Theorem (for angles)

(1) m(Ø) = 0
(2) m is additive, \( P = \bigcup_{k=1}^{n} P_k \), let \( m(P) := \sum_{k=1}^{n} m(P_k) \) for \( k \neq j \)
(3) m is additive, \( P \cup P_j = \emptyset \) for \( k \neq j \)

The area of a rectangle is its measure:

\[ m((a,b) \times (c,d)) = (b-a)(d-c) \]

- measure of planar sets
- measure of a generalization of length, area, volume

Ex: \( m([a,b]) := t(b) - t(a) \) (may be 0)

\( (x,y) \in \mathbb{R}^2 \) if \( a \leq x \leq b, \ c \leq y \leq d \)

Book: Introductory Real Analysis, A. Kolmogorov & S. Fomin

Real and Complex Analysis, W. Rudin

Dmitri Ryabogin, or "Dima"

Grishin
2. \( A \cup B = p \setminus \left( (p \setminus A) \cup (p \setminus B) \right) \)

If \( A \cap B = \emptyset \) then \( A \cup B = A \cup B \)

There exists a set containing both of them and

Finite union

\[ p \setminus (A \cup B) \]

Remark: \( p \setminus \) (elementary) = elementary set

If \( A = \bigcap_{k=1}^{m} p_{k} \) then \( A \cup B = \bigcup_{k=1}^{m} p_{k} \cap (p \setminus A) \)

Proof: If \( A \cup B \) are rectangles, then \( A \cup B \) is also a rectangle.

Diagram: Intersections of elementary sets are also elementary.

In short, if \( A = \bigcap_{k=1}^{m} p_{k} \) with \( A \cap p_{i} = \emptyset \) for \( i \neq j \) and \( A = \bigcup_{k=1}^{m} p_{k} \), then \( A \setminus \emptyset \) is an elementary set if \( p_{i} \) rectangles.
\[ \frac{1 + \sqrt{2}}{3} \]

Find the subcollection \( A_n \) and show that \( m^*(A) = \sum_{n=1}^{\infty} m^*(A_n) \). Let \( \mathcal{A} \) be an open \( \mathcal{G} \)-collection of all elementary sets. We must reduce the \( \mathcal{G} \)-collection to a finite set.

If \( m(A) > 0 \), \( m(A) = \sum_{n=1}^{\infty} m(A_n) \). Since \( \mathcal{P} \cup \mathcal{Q} \) is a rectangle, we have:

\[ A = \bigcap_{n=1}^{\infty} P \]

The measure of decomposition. The definition is independent of \( A \).

\[ m(A) = \sum_{n=1}^{\infty} m(P_n) \]
\[ m(A) = \frac{1}{2} \cdot m(A) \cdot m(A) = \frac{1}{2} \]

\[ A \subset \mathbb{R}, \quad A \neq \emptyset \]

Define by a a collection of all measurable sets.

The definition of the measure is as follows:

\[ m(A) = \lim_{n \to \infty} m(A_n) \]

where \( A_n \to A \).

Note that \( m(A) > 0 \) if and only if \( A \neq \emptyset \).

Then, \( m(A) = \sum_{n=1}^{\infty} m(A_n) \).

Let \( A \) be an arbitrary set, and let \( A_n \) satisfy:

\[ \bigcup_{n=1}^{\infty} A_n = A \]

(Continued)
\[ \text{Covering } \not\exists \text{ since } \not\exists \text{ finite subcollection } \not\exists \text{ of } A \]

\[ A \subseteq \bigcup_{i=1}^{n} A_i \]

\[ A \neq 0 \land A \subseteq E \]

\[ A \cap m \cap \{p_i\} \neq \emptyset \]

\[ \frac{a}{2} \]

\[ \prod_{i=1}^{n} p_i \]

\[ f = 1 \]
\[ m(A) = \sum_{i=1}^{n} m'(A_i) + \varepsilon \]

Corollary:  \[ A \in \mathcal{S}, \quad \forall \varepsilon \in \mathbb{R}, \quad A = \bigcup_{i=1}^{n} A_i \Rightarrow m(A) = \sum_{i=1}^{n} m(A_i) \]

To show these are equal (Standard way)

Well start from there: \[ m'(A) \geq m'(C_{j=1}^{N} A_j) = \sum_{j=1}^{N} m'(A_j) \]

\[ A \subset \bigcup_{j=1}^{N} A_j \rightarrow \text{limit} \]

\[ \sigma \text{-additivity} \]

\[ \sigma \text{-sets in the plane?} \]

Maybe it is possible to extend \( m \) onto all \( \sigma \text{-sets in the plane?} \)
We define \( \lambda^* (A) = \inf \{ \frac{1}{n} m_i (A_i) \} \), \( A \subseteq \mathbb{R} \).

**Definition:** Outer measure

For simplicity, at the beginning:

\[ \lambda^* (A) = \sup \{ \sum_{i=1}^\infty m_i (A_i) \} \quad \text{if} \ A = \bigcup_{i=1}^\infty A_i \]

\[ m (A) = \lim_{N \to \infty} (\frac{1}{N}) \sum_{i=1}^N m(A_i) \]

w.r.t.

\[ m \geq 0 \]

It will turn out that \( m \) extends \( m \) to \( \mathcal{H} \).

**Proposition:**

If \( A \subseteq \mathbb{R} \), then \( \lambda^* (A) = 0 \) if and only if \( \lambda (A) = 0 \).

Now, let \( n \geq 1 \) be and the set \( \mathcal{A} \) of all bounded intervals in \( \mathbb{R} \) with rational endpoints.

The union of an enumeration of \( \mathcal{A} \) is called-mto the outer measure.

First we need to read

\[ f (x) = \frac{p}{q} \]

This gives the idea;

\[ f (x) = \frac{p}{q} \] for integers

\[ A \notin \mathbb{R} \]
But then $A \in \bigcup_{p \in P} A(p)$, so $A \not\in A(p)$ for some $p \in P$. This is a contradiction, which is what we wanted to show.

Proof:

First, we define $\mathcal{A} = \{ A \in \mathcal{P}(\mathbb{N}) \mid \forall n \in \mathbb{N}, \exists p \in P : A(p) \subseteq \mathbb{N} \}$.

By the diagonalization argument, we can construct a function $f : \mathbb{N} \to \{0, 1\}$ such that $f(n) = 1$ if and only if $n \not\in A(n)$ for all $A \in \mathcal{A}$.

In particular, if $A = A(f)$, then $A(f) \not\in A(f)$.

Then, if $A \subseteq \mathbb{N}$, then $A \not\in A(f)$.

To prove this, let $A = \bigcup_{p \in P} A(p)$.

To prove $\nabla f$, let $A \in \mathcal{A}$.

This is much harder (why?...)

Not like Erwin Schrödinger's wavefunction!
No. All sets measurable.

\[ \mathcal{M} = \mathcal{M}^* \]

\[ \mathcal{M} = \text{collection of measurable sets} \]

\[ \emptyset \neq B = \mathcal{A} \]

(Hint: take \( B = 0 \) is measurable, \( \mathbb{E} \) any subset of \( A \).

\[ \exists \mathcal{A} \text{ is not enough} \]

\[ A \cap B \]

\[ \mathcal{M}(A \cap B) = \mathcal{E} \]

\[ \mathcal{A} \in \mathcal{M}, \mathcal{B} \in \mathcal{M} \]

\[ \mathcal{M}(A \cup B) = \mathcal{E} \]

\[ m(A) = \frac{1}{n+1} \left[ m(A) + m(B) \right] \]
Define \( m \) measurable sets \( A \subset \sigma \)-algebra \( \sigma \subset \mu \). If \( A = \bigcup_{n} m \implies \mu(A) = \sum_{n} \mu(A_n) \). The measure that we are talking about is invariant under translation. Very specific and important

\( \forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \forall n \in \mathbb{Z} \), the proof is not known. It

\[ X_{n} = \{ x + \alpha n \}, \ x \in \mathbb{R} - \mathbb{Q}, \ n \in \mathbb{Z} \]
Should we only measure or mean measure?

\[ \mathbb{R} \setminus \{ \phi \} = \bigcap_{k=1}^{n} A_k \]

\[ \mathbb{R} \cup \{ 0 \} = \bigcup_{k=1}^{n} A_k \]

Assume that \( (\omega, \bar{\mu}, \mu) \) is measureable.\( \exists \) an \( \alpha \) such that

\[ S = \prod_{x \in \alpha} \bigcup_{y \in \alpha} \phi = \bigcup_{y \in \alpha} \bigcup_{x \in \alpha} \phi \]

Define \( \alpha \) = \( \alpha + \text{not} \). For each \( x \in \alpha \)

Take one point from each \( x \)

\( \exists \) \( x \in \alpha \) such that \( y \in \alpha \) and \( x \neq y \)
We need to show that $R*(A, B) = R*(A, B)$.

Let $A, B \in R$.

For $R = \{ (a, b) \mid a \leq b \}$,

$$\forall n, \forall a, b \in \mathbb{R}, a \leq b \iff a + n \leq b + n$$

We have $A \cup B = \{ a \mid a \in A \lor a \in B \}$.

We also have $\forall A, B \in R, A \cup B = A \cup B$.

To prove that $R*(A, B) = A \cup B$.

We have $R*(A, B) = \{ (a, b) \mid a \leq b, a \in A, b \in B \}$.

For $R = \{ (a, b) \mid a \leq b \}$,

$$\forall a, b \in \mathbb{R}, a \leq b \iff a + n \leq b + n$$

We conclude that $R*(A, B) = A \cup B$.

Lemmas:

1. $\forall A, B \subset \mathbb{R}, A \cup B = \bigcup_{a \in A} \bigcup_{b \in B} (a, b)$
We need to show \( \mathcal{L}(A) \geq 6 \) for any \( A \neq \emptyset \). Let \( A \neq \emptyset \) and \( B \neq \emptyset \) be any two nonempty sets.

Claim: \( A \cup B \in \mathcal{L} \).

Proof: \( A \cup B \in \mathcal{L} \) is obvious to consider \( \mathcal{L} \).

Lemma 3: \( A = \bigcup_{n=1}^{\infty} A_n \rightarrow A \cap A_\infty \in \mathcal{L}(A) \).

Proof: \( A_\infty = A - (A - A_\infty) \). For \( A \in \mathcal{L} \), \( A \cap A_\infty \in \mathcal{L} \).

Corollary: \( A \cup A_\infty \in \mathcal{L} \).

What?

By Lemmas 1 and 2.

For inclusions, \( A \cap A_\infty = A - (A - A_\infty). \)
\[ \mu^*(A) \leq \mu^*(B) \leq \mu^*(A \Delta B) + \mu^*(B \Delta A) \]

By the claim, we have:

\[ \mu^*(A \Delta B) < 2\varepsilon \]

This implies:

\[ \mu^*(A) + \mu^*(B) - \mu^*(A \Delta B) > \mu^*(A) + \mu^*(B) - 2\varepsilon \]

Observe that:

\[ A \Delta B \subseteq (A \Delta B) \cup (B \Delta A) \]

Finally, we get:

\[ \mu^*(A) \geq \mu^*(B) - 2\varepsilon \]

and

\[ \mu^*(B) \geq \mu^*(A) - 2\varepsilon \]

Thus, we have:

\[ \mu^*(A) \geq \mu^*(B) - 2\varepsilon \]

Consequently,

\[ A \subseteq B \]

Finally, the claim is proved.
Lemma 1

\( \text{Let } A = \bigcup_{n=1}^{\infty} A_n \Rightarrow \text{Put } A' = A - \bigcup_{k=1}^{\infty} A_k \)

This is an important idea called "cut the tail."

By Lemma 3, \( \mu \left( \bigcup_{n=1}^{\infty} A_n \right) = \mu^* \left( \bigcup_{n=1}^{\infty} A_n \right) \leq \sum_{n=1}^{\infty} \mu^*(A_n) \)
\[
3 = \frac{\theta_1}{3} + \frac{\theta_2}{3} \Rightarrow \\
(n, N) \Rightarrow (n, A \lor B) \Rightarrow (n, (A \lor B^c) \lor (A \lor B^c)) \\
\text{Since } A \lor B^c \subseteq (A \lor B^c) \lor (A \lor B^c) \lor \emptyset \\
\frac{\theta_1}{3} \Rightarrow \bigwedge_{n=1}^{N} (A_n \lor B_n^c) \Rightarrow \bigwedge_{n=1}^{N} (A_n \lor B_n^c) \lor A_n \lor B_n^c \\
\text{Moreover, we know } AN \cap N \lor A_n \lor B_n^c \\
\frac{\theta_2}{3} \Rightarrow \bigwedge_{n=1}^{N} (A_n \lor B_n^c) \Rightarrow \bigwedge_{n=1}^{N} (A_n \lor B_n^c) \lor A_n \lor B_n^c \\
\text{To forget about the } \theta_1 \text{ and } \theta_2, \\
\exists \theta \in \mathbb{R}^3: A \lor B^c \Rightarrow (A \lor B^c) \lor c \\
\frac{\theta_1}{3} \Rightarrow \bigwedge_{n=1}^{N} (A_n \lor B_n^c) \Rightarrow \bigwedge_{n=1}^{N} (A_n \lor B_n^c) \lor A_n \lor B_n^c \\
\text{The small } 0 < 3 \text{ A}
\[ \bigcap_{n=1}^{\infty} A_n \subseteq B \]

Proof:

For \( A_n \setminus B \neq \emptyset \), let \( a \in A_n \setminus B \). Then \( a \in A_n \) and \( a \notin B \).

Since \( a \in A_n \), there exists \( k \) such that \( n \leq k \).

Thus, \( a \in A_k \) and \( a \notin B \), which implies \( a \in A_k \setminus B \).

Therefore, \( A \setminus B = \bigcup_{n=1}^{\infty} (A_n \setminus B) \).

For intersections:

\[ A \cap E \cap (\bigcup (E \setminus A_n)) \]

For integrals:

\[ \int 1 \]
If \( A_1 \supset A_2 \supset A_3 \supset \ldots \), \( A_n \), \( A = \bigcap_n A_n \)

\[
\implies \mu(\bigcap_n A_n) = \lim_{n \to \infty} \mu(\bigcap_n A_n) = 0
\]

We may assume that \( A = \emptyset \) and we want to prove that \( \lim_{n \to \infty} \mu(A_n) = 0 \).

\[
A_1 = (A_1-A_2) \cup (A_1-A_3) \cup \ldots
\]

\[
A_2 = (A_2-A_3) \cup (A_2-A_4) \cup \ldots
\]

\[
A_n = (A_n-A_{n+1}) \cup (A_n-A_{n+2}) \cup \ldots
\]

\[
\mu(A_n) = \sum_{k=1}^{\infty} \mu(A_n-A_{n+k}) < 3 \quad \forall \epsilon > 0
\]

As \( \epsilon \to 0 \),

\[
\mu(A_n) = \lim_{n \to \infty} \mu(A_n) = 0
\]

\[
\left( \lim_{n \to \infty} \mu(A_n) \right) = 0
\]

Corollary: \( A \subseteq A_2 \subseteq A_3 \subseteq \ldots \)

\[
\mu(\bigcap_n A_n) = \lim_{n \to \infty} \mu(A_n)
\]
is called measure if

\[ m(A) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} 1_{A}(k) \]

where

- \( 1_{A}(k) \) is 1 if \( k \in A \)
- \( 1_{A}(k) \) is 0 if \( k \notin A \)

Some examples:

1. \( A \subset \mathbb{N} \)

2. \( A \subset \mathbb{R} \)

3. \( A \subset \mathbb{R}^2 \)

4. \( A \subset \mathbb{C} \)

5. \( A \subset \mathbb{Z} \)

Let's deal with the unit square.
Subring

1) \( A, B \subset S \Rightarrow A \cap B \subset S \)
2) \( A \setminus B = \bigcap_{j=1}^{N} C_j, \quad C_j \subset S \)

- Simplest structure that will help us define the measure

Unions are not in subrings

\[
\begin{align*}
(0,0) & \quad a \quad b \\
(1,0) & \quad (1,1) \\
0 & \quad c \\
\end{align*}
\]

\( \mu(\emptyset) = 1 - a \)

\( \mu(\square) = d - c \)

Assume you have a subring

You start with a set of rectangles
with width/length 1

2 extensions: \( m' \) natural (write what I see)

\[
m''(A) = m''(A \cup \square) = \frac{\mu(A \cap \text{diagonal})}{\sqrt{2}}
\]

Does it make sense?
We use subrings to keep track of structure and uniqueness.

and uniqueness.

measure of subring known is needed for extending $A.

1. $\mu \geq 0$ needed for nontriviality of $\mathbb{Z}$.
Since $A_1$'s are disjoint, $(x,y) \in A_1$, imply

\[ \forall (x,y) \in A_1 \] \[ \exists \eta \ni d(x,y) \geq \eta \]

Check $f$ is a measure.

\[ f(A) = \int f(\eta) \, d\eta \]

Let's do this.

Recall, in $\mathbb{R}$

\[ d(\eta, \bar{\eta}) = 0 \right\{ \eta, \bar{\eta} \in \mathbb{R} \] \[ d(\eta, \bar{\eta}) = \frac{|\eta - \bar{\eta}|}{\eta} \left\{ \eta, \bar{\eta} \in \mathbb{R} \] \[ d(\eta, \bar{\eta}) = \frac{|\eta - \bar{\eta}|}{\eta} \left\{ \eta, \bar{\eta} \in \mathbb{R} \]

as follows:

The measure space $\mathbb{R}$

\[ f(x,y) \in \mathbb{R} \]

for

\[ (x,y) \in \mathbb{R} \]

This gives rise, to, the measure

\[ \int_{[a,b]} \]

Example: $f(x, y) = xy$, volume,

\[ \int_{[a,b]} \frac{1}{h} \]

Def. A function $A$ endowed with a structure $\mathcal{A}$.
This is not a B-additive measure (why not?)

\[ m([a, b]) = b - a \]

Consider a sequence of

\[ x \in E \setminus \bigcup_{i} A_i \]

where each \( A_i \) is a countable subset of \( E \) with \( m(A_i) = \frac{1}{2^n} \) for \( n = 1, 2, 3, \ldots \).

Thus are all \( \sigma \)-additive, not only additive.

\[ m = \frac{1}{2^n} \]

To each \( x \in E \)

\[ m(x) = \frac{1}{2^n} \]

\[ x \mapsto \frac{1}{2^n} \]

From countable union of \( \frac{1}{2^n} \)

\[ \sum \frac{1}{2^n} = 1 \]

Thus product measure

Think product measure

Think sequence of \( \times \frac{1}{2^n} \)

\[ x = x_1, x_2, \ldots \]

\[ m = \frac{1}{2^n} \]

\[ x \in X \]

\[ m(x) = \frac{1}{2^n} \]
\[ u(\text{I}0\prime) \text{ do begin} \]

\[ \text{Universal on } \mathbb{R} \text{ to generate measures} \]

\[ \text{Borel measurable on } \mathbb{R} \text{ as a measure} \]

\[ m(A) = \int f(x) \, dx \]

\[ f \geq 0 \text{ on } \mathbb{R}_+ \times \mathbb{R} \]

\[ \exists \lambda \text{ let } f \text{ be a non-negative measurable function on } \mathbb{R}_+ \times \mathbb{R} \]

\[ \text{additive} \]

\[ \mathbb{P} \text{ additive is a restriction on } \mathbb{P} \]

\[ \mathbb{P}(X) = \frac{1}{\beta} \mathbb{P}(X) \]

\[ \mathbb{P}(X) \text{ is countable} \]

\[ \mathbb{P}(X) \text{ is a point} = 0 \]

\[ \text{Prove Borel sets } \{(x,+}\)
Next Page

From Figures: 4 corners measure or pixel measure

Once the measures, say, at $x$:

What function does correspond to $x$?

$g^*(a, b) = g(a) - g(b)$

Ex. $g^*([0, 1], [-1, 2]) = 1 - (-1) - (2) = -2$

Could be $g(x) = x$ or $g(x) = x^2$

$g(0) = 0$

For $x > 0$, $g(x) = 1$

You pick a function.

$R \rightarrow [-1, 0]$
\[
\begin{align*}
\text{Init:} & \quad n = 1 \\
\text{Cond:} & \quad n = 1 - 1 \\
\text{Jump:} & \quad n = 1 \\
\text{Count:} & \quad n = 0 \\
\end{align*}
\]
\[
\{ (a, b) : (a, b) \in \mathbb{R}_+ \times \mathbb{R}_+ \} = \mathbb{R}_+^2 \tag{42.1}
\]

Continued...

We define a function \( \phi \) on \( \mathbb{R}_+^2 \).

\[
\phi(x, y) = \frac{1}{1 + e^{x+y}}
\]

Now, \( \phi(x, y) \geq 0 \) and \( \phi(x, y) \leq 1 \).

Furthermore, \( \phi \) is continuous on \( \mathbb{R}_+^2 \).

Ex. 9. Your set is \( C([a, b]) \), continuous functions on \([a, b]\).

Ex. 10. \( \int_{[a, b]} f(x) dx = \int_{[a, b]} \phi(x) f(x) dx \)

Thus, \( \phi \) is a measure.

Consider the continuity of \( f(x) = \log_2 \left( \frac{1+x}{1+x} \right) \) at \( x = 0 \).

**Exercise:**
If \( \alpha \in \mathbb{C} \) and \( \mathbf{A} \in \mathcal{D} \), \( M(\mathbf{A}) = \left( \alpha \right) \).

**Definition.** If \( \Gamma \) is a graph, \( x \) a measure on \( \mathcal{D} \), then

\[
\mathrm{sub}(x) \leq x
\]

no sub. = minimal function.

**Example.** A measure on \( \mathbb{R} \)

\[
\text{enough example}
\]

Lots of different measures.

Then measure zero. With no differentiable

Wierner measure

\[
\int_{1}^{2} \int_{2}^{3} \cdots \int_{n}^{d} x \, dx \, \cdots \, dx = \frac{n!}{2! \cdots (n-1)!} \cdot \frac{\epsilon}{2^{n-1}}
\]

\[
\sum_{n=1}^{\infty} x \frac{n!}{2^{n-1}} = -\frac{2}{u-1}
\]

\[
\Lambda (X (\lambda + \cdots + \lambda, \lambda \cdot \cdots \lambda) = \frac{2}{u-1}
\]
For subring, intersection must be in.

\[ A \cap A_2 = (A_1) \cap (A_2, A_3, \ldots) \]

Let \( A = (A_1) \) be a subring of \( R \), and \( A \leq \phi(R_n) \).

We would like to extend from one of \( A \) to more.

Theorem: \( \phi \) is an extension of \( \mu \), where \( \mu \) is a \( \mu \) on \( T_m \). The minimal map

\[ \phi = f \]

\[ f \mid \mathbb{Z} \]

Need 3 statements before the proof.
Proposition 1. Let $A = \bigcup_{i=1}^{n} A_i$. Also, does $m(A) > B$?

We get $m(A) = \bigcup_{i=1}^{n} (A_i)$. Also, $A \in \mathbb{R}(A)$.

Proof: (i) $A \in \mathbb{R}(A)$.

$A \in \mathbb{R}(A)$.

Lemmas 2 and 3. $A \subseteq B$.

For $A \subseteq B$, let $A \subseteq A_1 \subseteq \ldots \subseteq A_n$. A and $A_i$ satisfy $A \subseteq B$.

Solve for $A$ and $A_i$ by setting $A \subseteq B$.

Lemmata 1, 2, and 3. A and $A_i$ satisfy $A \subseteq B$. Find $A$ and $A_i$.
\[ \mathbb{R}_2 \text{ is a collection of all } r \in \mathbb{R} \text{ s.t. } 0 \leq r \leq 1. \]

Take all rings \( R \subseteq \mathbb{R}_2 \).

Maybe these are smaller rings. How are the found?

**Proposition:** The set \( \mathbb{R}_2 \) cannot contain a ring that is not simply add.

**Proof:** Consider \( X = \bigcup A \text{ and consider } (\bigcup A, \circ). \)

If there are 2 such minimal rings \( (A, \circ) - (A, \circ) \) that are isomorphic.

- **Claim:** There is another extension.

\[ (A, \circ) \cong (B, \circ) \]

\[ B = \begin{cases} 0 & \text{if } r < 0 \\ \mathbb{R}_2 & \text{if } r = 0 \\ 1 & \text{if } r > 0 \end{cases} \]

Consider \( A = \mathbb{R}_2 \) and \( B = \mathbb{R}_2 \).

**Definition of Extension.**
\[ A = \bigcup_{i=1}^{n} A_i \cup A \cup \bigcup_{i=1}^{n} B_i \cup B \]

Let \( B_0 = A \) and \( B'_0 = A \cup B \).

At \( k \geq 2 \):

\[ B_k = \begin{cases} B_{k-1} & \text{if } k \text{ is odd} \\ B_{k-1} & \text{if } k \text{ is even} \end{cases} \]

By construction:

\[ A = A \cup A_1 \cup A_2 \cup \cdots \cup A_n \cup B \]

Exercise: Prove for \( n = m + 1 \):

Assume the statement is true for \( n = m \).

If \( n = 2 \):

The case \( n = 1 \) follows from the above diagram.

The case \( n \geq 1 \)

Prove of Lemma 2 (by induction):
\[ A \Delta B = (A \cup B) \setminus (A \cap B) \]

\[ A \cup B = \bigcup_{i=1}^{n} (A_i \cup B_i) = \bigcup_{i=1}^{n} A_i \cap B_i \]

\[ B = \bigcup_{i=1}^{n} \mathbb{C}_i \cup \bigcap_{i=1}^{n} \mathbb{D}_i \]

\[ \text{Lemma 2: } A \cap \bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} (A \cap A_i) \]

Thus, \[ A \cap B = C \cap \mathbb{E} \cap D \]

\[ \Rightarrow A \cap B = C \cap \mathbb{E} \cap D \]

Let \( A, B \in \mathbb{F} \). We have to show that

\[ w \in A \iff w \in B \]

For \( w \in A \):

\[ \text{we know } w \in A \]

\[ \text{and } w \in B \]

\[ \text{it follows } w \in K(0) \]

\[ \text{and } w \not\in K(0) \]

Proof: \( w \in K(0) \), \( w \not\in K(0) \)

\[ A \not\in K(0) \]

\[ A \in K(0) \]

Thus, \( w \in A \iff w \in B \)
Thm: Let \( A_1, A_2, \ldots, A_n \) be a ring. Then:

\[ m(A) = \sum_{k=1}^{n} m(A_k) \]

\( A = (A \cup \bigcup_{k=1}^{n} A_k) \cup A \setminus (A \cup \bigcup_{k=1}^{n} A_k) \)

\( m(A) = m(A \cup \bigcup_{k=1}^{n} A_k) + m(A \setminus (A \cup \bigcup_{k=1}^{n} A_k)) \)

\( m(A) \leq \sum_{k=1}^{n} m(A_k) \)

Proof:

I. \( U \cup A_k \subseteq A \Rightarrow \sum_{k=1}^{n} m(A_k) \leq m(A) \)

II. \( A \supseteq (U \cup A_k) \cup A \setminus (A \cup \bigcup_{k=1}^{n} A_k) \)

\( m(A) = m((U \cup A_k) \cup A \setminus (A \cup \bigcup_{k=1}^{n} A_k)) \)

\( m(A) = m(U) + m(A \setminus (A \cup \bigcup_{k=1}^{n} A_k)) \)

\( m(A) \leq \sum_{k=1}^{n} m(A_k) \)

For \( n = 2 \):

\( m(A, U, A_2) = m(A_1) + m(A_2) - m(A \cup A_2) \)

\( m((A, U, A_2) \cup A \setminus (A \cup A_2)) \)

\( m(A, U, A_2) = \sum_{k=1}^{n} m(A_k) \)
To extend this to infinite \( \sigma \)-algebras:

\[
A = \bigcup_{j=1}^{\infty} A_j, \quad m(A) = \sum_{j=1}^{\infty} m(A_j)
\]

Let \( \mu \) be an \( \sigma \)-additive measure on \( \mathcal{F} \).

**Theorem:** Let \( m \) be defined on \( \mathcal{F} \), and \( \mu \) defined on \( \mathcal{F}(\mathcal{F}) = \{ A \subset \mathcal{F} : A \text{ is finite} \} \).

We will prove \( \sigma \)-additivity on \( m \).

\[
\mu \bigg|_{\mathcal{F}(\mathcal{F})} \quad \text{is} \quad \sigma \quad \text{-additive}.
\]

**Proof:**

Let \( A, B_n \in \mathcal{R}(\mathcal{F}) \), \( A = \bigcup_{j=1}^{\infty} A_j \), \( B_n = \bigcup_{j=1}^{n} B_{n,j} \), and \( B_{n,j} \) are pairwise disjoint. Moreover, \( A_j \cap C_{n,j} = \emptyset \), \( C_{n,j} \cap C_{n',j'} = \emptyset \).

Then \( C_{n,j} = B_{n,j} \setminus A_j \)

\[
m(A) = \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} m(B_{n,j})
\]
\[ \mu(A) = \sum_{j=1}^{\infty} \mu(A_j) = \sum_{n=1}^{\infty} \frac{1}{2^n} m(A_n) = \sum_{n=1}^{\infty} \frac{1}{2^n} (B_n) \]

\[ \mu(A) = \sum_{n=1}^{\infty} \mu(B_n) \]

The proof is the same as before with \( I_n A_k \rightarrow A \Rightarrow \exists m(A_k) \)

\[ \sum_{k=1}^{\infty} m(A_k) \]

Only need \( \sigma \)-additivity for part II.

pf. \( \cap_{n \rightarrow \infty} A_n = A \cap A \)

\[ \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} m(A_k) = m(A) \]
Define (main trick) $B_n = \Lambda A \setminus \bigcup_{k=1}^{n} A_k$.

\[ m_k(A) = \sum_{n=1}^{\infty} m(B_n) \leq \sum_{n=1}^{\infty} m(A_k) \]

La Béqne integral identity $E$.

Outer measure $E$ is defined as $E = \inf_{\mathcal{E}} \sum_{\mathcal{E} \in \mathcal{E}} m(B_n)$.

Def: The outer measure $\mu$ of any subset $A$ is given by $A \in \mathcal{L}_B$, $A = \bigcup_{n=1}^{\infty} B_n$.

Proof: Assume that $\mu^*(A) = \sum_{n=1}^{\infty} \mu^*(B_n)$.

Then $A \subseteq \bigcup_{n=1}^{\infty} A_n$.
Def. M is a measurable set if \( m^*(A \cup B) = m^*(A) + m^*(B) \) for all \( A, B \in \mathcal{M} \).

Theorem: If \( M \) is a measurable set, then \( M^c \) is also measurable.

Proof: Let \( A \subseteq M \) be measurable. Then \( M \setminus A = A^c \subseteq M^c \) is measurable. If \( M^c \) is measurable, then \( M \) is measurable.

Theorem: If \( A \subseteq M \) and \( m^*(A) = 0 \), then \( m^*(M \setminus A) = m^*(M \setminus (A \cup B)) = m^*(M \setminus A) + m^*(B) \).

\[ m^*(A \cup B) = m^*(A) + m^*(B) \]

\[ m^*(M \setminus A) = m^*(M \setminus (A \cup B)) = m^*(M \setminus A) + m^*(B) \]
The number of succeeds $\mathbb{C} \to \mathbb{C}$
called $\mathbb{C}$.

which is real uncountable many.

The cardinality of $\mathbb{C}$ is the same as $\mathbb{R}$.

$$m(C) = 0 = \frac{1}{2} - \frac{3}{4} - \cdots$$

$$\sum_{n=0}^{\infty} C \subseteq \bigcup_{n=0}^{\infty} C_n$$

There are a lot of Lebesgue measurable sets.