Linear (fractional) Transformations

\[ W = S \cdot Z = \frac{aZ + b}{cZ + d} \quad \text{for} \quad a \cdot d - c \cdot b \neq 0 \]

Recall: Cross ratio
\[ W(Z, Z_2, Z_3, Z_4) = \frac{Z - Z_3}{Z - Z_4} \cdot \frac{Z_2 - Z_4}{Z_2 - Z_3} \]
\[ \text{is the most general form. (There are others.)} \]

Symmetry of Lin. Transf. (which is preserved under conformal mappings)

\[ Z \quad \text{and} \quad \overline{Z} \quad \text{are sym. w.r.t.} \quad TR \]

Consider a circle:

\[ \text{and a transf. } T: R \rightarrow C \text{ (a circle or a line)} \]
\[ \text{We say } T \cdot Z = W \quad \text{and} \quad W^* = T \overline{Z} \text{ are sym. w.r.t.} \quad C. \]

Remark: a) Sym. w.r.t. C does not depend on T. What does this mean?
Indeed, consider

\[ T \]
\[ \text{Let } Z = S^{-1}(TZ) \quad \text{and} \quad Z^* = S^{-1}(TZ^*) \]

Consider \( U = S^{-1}T \) which is a lin. fract. transf. since composition is closed for the set of lin. fract. transf. But \( U: R \rightarrow R \). Thus, \( a, b, c, d \in R \) (see HW problem #24), i.e. \( U = \frac{aZ + b}{cZ + d} \).

Now \( U: Z \rightarrow 3 \) and \( U: \overline{Z} \rightarrow \overline{3}^* \), so \( \overline{3}^* = \overline{3} \Rightarrow 3^* = 3 \). So lin. fract. transf. preserve this symmetry.
Def. $z$ and $z^*$ are said to be sym. wrt $C$ (a line or circle) through $z_1, z_2, z_3$ if $(z, z_1, z_2, z_3) = (z^*, z_1, z_2, z_3)$.

Rem. $b$) Defn does not depend on $z_1, z_2, z_3$.

c) $z \rightarrow z^*$ is called a reflection wrt. $C$.

d) If $z \in C$, then $z^* = z$.

Consider $(z, z_1, z_2, z_3) = (z^*, z_1, z_2, z_3) \Rightarrow (z, z_1, z_2, z_3) = (z^*, z_1, z_2, z_3)$.

Ex. $C$ is a line $z = \infty$ (so special case).

\[
\frac{z^* - z_2}{z_1 - z_2} = \frac{z - z_2}{z_1 - z_2}.
\]

Using Modulus, $|z^* - z_2| = |z - z_2|$.

\[
\Rightarrow z \text{ and } z^* \text{ are sym. wrt. } z \in R
\]

equidistant to

\[
\text{Case 1: } \text{Im} \frac{z - z_2}{z_1 - z_2} = -\text{Im} \frac{z^* - z_2}{z_1 - z_2}.
\]

\[
\text{Case 2: } \text{Im} \frac{z - z_2}{z_1 - z_2} = 0 \Rightarrow z \in R \Rightarrow z^* = z
\]

\[
\text{Case 2: } \text{Im} \frac{z - z_2}{z_1 - z_2} \neq 0 \Rightarrow z \neq z^* \Rightarrow C \text{ is the bisecting normal to } [z, z^*].
\]
Ex: \( C = \{ z \mid |z - a| = r, \ a \in \mathbb{C}, \ r \in \mathbb{R}, \ r > 0 \} \) 
\( \mathbb{R}^2, \ R > 0 \)

We will use the fact that the cross ratio is invariant under M.T., i.e., \( (z, z_1, z_2, z_3) = (Tz_1, Tz_2, Tz_3, Tz) \)

Given \( z \), to find \( z^* \) we need to solve:

\[
\begin{align*}
\frac{(z, z_1, z_2, z_3)}{(Tz = z - a)} &= (z^*_1, z_1, z_2, z_3) \\
= (z - a, z_1 - a, z_2 - a, z_3 - a)
\end{align*}
\]

Consider \( z_1 - a = Re^{i\theta} \)

\[
\begin{align*}
\frac{z_1 - a}{z_1 - a} &= \frac{R}{e^{i\theta}} = \frac{R}{(\frac{z_1 - a}{R})} \\
&= \frac{R^2}{z - a}
\end{align*}
\]

Now use the M.T. \( z \rightarrow R^2, \frac{1}{z} \)

\[
\begin{align*}
(z_1 - a, z_2 - a, z_3 - a) &\quad \text{and} \quad (z = z + a) \\
\Rightarrow \quad z^* &\quad (\text{the reflection of } z \text{ in a circle}) = a + \frac{R^2}{z - a}
\end{align*}
\]

\[
\begin{align*}
\Rightarrow z^* - a &= \frac{R^2}{z - a} \\
\frac{z^* - a}{z - a} &= \frac{R^2}{|z - a|^2} \in \mathbb{R}^+ \\
\Rightarrow \text{Im} \frac{z^* - a}{z - a} &= 0 \Rightarrow \arg(z^* - a) = \arg(z - a) \Rightarrow z \text{ and } z^* \text{ are on the same central ray}.
\end{align*}
\]

\[
\begin{align*}
|z^* - a| &= \frac{R^2}{|z - a|} \\
&= \frac{R^2}{|z - a|} \Rightarrow |z^* - a| = |z - a| = R^2.
\end{align*}
\]
Thm: (The symmetry property)
If a linear transformation carries a circle or a straight line $C_1$ into a circle or a straight line $C_3$ then it transforms any pair of symmetric points w.r.t. $C_1$ into a pair of symmetric points w.r.t. $C_2$.

Suppose $z$ and $z^*$ are sym. w.r.t. $C_1$.

$W = Tz$, $W^* = Tz^*$

we claim $W$ and $W^*$ are sym. w.r.t. $C_2$.

Indeed, let $z_1, z_2, z_3 \in C_1$ (distinct) with $W_j = Tz_j \in C_2$, $j = 1, 2, 3$

Let $T_1 z = (z_1, z_2, z_3, z_3)$

Consider $U = T_2 T_1^{-1}$, $1 \rightarrow 1$, $0 \rightarrow 0$, $\infty \rightarrow \infty$

We know that any conformal mapping which preserves 3 pts. is the identity mapping. Thus, $U = I$ and $U z = z$.

Since $U = T_2 T_1^{-1} = I$, then $T = T_2^{-1} T_1$

$\Rightarrow W$ and $W^*$ are sym. w.r.t. $C_2$.

Ex. Find a conformal mapping $W = f(z)$ of $H = \{z | \text{Im } z > 0\}$ onto
$D = \{w | |w| < 1\}$ such that for a given $\alpha \in H$, $f(\alpha) = 0$

Sol'n

$\frac{\alpha - \alpha}{\alpha} \rightarrow 0$, $\frac{\alpha}{\alpha} \rightarrow \infty$

$W = f(z) = \frac{\alpha - \alpha}{\alpha} = \frac{z - \alpha}{\alpha - \alpha}$

For $z$ on line: $z \in \mathbb{R}$

$|w| = 1$ (on circle)

$|\lambda| = 1$

$\Rightarrow |\lambda| = 1$
Ex: \( D = \text{unit disk} \) \( a = \frac{1}{z} \) \( f \)

For a given \( a \), \( a^+ = \frac{1}{a} \)

\( f : D \rightarrow D \), \( f(a) = 0 \)

so \( a^+ = \frac{1}{a} \) \( f \Rightarrow \infty \)

\[ W = \lambda \frac{z - \alpha}{z - \frac{1}{\alpha}} \]

\[ = \mu \frac{z - \alpha}{1 - z \bar{\alpha}} \quad \text{where} \quad \mu = -\frac{\alpha}{\lambda} \]

if \( z \in D \), \( |W| = 1 = |\mu| \left| \frac{z - \alpha}{1 - z \bar{\alpha}} \right| = |\mu| \left| \frac{e^{i\theta} - \alpha}{e^{i\theta}(e^{-i\theta} - \bar{\alpha})} \right| \]

\[ = |\mu| \left| \frac{e^{i\theta} - \alpha}{e^{i\theta} - \bar{\alpha}} \right| \left| \frac{1}{e^{i\theta}} \right| \]

\[ = |\mu| \cdot 1 \cdot 1 \]

\[ \therefore W = f(z) = \mu \frac{z - \alpha}{1 - z \bar{\alpha}} \quad \text{where} \quad \mu \in \mathbb{C}, \quad |\mu| = 1. \]
Let's look at analytical lens from a geometric point of view. As far as conformal mappings:

**Elementary conformal mappings**

\[ W = f(z) \], analytic, with domain \( D \subset \mathbb{C} \)

\[ z = x + iy \]

\[ \mathbb{C} \]

\[ u = u_0 + iv \]

Consider what happens to lines \( x = x_0, y = y_0 \subset D \), etc.

or look at level sets of \( f \):

\[ \{ z \mid \text{Re}(f(z)) = u_0 \} \]

\[ \{ z \mid \text{Im}(f(z)) = v_0 \} \]

**Example:**

Consider \( W = z^2 \)

\[ z = x + iy \]

\[ W = (x^2 - y^2) + 2ixy \]

\[ x = x_0 \Rightarrow u = x_0^2 - y^2 \]

\[ v = 2x_0y \]

\[ u = x^2 - y^2 \]

\[ v = 2xy \]

\[ \Rightarrow v^2 = 4x^2y^2 = 4x_0^2(x_0^2 - u) \]

\[ v^2 = 4x_0^2(x_0^2 - u) \]
For \( f(z) = z^2 \), let \( y = y_0 \) \( \Rightarrow u = x^2 - y_0^2 \) \( v = 2xy_0 \) \( v^2 = 4y_0^2(y_0^2 + u) \).

\[
\begin{align*}
\text{Let } u &= v \text{ be fixed.} \\
u &= u_0 \Rightarrow u_0 = x^2 - y^2 \\
u &= 0 \Rightarrow |x| = |y|
\end{align*}
\]

\( u_0 > 0 \) \( \Rightarrow \) \( x > 0, y > 0 \)

\( u_0 < 0 \) \( \Rightarrow \) \( x < 0, y < 0 \)

\( v = v_0 > 0 \)

\( v = v_0 < 0 \)

\( v = 0 \)

For \( f(z) = z^2 \)

\[ f'(z) = 2z \neq 0 \text{ if } z \neq 0 \Rightarrow \text{conformal } \Rightarrow \text{angles are preserved.} \]

Let's try this looking at polar coordinates.
\[ w = z^\alpha = e^{\alpha \log z}, \quad z \in \mathbb{C}, \quad \alpha > 0 \]

\[ z = e^{\|z\| + i\theta} = r e^{i\theta}, \quad w = r e^{i\phi} = r e^{i\alpha \theta} \]

In general,

\[ \arg w = \alpha \arg z \]

\[ \text{Defn. A sector} \quad S_\theta \{\theta_1, \theta_2\} = \{ r e^{i\theta} \mid \theta_1 < \theta < \theta_2 \} \quad \text{where} \]

\[ \text{origin, center of the sector.} \]

If \( z \in S_\theta \{\theta_1, \theta_2\} \), then \( w = f(z) \) is a single valued fn.

\[ (z^\alpha)' = (e^{\alpha \log z})' = e^{\alpha \log z} (\alpha \frac{1}{z}) = \alpha \frac{z^\alpha}{z} = \alpha z^{\alpha-1}. \]

\( f(z) \) analytic, and \( f'(z) \neq 0 \Rightarrow z \neq 0, \Rightarrow f \) conformal. ⇒ angles preserved.

Ex \[ w = e^z \quad \text{z = x + iy} \Rightarrow w = e^{x+iy} = e^x (\cos y + i \sin y). \]

\[ |w| = e^x \quad \text{arg} w = y + 2\pi k, \quad k \in \mathbb{Z} \]
Def. For $0 < \gamma_1 - \gamma_2 < 2\pi$, 
\[ \Sigma_1(\gamma_1, \gamma_2) = \{ z \mid z \in \mathbb{C}, \ \gamma_1 < \text{Im} z < \gamma_2 \} \]

Hence, $f(z) = e^z$ maps any horizontal strip into a sector, and it is 1-to-1 in the strip as the domain, i.e., $f : \Sigma(\gamma_1, \gamma_2) \to S_1(\gamma_1, \gamma_2)$ is injective.

Consider $f$ analytic in $D$ and $f$ is one-to-one. These are called univalent or schlicht in German.

\[ \implies \exists f^{-1}(z) = g(z); \ g : f(D) \to D \text{ and } g \text{ is analytic} \]

b/c $f$ analytic \[ \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \text{ exists and } \neq 0 \implies \lim_{w \to w_0} \frac{z - z_0}{w - w_0} = \frac{1}{f'(z_0)} \]

Thus, if $f$ is analytic and univalent, then $f^{-1} = g$ is analytic and univalent, so $f$ and $g$ are both called a topological mapping; i.e., if $f(A) = B$ and $A$ is a domain, then $B$ is a domain, an open connected set. Indeed, the preimage of an open set, is an open set, b/c of continuity. (b) If $A$ is open, then $B$ is also open.

(c) If $A$ is connected, then $B$ is connected.

Proof: Suppose $A$ is connected, and $B$ is not connected. \[ B_1, B_2 \text{ open, } B_1 \cap B_2 = \emptyset; \ A = f^{-1}(B_1) \cup f^{-1}(B_2) \]

open, nonempty, disjoint $\implies A$ is not connected.

$\implies \implies A$ is connected. 1-to-1 mappings preserve the most important topological properties. There are huge areas of physics that use these mappings, and even math models, beyond conformal mappings.
Conformal mapping which maps \([-1, 1] \rightarrow \{ z \mid |z| \leq 1 \}.

We will look at a chain of steps: A typical multiple step approach.

1. \( z \): line \( \rightarrow \) line \( \rightarrow \) linear fractional
   \[ z \rightarrow 0, \quad \frac{z}{z+1} \rightarrow \infty \quad \text{and} \quad z = \frac{z-1}{z+1}, \quad \text{so} \quad 0 \rightarrow 1. \]

2. \( z_L \): sector \( \rightarrow \) sector \( \rightarrow \) power function
   \[ \frac{z_L}{z_L+1} \rightarrow \frac{1}{z_L}, \quad \frac{z_L}{z_L+1} \rightarrow \frac{1}{z_L}, \quad \frac{z_L}{z_L+1} = \frac{z_L}{z_L+1}, \quad \text{so} \quad 0 \rightarrow 1. \]

3. \( w = z_3 \): line \( \rightarrow \) circle \( \rightarrow \) linear fractional
   \[ w = \frac{z_L-1}{z_L+1}. \]

Inverse: \( z_3 = z_2^2 \)

\( z = \frac{z_3+1}{z_3+1} \quad \text{and} \quad z_L = \frac{w+1}{-w+1} \)

\[ z = \frac{(w+1)^2}{-w+1} + 1 = \frac{(w+1)^2 + (1-w)^2}{-(w+1)^2 + (1-w)^2} = \frac{2(w^2+1)}{-4w} = -\frac{1}{2}(w + \frac{1}{w}) \]

Replace \( z \) with \(-z\)

\[ z = \frac{1}{2}(w + \frac{1}{w}) \]

The geometry of this transformation:

Let \( p = p_0 \) be fixed: \( \cos^2 \theta + \sin^2 \theta = 1 \Rightarrow \)

\[ \frac{x^2}{(p+\frac{1}{p})^2} + \frac{y^2}{(p-\frac{1}{p})^2} = 1 \rightarrow \text{circles} \rightarrow \text{ellipses} \]

Classical Sonkovskyy transformation

\( z = x + iy, \quad w = p e^{i\theta} \)

\[ z = \frac{1}{2} \left( p e^{i\theta} + \frac{1}{p} e^{-i\theta} \right) \]

\( x = \frac{1}{2} \left( p + \frac{1}{p} \right) \cos \theta \)

\( y = \frac{1}{2} \left( p - \frac{1}{p} \right) \sin \theta \)
Fixed $\Theta = \Theta_0 \quad \frac{x^2}{\cos^2 \Theta_0} - \frac{y^2}{\sin^2 \Theta_0} = \left( \frac{1}{2} \left( \rho + \frac{1}{\rho} \right) \right)^2 - \left( \frac{1}{2} \left( \rho - \frac{1}{\rho} \right) \right)^2 = 1$

\[ \text{lim (ray)} \rightarrow \text{hyperbola} \]

2/24/09 How to get domains: using $f(z)$

Elem. $f(z)$  

a) $w = \frac{az+b}{cz+d}$  
ad $-bc \neq 0$

b) $W = z^\alpha, \quad \alpha > 0 \rightarrow \text{will help open (or close $\alpha < 1$) angles}$

c) $W = e^z$

d) $W = \log z$

Ex 1: $\mathbb{Z}$

Think $z = r \cos \Theta \rightarrow r^2 e^{i \Theta}$

So $r \in [0, \infty) \rightarrow r^2 \in [0, \infty)$
Ex 2:

Look at what happens to the boundary. On the diameter, reals are mapped to reals. Not possible.

Linear transforms maps circle to circle or line. (z)

\[ f(z) = \frac{1+i}{1-i} = \frac{2i}{2} = i \]

\[ f\left(\frac{1+\sqrt{3}i}{2}\right) = \frac{\frac{3}{2} + \sqrt{3}i}{(1-\sqrt{3}i)(1+\sqrt{3}i)} = \frac{3+4\sqrt{3}i}{4} = \sqrt{3}i \]

\[ f(\infty) = -1 \quad \text{(outside the region)} \quad \text{also } f(2) = -3 \]

Ex 3:

Complicated \rightarrow Simple

\[ W = f_3(f_2(f_1(z))) = \sqrt{z} \]

\[ f_2(z) = z^2 + 3 \quad W = \left(\frac{1+z^2}{1-z^2}\right)^2 \]

\[ f_1(z) = \frac{1+z}{1-z} \quad z \rightarrow \frac{1+\sqrt{3}}{1-\sqrt{3}} \]

So for complicated conformal mappings, split it into simpler steps and take the composition.

Ex 4:

0 < \alpha < \pi

\[ f_1(\alpha) = e^{i\alpha} = r_1 e^{i\theta} \]

\[ f_2(z) = z + c \quad \text{on real axis} \quad \text{use a linear fractional: } \]

\[ W = \frac{z-1}{z+1} \quad z \rightarrow 0 \]

\[ b \rightarrow \text{read to real} \]
\[ f(e^{ia}) = \frac{e^{ia} - 1}{e^{ia} + 1} \quad \ast \quad \frac{(e^{-ia} + 1)}{e^{-ia} + 1} = \frac{2i \text{Im}(e^{ia})}{|e^{ia} + 1|^2} = i b, \quad b > 0 \]

(c) \[ w = \frac{1}{b} Z (-i) \]

0 \rightarrow 0
\[ b \rightarrow 1 \]
\[ b \rightarrow -\frac{b}{i} = -i \]

(f) \[ w = \sqrt{Z} \]

(e) \[ w = Z - 1 \]

(g) finally

\[ f = i Z \]

\[ f_1(z) = e^z \]
\[ f_2(z) = \frac{z - 1}{z + 1} \]
\[ f_3(z) = \frac{-i}{b} z \]
\[ f_4(z) = z^2 \]
\[ f_5(z) = z - 1 \]
\[ f_6(z) = \sqrt{Z} \]
\[ f_7(z) = i Z \]

So

\[ b = \frac{2 \text{Im}(e^{ia})}{|e^{ia} + 1|^2} \]

\[ f(z) = i \sqrt{\left(\frac{-i}{b} \cdot \frac{e^{2z} - 1}{e^{2z} + 1}\right)^2} - 1 \]

\[ z \rightarrow z^{\frac{1}{3}} \]
Complex integration theory: He will assume we know about Riemann integrals. The more complicated ones (such as Lebesgue integrals) will not be needed since our f u n s. (analytic) are not complicated.

Complex integration: Given: \( f(t) = u(t) + i v(t) \), \( a \leq t \leq b \)

Definition: \[ \int_{a}^{b} f(t) \, dt = \int_{a}^{b} u(t) \, dt + i \int_{a}^{b} v(t) \, dt \]

Remark: 1) All properties of real integrals hold. Example: \[ \int_{a}^{b} cf(t) \, dt = c \int_{a}^{b} f(t) \, dt, \quad c \in \mathbb{C} \]

2) The case of \( b < a \) is covered.

3) \( |\int_{a}^{b} f(t) \, dt| \leq \int_{a}^{b} |f(t)| \, dt \). Indeed, for \( \theta \in \mathbb{R} \), \( \text{Re}(e^{i\theta} \int_{a}^{b} f(t) \, dt) = \int_{a}^{b} \text{Re}(e^{i\theta} f(t)) \, dt \)

Proof:

a) \( \int_{a}^{b} f(t) \, dt = 0 \Rightarrow \) done (trivially true)

b) \( \int_{a}^{b} f(t) \, dt \neq 0 \), take \( \theta = \arg \int_{a}^{b} f(t) \, dt \), then \( \text{Re}(e^{i\theta} \int_{a}^{b} f(t) \, dt) = |\int_{a}^{b} f(t) \, dt| \leq \int_{a}^{b} |f(t)| \, dt \).
Integrals on a curve or arc

Recall an arc $\gamma: z = z(t)$ for $a \leq t \leq b$

Let $\gamma$ be a piecewise differentiable arc, i.e., the derivative exists for all but a finite number of values of $t$. Consider $f$ on $\gamma$, $f$ continuous on $\gamma$.

**Defn.** $\int_{\gamma} f(z) \, dz = \int_{a}^{b} f(z(t)) \, z'(t) \, dt$

Integral on an arc. We will see other ways to deal with these, but this is the best.

**Properties**

1. Given $\gamma$ (the arc is really not dependent on the arc)

\[
\int_{\alpha}^{\beta} f(z) \, dz \text{ does not depend on a change of the variable.}
\]

\[
\int_{\gamma} f(z) \, dz = \int_{\alpha}^{\beta} f(z) \, z'(t) \, dt = \int_{\alpha}^{\beta} f(z(t)) \, z'(t) \, t'(\tau) \, d\tau
\]

the chain rule

but $z(t(\tau)) = z(\tau)$

\[
= \int_{\alpha}^{\beta} f(z(\tau)) \, z'(\tau) \, d\tau
\]