_4} = \infty \Rightarrow z - z_4 \text{ in denominator } \quad S_{z_3} = 1 \Rightarrow S = \frac{z - z_3}{z - z_4} \frac{z_2 - z_3}{z_2 - z_4} \]

Since $S$ is unique, this is $S_z$ for the finite case.

(b.) $z_2 = \infty$: $S = \lim_{z \to \infty} \frac{z - z_3}{z - z_4} \frac{z_2 - z_3}{z_2 - z_4} = 1 \Rightarrow S = \frac{z - z_3}{z - z_4} \frac{z_2 - z_3}{z_2 - z_4}$

(c.) $z_3 = \infty$: $S = \lim_{z \to \infty} \frac{z_2 - z_4}{z - z_4} \frac{z - z_3}{z - z_3} = 1 \Rightarrow S = \frac{z_2 - z_4}{z - z_4} \frac{z - z_3}{z - z_3}$

(d.) $z_4 = \infty$: $S = \frac{z - z_3}{z - z_3} \frac{z_2 - z_3}{z_2 - z_3}$ since $\lim_{z \to \infty} \frac{z - z_3}{z - z_4} = 1$

**Thm:** If $z_1, z_2, z_3, z_4 \in \mathbb{C}$ are distinct pts. and $T$ is any linear transformation, then $(Tz_1, Tz_2, Tz_3, Tz_4) = (z_1, z_2, z_3, z_4)$

**Pf:** Let $S = (z_1, z_2, z_3, z_4)$

\[ ST^{-1}(Tz_1) = ST^{-1}(0) = S_{z_3} = 0 \quad \text{and } ST^{-1}(Tz_4) = ST^{-1}(\infty) = S_{z_4} = \infty \]

\[ ST^{-1}(Tz) = S \Rightarrow (z_1, z_2, z_3, z_4). \text{ Uniqueness } \Rightarrow Tz = S z, \text{i.e., } (Tz_1, Tz_2, Tz_3, Tz_4) \]
Rem The linear tr-n $Tz_j = W_j$ for distinct pts. $z_j$ and $w_j$ satisfies $(w_1, w_2, w_3, w_4) = (z_1, z_2, z_3, z_4)$

Thm $(z_1, z_2, z_3, z_4)$ is real iff the 4 pts. lie on a circle or a straight line.

Pf Consider $(z, 1, 0, \infty) = z$. This is real iff $z \in \mathbb{R}$. In order to prove the theorem it is enough to show $T(\mathbb{R})$ is a circle or a straight line. The image for any linear tr-n $T$, since $T$ will preserve the cross ratio since $T(\mathbb{R}) = (Tz, T1, T0, T\infty) = (z, 1, 0, \infty)$

If $w = Tz$, $z = T^{-1} w$

$z = \frac{aw + b}{cw + d}$

Since $z \in \mathbb{R}$, $z = \overline{z}$ implies

$a\overline{w} + b = \frac{aw + b}{cw + d}$

$c w \overline{w} + b c w + \overline{a} d w + \overline{b} d = a \overline{c} |w|^2 + a \overline{d} w + b \overline{c} \overline{w} + b \overline{d}$

$(a c - a \overline{c}) |w|^2 + (b c - a \overline{d}) w + (a \overline{d} - b \overline{c}) \overline{w} + (b \overline{d} - b \overline{d}) = 0$

Pure Image

(a) Consider $a c - a \overline{c} = 0 \Rightarrow C = T(\mathbb{R})$ is a line.

(b) $a c - a \overline{c} \neq 0 \Rightarrow C = T(\mathbb{R})$ is a circle.