1. Prove identity (*) \( |z_1 + z_2|^2 + |z_1 - z_2|^2 = 2( |z_1|^2 + |z_2|^2) \)

\[
|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1 + z_2}) = (z_1 + z_2)(\overline{z_1} + \overline{z_2})
\]

So (*) = \[
\begin{align*}
& z_1 \overline{z_1} + z_1 \overline{z_2} + z_2 \overline{z_1} + z_2 \overline{z_2} \\
& + z_1 \overline{z_1} - z_1 \overline{z_2} - \overline{z_1} z_2 + \overline{z_2} \overline{z_2} \\
& = 2|z_1|^2 + 2|z_2|^2
\end{align*}
\]

2. Calculate all possible values of \( \int_C \frac{dz}{z(z^2 + 1)} \) for different positions of \( C \). \( C \) doesn't pass through any one of 0, 1, -1.

Solution:

Recall: \( \text{Res} (f, a) = \frac{P(a)}{Q'(a)} \), where \( f = \frac{P}{Q} \)

Let \( g = \frac{1}{z(z^2 + 1)} \). \( \text{Res} (f, a) = \frac{P(a)}{Q'(a)} \), where \( f = \frac{P}{Q} \)

So

\[
\begin{align*}
\text{Res} (g, 0) &= \frac{1}{3z^2 + 1} \bigg|_{z = 0} = -1, \quad \text{Res} (g, -1) = \frac{1}{2} \\
\text{Res} (g, 1) &= \frac{1}{2}. \quad \text{So possible values of integral are:} \\
& \{0, -1, \frac{1}{2}, 1, -\frac{1}{2}\} \cdot 2\pi i
\end{align*}
\]
3. Prove no matter how small \( p > 0 \), for large \( n \), all zeros of
\[
f_n(z) = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \cdots + \frac{1}{n!z^n}
\]
are situated inside \( |z| < p \).

Proof:
We know \( f_n(z) \to e^{\frac{1}{z}} \) as \( n \to \infty \).

Fix \( p \), \( |f_n(z) - f(z)| < |f(z)| \) on \( |z| = p \).

We have
\[
|f(z)| = |e^{\frac{1}{z}}| = |e^{\frac{z}{2z}}| = e^{\frac{x-i\cdot y}{2z}} = e^{-\frac{y}{2z}} \text{ minimized when } x = -\rho
\]

\( \exists N \in \mathbb{N} \) s.t. \( |f_n(z) - f(z)| < e^{-\rho} \) \( \forall z \in \mathbb{C} = \{ z : |z| = p \} \)

So \( \forall n \geq N \) \( |f_n(z) - f(z)| < |f(z)| \). Using Rouche's Theorem (\( f \) does not have zeros in \( |z| = p \)), so \( f_n(z) \) doesn't have zeros in \( |z| \geq p \).
4. Do functions exist which are analytic at $z=0$ and satisfy

$$f\left(\frac{1}{n}\right) = f\left(-\frac{1}{n}\right) = \frac{1}{n^2},$$

$$g\left(\frac{1}{n}\right) = g\left(-\frac{1}{n}\right) = \frac{1}{n^3}.$$ 

Solution:

Take $f(z) = z^2$. For $g\left(\frac{1}{n}\right) = \frac{1}{n^3}$, $g(z) = z^3$ works.

But by Identity Theorem ????

$a_n \to a$ if $f(a_n) \to 0$ for all $n$ then $f \equiv 0$.

Take $\{\frac{1}{n}\}$. Then $\frac{1}{n} \to 0$ as $n \to \infty$. Here, $f(z) = g(z) - z^3$.

So no such $g$ exists.

5. Map whole plane with cuts along $[1,1i]$ and $[-i,i]$ onto exterior of unit circle.

$$z^2$$

$g^{-1}$, where $g^{-1} = z + \frac{1}{2}$
1. Show that any entire function $f$ satisfying both $f(z+1) = f(z)$ and $f(z+i) = f(z)$ for every $z$ must be a constant function. Are both conditions necessary? Why? Why not?

Both are necessary. Take $\sin(2\pi x)$ or $\sin(2\pi i z)$. Using induction, $f(z) = f(z+1) = f(z+1+1) = f(z+k)$ for all $k$.

Similarly, $f(z+i) = f(z) = f(z+ik)$ for all $k$.

So the behavior of $f$ in and on $S = [0,1] \times [0,1]$ repeats throughout $C$ with a period of 1 vertically and horizontally. By continuity of $f$ (since $f$ is entire), find the maximum modulus of $f$ on $S$.

This is true for all $z \in C$. By Liouville's, $f = c \in C$.

2. (a) Show $g(z) = \sum_{k=1}^{\infty} \frac{1}{k^2+z}$ is analytic in $\mathbb{R}(z) > 0$.

(b) Show $g$ is meromorphic in $C$.

Proof:

Let $S_n = \sum_{k=1}^{n} \frac{1}{k^2+z}$ and $z \in \mathbb{H} = \{z: \text{Re}(z) > 0\}$. Let $\epsilon > 0$ s.t. $C = \{1 \geq |z| \geq \epsilon \}$.

We have $\left| \frac{1}{k^2+z} \right| \leq \frac{1}{k^2}$ for all $z$, s.t. $k \geq 1$.

Weierstrass M-test: If $|u_n(z)| \leq M_n$, where $M_n$ is independent of $z$ in region $R$ and $\sum M_n < \infty$, then $\sum u_n(z)$ is uniformly convergent in $R$.

Since $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} < \infty$, using M-test gives $S_n \to g$. Since uniform convergence preserves analyticity, $g$ is analytic.
2(c) g meromorphic in \( C \)?

Poles of \( g \) occur at negative perfect squares on the real axis (these are isolated.) Use M-test again to get analyticity on \( C \) except at isolated poles.

3. With \( -\pi \leq \arg z \leq \pi \) and \( |z|=1 \), show that

\[
\arg \left( \frac{z-1}{z+1} \right) = \begin{cases} \frac{\pi}{2}, & \text{if } \Im z > 0 \\ -\frac{\pi}{2}, & \text{if } \Im z < 0 \end{cases}
\]

**Proof:**

Since \( |z|=1 \), \( z = e^{i\theta} \). So

\[
\frac{z-1}{z+1} = \frac{e^{i\theta} - 1}{e^{i\theta} + 1} \cdot \frac{e^{-i\theta} + 1}{e^{-i\theta} + 1} = \frac{e^{i\theta} - e^{-i\theta}}{\text{real} \ #} = \frac{2i\sin \theta}{2 + 2\cos \theta} = i\tan \frac{\theta}{2}.
\]

4. Let \( \Gamma \) be a fixed smooth simple closed contour. Use

\[
\Gamma(\lambda) = \frac{1}{2\pi i} \int_{\Gamma} \frac{(f+\lambda g)'}{f+\lambda g} \, dz,
\]

\( 0 \leq \lambda \leq 1 \)

to prove Rouche's Theorem. Suppose \( f, g \) are analytic and

By Argument Principle,

\( \Gamma(1) = (\# \text{zeros of } f+g) - (\# \text{poles of } f+g) \) and

\( \Gamma(0) = (\# \text{zeros of } f) - (\# \text{poles of } f) \).

Let \( F = g/f \). Then \( g = Ff \). **
So \( J(1) - J(0) = \frac{1}{2\pi i} \left( \int \frac{f'}{f + Ff'} \, dz - \int \frac{f'}{f} \, dz \right) \)

\[
= \frac{1}{2\pi i} \left( \int f' \left( 1 + \frac{f'}{f(1+F)} \right) \, dz - \int \frac{f'}{f} \, dz \right)
\]

\[
= \frac{1}{2\pi i} \left( \int \frac{f'(1+F)}{f(1+F)} \, dz \int \frac{f'}{f(1+F)} \, dz - \int \frac{f'}{f} \, dz \right)
\]

\[
= \frac{1}{2\pi i} \int \frac{F'}{1+F} \, dz = \frac{1}{2\pi i} \int F' \left( \sum_{n=0}^{\infty} (-1)^n F^n \right) \text{ which converges uniformly.}
\]

(Note that \(|F| < 1\).)

By uniform convergence,

\[
= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int \frac{F'}{(-1)^n F^n}
\]
5. Show that \( \int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2} \, dx = \pi \frac{e^{-a}}{a} \), for \( a > 0 \).

Proof:

Let \( f(z) = \frac{e^{iz}}{z^2 + a^2} \). So \( f \) has a pole \( ia \) in the upper half-plane. And \( \text{Res}(f, ia) = \frac{e^{i(ia)}}{ia + ia} = \frac{e^{-a}}{2ia} \).

Consider \( \delta \) given by:

Thus \( \int_{\delta_R} f(z) \, dz = 2\pi i \left( \frac{e^{-a}}{2ia} \right) = \pi \frac{e^{-a}}{a} \).

But \( \int_{\delta_R} f(z) \, dz = \int_{-R}^{R} \frac{e^{ix}}{x^2 + a^2} \, dx + i \int_{0}^{\pi} f(Re^{i\theta}) \overline{Re^{i\theta}} \, d\theta \).

\[
| \int_{0}^{\pi} f(Re^{i\theta}) \overline{Re^{i\theta}} \, d\theta | \leq R \int_{0}^{\pi} | f(Re^{i\theta}) e^{i\theta} | \, d\theta \leq R \int_{0}^{\pi} \frac{e^{-R\sin \theta}}{R^2 - a^2} \, d\theta \leq \frac{\pi R}{R^2 - a^2} \rightarrow 0 \text{ as } R \rightarrow \infty.
\]

Thus, \( \pi \frac{e^{-a}}{a} = \int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2} \, dx + i \int_{-\infty}^{\infty} \frac{\sin x}{x^2 + a^2} \, dx \).

0 since integrand is odd.