Seperability

55) Show that if $K$ is a separable extension of $F$ and $L$ is a field with $F \subseteq L \subseteq K$, then $L$ is a separable extension of $F$ if $K$ is a separable extension of $L$.

- $K$ sep over $F \Rightarrow$ every elt of $K$ is the root of a sep poly in $F$
  - (min poly over $F$ of every element of $K$ is sep).

\[ \text{Proof} \]

Since every element of $L$ is an element of $K$, then every element of $L$ is the root of a separable polynomial in $F$. Hence, $L$ is a separable extension of $F$.

Let $\alpha \in K$. $m_{\alpha,F}(x)$ is separable since $K$ is separable over $F$. Since $F \subseteq L \subseteq K$,

$\text{gcd}(m_{\alpha,F}(x), m_{\alpha,L}(x)) | m_{\alpha,L}(x)$, $\text{since } m_{\alpha,L}(x)$ is separable and $m_{\alpha,F}(x)$ is separable.

Therefore, $L$ is a separable extension over $L$. \[ \square \]
57) Show that if \( K \) is a finite dimensional separable extension of \( F \), then \( K = F(u) \) for some \( u \in K \).

Pg. 594 Prop. 24: Let \( K/F \) be a finite extension. Then \( K = F(\theta) \) iff there exist only finitely many subfields of \( K \) containing \( F \).

- See this proof. (Need induction.)
Let $F$ be a field and let $f(x) = x^2 - x \in F[x]$. Show that if $\text{char}(F) = 0$ or $\text{char}(F) = p$ and $p \nmid n-1$, then $f$ has no multiple root in any extension of $F$.

$$D_x f(x) = nx^{n-1} - 1$$

**Proof:**

$f(x) = x^n - x = x(x^{n-1} - 1)$. So the roots of $f(x)$ are 0 and the $n-1$ roots of unity.

$D_x f(x) = nx^{n-1} - 1$. Clearly, 0 is not a root of $D_x f(x)$.

- If $\text{char}(F) = p$ and $p \nmid n-1$, then $n \neq 1$ in $F$, so $D_x f(x) = nx^{n-1} - 1$; if $x^{n-1} = 1$, $D_x f(x) = p(1) - 1 \neq 0$.
- If $\text{char}(F) = 0$, if $x^{n-1} = 1$, $D_x f(x) = n - 1$.

From #56: if $f(x)$ and $D_x f(x)$ have a common root $a$, then $f(x) = (x-a)^n g(x)$, where $n \neq 1$.

$\Rightarrow (f(x), D_x f(x)) = 1$

By #56, $f(x)$ has no multiple root in any extension of $F$. $\blacksquare$
84) Let \( K \) be a finite normal extension of \( F \) & let \( E \) be the fixed field of the group of all \( F \)-automorphisms of \( K \). Show that the minimal polynomial over \( F \) of each element of \( E \) has only one distinct root.

Normal extension - an algebraic extension which is a splitting field.

WTS: \( \forall \alpha \in E, \ m_{\alpha, F} \) has only one root.

Proof:
Suppose not. Let \( \beta \neq 2 \) be a root of \( m_{\alpha, F} \).
We know \( m_{\alpha, F} \) is irreducible. Let \( \phi, \beta \in \text{Aut}(E) \) where \( \phi : F \rightarrow F \) map \( \alpha \rightarrow \beta \).
So \( \alpha \) is not fixed by \( \phi \), \( \beta \in \text{Aut}(E) \). \( \Box \)
(24) Let \( \mathbf{A} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \)

\[(x-2)(x-1)^3 = \text{char } \mathbf{A} \]

\[\lambda = 1; \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \]

\[w = 0 \quad z = -2w \quad \Rightarrow w = 0 \]

\[x, y \text{ arbitrary} \]

\[\lambda = 2; \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \]

\[x = 0 \quad 2w - y = 0 \]
\[z = 0 \quad 2w = y \]

Dim Eigenspace # of 1-blocks: 2
# of 2-blocks: 1

Jordann Form

\[ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \]

- Eigenvalues on main diagonal, within each block diagonal