From last time:

If abs. cont. on $[a,b]$ and continuous on the range $f([a,b])$ implies $g$ of $f$ is AC?

$$f(x) = \begin{cases} 
\sqrt{x} & \text{for } x \in [0,1] \\
0 & \text{for } x \in (0,1]
\end{cases}$$

Abs. cont. by Lipschitz (deriv. is Abs. bdd.)

$$g(x) = \sqrt{x}, \quad x \in [0,1]$$

$$\sqrt{x} = \int_0^x \frac{dt}{2\sqrt{t}}$$

$$g(f(x)) = \begin{cases} 
\sqrt{\sqrt{\text{rel. min} f(x)}}, & x < 0 \\
0, & x = 0
\end{cases}$$

$\mathbb{B} \not\subseteq AC([0,1])$

On Assignment #0

$f \not\in AC([0,1])$, but $f' \not\in AC([\text{range } f])$ on $[0,1]$ construct $W$: $W = (0,1)$, $0 < \mu(W) < 1$

$W = U W_n$ with $W_n = (r_n - \frac{1}{2^{n+2}}, r_n + \frac{1}{2^{n+2}})$, $r_n \in \mathbb{Q} \cap [0,1]$

$\{0,1\} = \bar{\mathbb{Q}} \subseteq \bar{W}$

$\bar{W} \subseteq (0,1)$ since $W \subseteq [0,1]$
\[ \Phi(x) = \varphi_1(x) + \varphi_2(x) \]

for all \( x \in [0,1] \quad \text{and} \quad \Phi(x) = 0 \quad \text{for} \quad x \notin [0,1] \]

\[ \int_0^{\gamma} f(t) \, dt \]

\[ f(x) \in AC([0,1]) \quad \text{since} \quad f(x) = \varphi(x) \quad \forall x \in [0,1] \]

Also, \( \Phi(x) = 0 \) on \([0,1] \setminus W\) with \( \mu([0,1] \setminus W) > 0 \)

\[ \infty = \left( f^{-1}\right)' = \frac{1}{f'} = 0 \]

Now the lesson: (Change of Variables)

Thm: \( \Phi \uparrow \) on \([p,q]\), \( \Phi \in AC([p,q]) \), \( f \in L([a,b]) \)

where \([a,b] = \left[ \Phi(p), \Phi(q) \right] \). Then
\[ \int_a^b f(x) \, dx = \int_p^q f(\Phi(t)) \Phi'(t) \, dt \]

This is familiar for Riemann integrals.
\textbf{Pf:} The main difficulty is that } f(\varphi(t)) \text{ might be non-measurable, but } f(\varphi(t)) \varphi'(t) \text{ is measurable.}

\textbf{Pf:} \quad f \in C ([a, b]) \quad \Rightarrow \quad \int_a^b f(x) \, dx = \int_0^1 \varphi'(t) \, dt = \varphi(b) - \varphi(a)

\psi(t_k) = \chi_{k+1} \quad \overset{d}{\Rightarrow} \quad M_k = \max f(x), \quad x \in (x_k, x_{k+1})

\psi(t_k) = \chi_k \quad \overset{d}{\Rightarrow} \quad m_k = \min f(x), \quad x \in (x_k, x_{k+1})

\chi_{k+1} - \chi_k = \int_{t_k}^{t_{k+1}} \varphi'(t) \, dt \quad \Rightarrow \quad m_k (x_{k+1} - x_k) \leq \int_{x_k}^{x_{k+1}} f(x) \, dx \leq M_k (x_{k+1} - x_k)

\overset{\text{Calc 1}}{\implies} \quad m_k \int_{t_k}^{t_{k+1}} \varphi'(t) \, dt \leq \int_{t_k}^{t_{k+1}} f(\varphi(t)) \varphi'(t) \, dt \leq M_k \int_{t_k}^{t_{k+1}} \varphi'(t) \, dt = M_k (\chi_{k+1} - \chi_k)

m_k (x_{k+1} - x_k)

\text{Now we sum these up!}

\sum m_k \int_{t_k}^{t_{k+1}} \varphi'(t) \, dt \leq \sum f(\varphi(t)) \varphi'(t) \, dt \leq \sum M_k \int_{t_k}^{t_{k+1}} \varphi'(t) \, dt

\overset{\text{lower Darboux sum}}{\Rightarrow} \quad \sum m_k (x_{k+1} - x_k)

\overset{\text{upper Darboux sum}}{\Rightarrow} \quad \sum M_k (x_{k+1} - x_k)

\text{converge to the same thing}

\text{So if } f \text{ is cont., } \int_a^b f(x) \, dx = \int_0^1 f(\varphi(t)) \varphi'(t) \, dt
If \( f \) is not necessarily continuous, but \( f \) is measurable and bounded, say by \( M \), then by LUSIN: \( \exists g_n \in C([a, b]) : g_n (x) \xrightarrow{a.e.} f(x) \) with \( |g_n(x)| \leq M \).

\[
\lim_{n \to \infty} \int_{a}^{b} g_n(x) \, dx = \lim_{n \to \infty} \int_{p}^{b} g_n(\varphi(t)) \varphi'(t) \, dt
\]


\[
\int_{a}^{b} f(x) \, dx
\]

true, but painful

measurable blc

it is the limit of a seq. of mea. funs.

Aside:

**3** (Pit.): \( f(x) = \lim_{n \to \infty} g_n(x) \) a.e. on \([a, b] \)

If \( f(\varphi(t)) \varphi'(t) = \lim_{n \to \infty} g_n(\varphi(t)) \varphi'(t) + \text{H}d \)

\[
\lim_{n \to \infty} g_n(\varphi(t)) \varphi'(t) + \text{H}d \]

two things to show

what is left to show?

\[ \Rightarrow f \text{ is measurable and } f(\varphi(t)) \varphi'(t) \leq M \varphi'(t) \epsilon[\log]
\]

What we know: \( f(x) = \lim_{n \to \infty} g_n(x) \).

let \( e_x = \{ x \in [a, b] : \text{\( (3) \) is not true} \} \)

\( \mu(e_x) = 0 \)

\( \varphi^{-1}(e_x) = e_t \)

If \( t \notin e_t \), then \( (3) \) is true.

\( \varphi^{-1}(e_x) = e_t \)

\( \varphi(t) \in e_x \Rightarrow (3) \text{ is true.} \)
Now we will split $e_t$

What is left is to show $\exists$ is true for $t \in e_t$

Define $e_t^* := \{ t \in e_t : \varphi'(t) \neq 0 \}$

Since $\varphi \uparrow \varphi'(t) = \left\{ \begin{matrix} +\infty \\ 0 \end{matrix} \right.$

Let us show: $\mu(e_t^*) = 0$ Another home assignment.

**Lemma** Let $e_x \subset [a, b] : \mu(e_x) = 0$ and let $e_t = \varphi(e_x)$. Then $\mu(e_t^*) = 0$ where $e_t^* = \{ t \in e_t : \varphi'(t) \neq 0 \}$

**PF:** Observe that we don't know whether $e_t$ is measurable or not.

Tomorrow we will finish. Go through this. It is the last exercise on the homework assignment.

Consider today:

Problem: We want to define a $\mu$-measure $\mu : \mathbb{R}^n \rightarrow [0, \infty]$ such that

1) $\mu([0, 1]) = 1$

2) $\mu(A) = \mu(B)$ if $B$ is a "shift" of $A$.

3) $\sum_{j=1}^{\infty} \mu(A_j) = \mu(A)$ (why?"

4) and $\mu$ to be defined on every bounded set.

Balick: proof. We can't for $\mathbb{R}^1$, $\mathbb{R}^2$. We can for $\mathbb{R}^n$ if $n \geq 3$. 
Thus 11/4/18 We are still proving.

\[ \phi \in AC([a,b]) \quad \phi'(p) = a, \quad \phi'(q) = b, \quad \phi \text{ increasing} \]

**Lemma** Let \( e_x \subseteq [a,b] \), \( \mu(e_x) = 0 \), and let \( e^*_t := \{ t \in e_x : \phi'(t) = 0 \} \)

Then \( \mu(e^*_t) = 0 \), \( e^*_t = \{ t \in e_x : \phi'(t) = 0 \} \)

\( (\phi'(t) = \infty, \phi'(t) < 0, \text{ or } \phi'(t) \to 0) \)

**Lemma (Ex. 5a.)** \( E_t \subseteq [a,b] \), \( E_x := \phi(E_t) \) then

\[ \int_{E_t} \phi'(t) \, dt = \mu(E_x) \]

If \( E_t = (\alpha, \beta) \Rightarrow E_x = (\phi(\alpha), \phi(\beta)) \)

then \( \mu(E_x) = \phi(\beta) - \phi(\alpha) = \int_{E_t} \phi'(t) \, dt = \int_{E_t} \phi'(t) \, dt \)

\( \Rightarrow \) countable union of open intervals \( \Rightarrow \) open subset of \( [a,b] \) is true \( \Rightarrow \) for any closed subset (as a complement) \( \Rightarrow ? \) approx.

\[ C_t \subseteq E_t \subseteq O_t \quad \text{on assignment} \]

\[ \mu(O_t \setminus E_t) \leq \varepsilon \]

\[ \mu(E_t \setminus C_t) \leq \varepsilon \]

**Pf:** We don't know (and don't need to know) whether \( E_t \) is measurable. We can assume \( E_x \subseteq (a,b) \)

Consider a sequence of open sets \((G^n_x)_{n=1}^{\infty} : (a,b) \supset G^1_x \supset G^2_x \supset \ldots \supset G^n_x \supset \ldots \supset E_x \)
\[ e_x = E_x \quad \text{and} \quad E_t = e_t \]

\[ \mu(G^n_x) \xrightarrow{n \to \infty} 0 \quad \text{by continuity of measure.} \]

\[ \mu \left( \bigcap_{n=1}^{\infty} G^n_x \right) = 0 \]

\[ E_x \supseteq e_x \]

\[ \mu(E_x) = \int_{E_t} \varphi(t) \, dt \quad \Rightarrow \quad \mu(E_t^*) = 0 \]

\[ \int_{E_t} \varphi(t) \, dt + \int_{E_t^*} \varphi'(t) \, dt > 0 \]

\[ E_t^* \uparrow E_t \backslash E_t^* \]

this is \( > 0 \)  \quad \text{this is} \ 0

\[ E_x \text{ is } \varphi \text{-delta (intersection of open sets)} \]

If \[ E_x = \bigcap_{n=1}^{\infty} G^n_x \]

\[ \varphi^{-1}(E_x) = \bigcap_{n=1}^{\infty} \varphi^{-1}(G^n_x) \]

\[ \text{always true} \]

\[ \text{this is } E_t \]

\[ \int_{E_t^*} \varphi'(t) \, dt + \int_{E_t} \varphi(t) \, dt \]

\[ \text{this is positive} \]

\[ \Rightarrow \mu(E_t^*) = 0 \]

\[ \text{since } e_t^* \subseteq E_t^*, \Rightarrow \mu(e_t^*) = 0 \]

\[ E_t^* = \left\{ t \in E_t \subseteq [p,q] : \varphi'(t) \neq 0 \right\} \]

\[ e_t^* = \left\{ t \in e_t = \varphi'(e_x) \text{ with } e_x \subseteq [a,b] : \varphi'(t) \neq 0 \right\} \]

\[ \text{the intersections of open sets are measurable.} \]
Recall \( (\phi^{-1})' = \frac{1}{\phi} \) \[ \phi \text{ is increasing} \]

**On home assignment**

\( f(\phi(t)) \) will not be measurable, with \( f(\phi(t)) = \frac{1}{E(t)} \text{ non-measurable} \).

but \( f(\phi(t)) \phi'(t) = 0 \)

This proof is the best exercise on measurability.

If \( f \text{ meas.}, g \text{ meas.} \implies fg \text{ meas.} \)

However \( f, g \text{ meas.} \implies f, \text{meas. and } g \text{ measurable.} \)

Now consider \( f \text{ unbounded}. \) We proved for \( f \text{ continuous, and for } f \text{ measurable and bounded.} \)

If \( f \geq 0, f = f_+ - f_- \).

Now, for any \( f \in L^2([a,b]) \) Consider the cutting for \( f_n(x) := \begin{cases} f(x), & f(x) \leq n \\ n, & f(x) > n \end{cases} \)

\[
\int_a^b f(x) \, dx = \lim_{n \to \infty} \int_a^b f_n(x) \, dx = \lim_{n \to \infty} \int f_n(\phi(t)) \phi'(t) \, dt
\]

\[
\lim_{n \to \infty} f_n(x) = f(x)
\]

\[
\lim_{n \to \infty} f_n(\phi(t)) \phi'(t) = f(\phi(t)) \phi'(t)
\]

now \( L \text{ is out of trouble} \)

\( \mu(t \text{ in trouble}) = 0 \)
\[ f_n(x) \leq f(x) \quad \forall x \in \mathbb{R} \]

Use Lebesgue's dominated convergence theorem, since \( f_n(x_0) \to f(x_0) \).

So, \[ \lim_{n \to \infty} \int_{\mathbb{R}} f(x) \, dx = \int_{\mathbb{R}} \lim_{n \to \infty} f_n(x) \, dx \]

The cutting off technique allows you to go from unbounded to bounded.

Radon-Nikodym Thm is the most important part of the course.

Let \( E \subseteq \mathbb{R} \).

Defn: \( x \) is a point of density of \( E \) if

\[ \frac{\mu(E \cap (x-h, x+h))}{2h} \to 1 \quad (h \to 0) \]

\( f(h) = \mu(E \cap [a, b]) \) means \( f'(h) \) exists?

Thm: Almost every \( x \) is a point of density.

Pf: \( \phi(x) = \int_a^x 1_E(t) \, dt \)

\( \phi(x) = \lim_{h \to 0} \frac{\phi(x+h) - \phi(x)}{h} \)

\[ \Rightarrow \phi'(x) = \lim_{h \to 0} \frac{\phi(x+h) - \phi(x-h)}{2h} \]

Thus,
\[ 1_E(x) \text{ a.e.} = \lim_{h \to 0} \left( \frac{\phi(x+h) - \phi(x-h)}{2h} \right) = \lim_{h \to 0} \frac{\int_{x-h}^{x+h} 1_E(t) dt - \int_{x-h}^{x+h} 1_E(t) dt}{2h} = \lim_{h \to 0} \frac{\int_{x-h}^{x+h} 1_E(t) dt}{2h} = \lim_{h \to 0} \frac{\mu(E \cap (x-h, x+h))}{2h} = \begin{cases} 
1, & x \in E \\
0, & x \notin E \end{cases} \]

Space \( X, \mathcal{M}, \mu \) and finite measure

Take \( f \geq 0 \) so \( f \in L^1(X, dm) \)

Define \( \phi(A) := \int_A f \, dm \)

A measure

\[ \mu(E) = 0 \Rightarrow \phi(E) = 0 \]

For every \( f \in BV, f = \phi + \chi + \text{jump} \)

Given \( \mu \), consider any measure \( \phi \) on the same space \( X \). Then:
1) $\emptyset$ is abs. cont. if $\mu(E)=0 \Rightarrow \phi(E)=0$.

2) $\emptyset$ is continuous if $\phi(\{x\})=0$

3) $\emptyset$ is discrete if supp $\emptyset$ is a countable set.

Aside: A compact set is called the support of $\emptyset$ if

$\emptyset(E)=0 \forall E \cap A = \emptyset$ (think Closed set)

Ex. $\delta_x(E) = \begin{cases} 1, & x \in E \\ 0, & x \notin E \end{cases}$

support $\delta_x = \{x\}$

4) $\emptyset$ is singular if $\mu(\text{supp. } \emptyset) = 0$

w.r.t. $\mu$

If $\mu(E)=0 \Rightarrow \phi(E)=0$

$\Rightarrow \exists f \in L^1(X, \mu)$:

$\forall A \in \mathcal{M}$

$\phi(A) = \int_A f \, d\mu$
Thm. 1) Subring $\mathcal{B}$, $\sigma$-additive measure (and additive)
Why we define the measure on a subring?
2) Outer measure and Lebesgue measurable sets
3) Continuity of the measure
4) Defn. of the Lebesgue integral
5) Chebyshev's inequality
6) Egorov's Thm.
7) Lebesgue Thm. on Dominated Convergence
8) Levi's Thm.
9) Sign-Changing measures
10) Radon-Nykodim.

Any func. $f \in BV([a,b])$ can be written as

$$f = \Phi + \chi + J$$

$\Phi$ absolutely continuous
$\chi$ singular
$J$ Cantor type

$J = \sum F_n$
$J' = 0$ a.e.

$\Phi(b) - \Phi(a) = \int_a^b \Phi'(t)dt$
Newton-Leibnitz-type

$\chi'(x) = 0$, but $\chi = \text{constant}$

$\Phi(b) - \Phi(a) = \int_a^b \Phi'(t)dt = \int_a^x f'(x)dx \leq f(b) - f(a)$
you need Fatou's Lemma to show this.
\[(X, \mathcal{M}, \mu), \mu(\infty) < \infty\]

\text{Radon-Nykodim let's you deal with the various types of measures.}

Let \(\phi\) be another measure on \((X, \mathcal{M})\).

After checking \(\phi\) is well-defined on \(\mathcal{M}\).

\text{Defn: } \phi \text{ is called absolutely continuous with respect to } \mu \text{ if } \mu(A) = 0 \implies \phi(A) = 0.

\text{N.B. } \mu \text{ and } \phi \text{ may or may not be Lebesgue-measurable.}

\text{Example: } \mu(x) \in L([0,1]) \text{ and } \phi(A) = \int_A f(x) \, dx \text{ or}

\[\phi(A) = \int_A f \, d\mu \text{ with}\]

\[\mu = \delta_\frac{1}{2} \text{ on } [0,1] ; \quad \delta_\frac{1}{2}(A) = \begin{cases} 1, \frac{1}{2} \in A \\ 0, \frac{1}{2} \notin A \end{cases} \]

\[A \subseteq [0,1] \]

\(\phi\) is absolutely continuous w.r.t. \(\delta_\frac{1}{2}\) iff

\[\int_A \frac{1}{2} \, d\mu = 0 \implies \phi(A) = 0 \implies \text{we will show } \phi = \text{const} \delta_\frac{1}{2}\]

If \(\frac{1}{2} \notin E\), then \(\phi(E) = 0\).

We will show \(\phi(A) = \int_A f(x) \, d\delta_\frac{1}{2}(x)\)

\[\int f(\frac{1}{2}), \frac{1}{2} \in A \]

\[0, \frac{1}{2} \notin A\]
We say \( \phi \) is "concentrated" on a set \( A \) (or supported by the set \( A \)) or "\( A \) is the support of \( \phi \)" if \( \forall E \in \mathcal{M}_\phi, \phi(E) = 0 \) \( \forall E \in A = \phi \).

Ex: \( \delta_{\frac{1}{2}} \) is concentrated at \( \{\frac{1}{2}\} = A \)

\[ \phi(A) := \int_A x \, dx \quad A \subseteq [0,1] \]

\[ A \subseteq [0,1] \quad \phi(A) = \text{"length"}(A) \]

\[ \text{supp}(\phi) = [0,1] \]

We say that \( \phi \) is singular with respect to \( \mu \) (\( \phi \) and \( \mu \) are mutually singular) iff \( \text{supp} \phi \cap \text{supp} \mu = \emptyset \)

\( \phi \) is singular w.r.t. \( \mu \) iff \( \mu(\text{supp} \phi) = 0 \)

"the support of \( \phi \) has small measure"

Ex: \( \delta_{\frac{1}{2}} \) is singular w.r.t. Lebesgue measure on \([0,1] \)

\[ \text{supp} \delta_{\frac{1}{2}} = \{\frac{1}{2}\} \quad \mu(\{\frac{1}{2}\}) = 0 \]

Ex: Consider the Cantor func. \( c(x) \) on \([0,1] \).

\[ \mu_c \left([a,b]\right) = c(b) - c(a) \]

\( \mu_c \) is singular w.r.t. Lebesgue measure on \([0,1] \)

\[ \text{supp} \mu_c = \text{Cantor set}, c \text{ and Lebesgue measure}(C) = 0 \]
Defn: \( \phi \) is \textit{continuous} (not absolutely cont.) if \( \phi(A) = 0 \forall A \subseteq \mathbb{R} \)

Ex: Let \( f \) be a "nice" function on \( \mathbb{R}^2 \)

\[ \phi \text{ on } \mathbb{R}^2: \phi(A) = \int f(x, 0) dx \]

where \( A \) intersects the \( \chi - \text{axis} \)

Ex: \( \int e^{-x^2} dx = \phi(A) \)

Supp \( \phi = \chi - \text{axis} \) (only place where \( \phi \) takes on values)

Lebesgue measure in \( \mathbb{R}^2 \) (\( \chi - \text{axis} \)) = 0

\( \phi \) is abs. cont. w.r.t. the Lebesgue measure on \( \mathbb{R} \) restricts to the \( \chi - \text{axis} \)

\[ \phi\mid_{(A \cap \chi - \text{axis})} = \int_{\chi - \text{axis}} e^{-x^2} dx \]

Some questions on support.
The support of \( \mu \) is the set \( A : \forall E \subseteq M, \mu(E \cap A) = 0 \) if \( E \cap A = \emptyset \).

Friday, 11/08: We are given \( (X, M, \mu) \) \( \emptyset \) on \( M \) which contains open sets.

- \( \emptyset \) is singular if \( \mu(\text{supp} \emptyset) = 0 \)
- \( \emptyset \) is supported by \( A \) if \( \emptyset(E) = 0 \) \( \forall E \subseteq A \)
- \( \text{Defn:} \) To define \( \text{supp} \emptyset \) we need a topological space. We do assume that \( X \) is a topological space.

- \( \mathbb{R}^n \) \( \text{supp} \emptyset \) is a compact set satisfying:
  \[ \forall x \in \text{supp}(\emptyset), \ \forall U_x \text{ open \ neighborhood \ of \ } x \text{, we have } \emptyset(U_x) > 0 \]

  The topology is not necessary.

  For example (from yesterday) \( \delta_A \) on \( [0, 1] \) with \( \text{supp} \delta_{\frac{3}{2}} = \{ \frac{3}{2} \} \).

We discussed that measure theory on topological spaces is very complex with little benefit to see in this class.

1) \( \emptyset \) is singular w.r.t. \( \mu \) if \( \mu(\text{support of } \emptyset) = 0 \)
2) \( \emptyset \) is absolutely continuous w.r.t. \( \mu \) if \( \mu(A) = 0 \Rightarrow \emptyset(A) = 0 \)

\( \emptyset \ll \mu \)
Let \( \{f_n\} \) be a sequence of measurable functions which converge a.e. to a measurable set \( E \) to a function \( f \). Then, given any \( \delta > 0 \), \( E \) a measurable set \( E_\delta \subseteq E \) such that: 1) \( \mu(E_\delta) > \mu(E) - \delta \) and 2) \( f_n \xrightarrow{n \to \infty} f \) on \( E_\delta \).

Proof: Let \( H = \{x \in E : \{f_n(x)\} \text{ converges to } f(x)\} \). By definition, \( \mu(E \setminus H) = 0 \) \( \Rightarrow \mu(E) - \mu(H) = 0 \Rightarrow \mu(H) = \mu(E) \).

Let \( H_{k,j} = \{x \in H : |f_n(x) - f(x)| < \frac{1}{k}, \forall n \geq j\} \). Note that for fixed \( k \), \( H_{k,1} \subseteq H_{k,2} \subseteq H_{k,3} \ldots \) and \( f(x) \) is measurable on \( H \).

\( H = \bigcup_{j=1}^{\infty} H_{k,j} \). Thus, \( \mu(H) = \lim_{j \to \infty} \mu(H_{k,j}) \). There exists \( N_k \) such that

\[ |\mu(H) - \mu(H_{k,j})| < \frac{\delta}{2^k} \quad \forall j > N_k. \]

Let \( F = \bigcap_{k=1}^{\infty} H_{k,N_k} \).

\( |f_n - f| < \frac{1}{k} \quad \forall n > N_k \) on set \( F \). Hence \( \{f_n\} \) converges uniformly to \( f \) on set \( F \). \( H \setminus F \subseteq \bigcup_{k=1}^{\infty} H \setminus H_{k,N_k} \).

\[ m(H \setminus F) \leq \sum_{k=1}^{\infty} (m(H \setminus H_{k,N_k}) < \sum_{k=1}^{\infty} \frac{\delta}{2^k} = \delta. \]

\[ m(H \setminus F) < \delta \]

\[ m(H) - m(F) < \delta \]

\[ m(E) - \delta < m(F) \]

Let \( F = E_\delta \).
The integral of a characteristic function is just the measure.

\[ \rho(A)(x \text{-axis}) = \int e^{-x^2} dx \]

\[ (x, M, \mu), \rho \text{ on } M \]

11-7-08

Def.: \( \rho \) is supported by \( X \) if \( \rho(E) = 0 \) \( \forall E \subseteq X \).

Def.: To define suppf, we need a topological space. We do assume that \( X \) is a topological space.

\[ \text{suppf is a compact set satisfying } \forall x \in \text{suppf} \quad \forall U \text{ we have } \rho(U_x) > 0. \]

Q) Let \( \rho \) be singular & abs. continuous w.r.t \( \mu \). Show \( \rho = 0 \).

Proof:
\[ \text{ac: } \mu_{\rho}(A) = 0 \Rightarrow \rho(A) = 0 \]
\[ \text{sing: } \mu_{\rho}(\text{suppf}) = 0 \]

Riesz-Nykodim: abs. cont. wrt \( \mu \)

Given \( (X, M, \mu) \), \( \rho \ll \mu \). Then \( \exists f \in L^1(X, \mu) \) \( \rho(A) = \int_A f d\mu \) for every \( A \in M \). (\( \mu \neq 0 \).)

\( f \) is called the derivative of \( \rho \) wrt \( \mu \).

\[ f = \frac{\rho}{\mu} \text{ a.e.}. \]

\[ \rho(B) - \rho(A) = \int_A^B f \ d\mu. \] (Hence \( \rho \) is absolutely continuous. Have \( \rho = \text{bg. abs. cont.} \) have \( \rho = \text{sgn} \) \( f \) \( f = 0 \Rightarrow \rho = \text{const.} \)

Main Lemma.

- Assume that \( \mu \) is a measure on \( (X, M, \mu) \) \( \mu(X) > 0 \).

Then \( \exists N \in \mathbb{N} \) & a set \( B = M \) such that \( \mu(B) > 0 \) &

\[ \frac{1}{N} \mu(A) \leq \mu(A), \forall A \subseteq X \]

(\( \rho \) monotonically increasing).
Warm up. Friday 11/7/2008

Let \( \varphi \) be singular and abs. cont. w.r.t. \( \mu \). Then
\( \varphi \equiv 0 \) (a.e. w.r.t. \( \mu \))

Singular \( \Rightarrow \mu(\text{supp } \varphi) = 0 \)
Abs. cont. \( \Rightarrow \varphi(\text{supp } \varphi) = 0 \)

but \( x \in \text{supp } \varphi \Rightarrow \varphi(x) \neq 0 \) \( \therefore \text{supp } \varphi = \emptyset \),
so \( \varphi(x) \equiv 0 \).

**Pf. of Radon-Nykodim**

\[(X, M, \mu), \varphi \ll \mu \quad (\mu(A) = 0 \Rightarrow \varphi(A) = 0)\]

Then \( \exists f \in L(X, \mu) \) such that
\[\varphi(A) = \int_A f \, d\mu \quad \forall A \in M.\]

\( f \) is called "the derivative" of \( \varphi \) w.r.t. \( \mu \)
\[f = \lim_{\mu(A) \to 0} \frac{\varphi(A)}{\mu(A)} \quad \text{if also recall} \quad \int_a^b \varphi'(t) \, dt \text{ with} \]
\[\varphi'(+) = 0 \Rightarrow \varphi = \text{constant}.\]
Main Trick of the Course

Proof (Main Lemma): Assume that $\lambda \ll \mu$ is a measure on $(X, \mathcal{M}, \mu)$ and $\lambda \neq 0$. Then $\exists n \in \mathbb{N}$ and a set $B \in \mathcal{M}$ or $B \in \mathcal{M}$ such that $\mu(B) > 0$ and $\frac{1}{n} \mu(A) \leq \lambda(A)$ for all $A \subseteq B$. 

$\forall A \subseteq B$, $(\frac{1}{n} \mu(E \cap B) \leq \lambda(E \cap B)$ $\forall E \in \mathcal{M}$.

Idea: 1) to look at the right set
2) to know that the extremal point of the set satisfies the condition.

1) We look at the set $K$, defined as follows:

$$K := \left\{ f \mid f \text{ on } X: f \geq 0, f \in L(X, \mu), \int_A f \, d\mu \leq \emptyset(A) \forall A \in \mathcal{M} \right\}$$

2) Let $M := \sup_{f \in K} \int_X f \, d\mu$. If $f$ satisfying the definition for $M$ above and for this $f$ we will have $\int_A f \, d\mu = \emptyset(A) \forall A \in \mathcal{M}$.

Start of the proof: $\exists f_0 \in K: M = \int_X f_0 \, d\mu$.

From the defn. of supremum it follows that $\exists (f_n)_{n=1}^\infty \in K: \lim_{n \to \infty} \int_X f_n \, d\mu = M$. 

...
Aside about sup.

Given \( \sup A = b \)

1. \( \forall a \in A; \ a \leq b \)
2. \( \forall \epsilon > 0, \ \exists a \in A : \ a > b - \epsilon \)

Set \( A \) \( b = \sup(A) \)

Back to proof:

Show \( f_0 = \lim_{n \to \infty} f_n = \lim_{n \to \infty} f_n \) always.

We will try to use Levi (lores monotonic fns.)

Let:\( g_n(x) = \max(f_1(x), f_2(x), \ldots, f_n(x)) = \max_{1 \leq j \leq n} f_j(x) \)

\( f_0 := \lim_{n \to \infty} g_n(x) \)

\( E_j \subseteq X : g_n(x) = f_j(x) \).

\( \forall E \in \mathcal{M} : \int_E g_n \, d\mu \leq \phi(E) \implies g_n \in K \)

Show this and \( g_n \in K \)

\( \int_E g_n \, d\mu = \sum_{j=1}^{n} \int_{E_j} f_j \, d\mu \leq \sum_{j=1}^{n} \phi(E_j) = \phi(E) \)

You made need to take differences to get these \( E_j \)'s disjoint
Thus, by the defn. of $g_n$,

\[ g_1(x) \leq g_2(x) \leq g_3(x) \leq \ldots \leq g_n(x) \leq \ldots \]

\[ \int g_n \, d\mu \leq \phi(x) = \text{constant (some finite value)} \]

since $g_n \leq K$

So we have the conditions for Levi's Thm.

\[ \Rightarrow \quad \exists \lim_{n \to \infty} g_n \text{ and } \lim_{n \to \infty} \int g_n \, d\mu = \lim_{n \to \infty} \int g_n \, d\mu \]

\[ \text{but this is } \int f_0 \, d\mu \text{ with } f_0 \in K \]

Claim: \[ \int_{f_0} \, d\mu = \phi(A) \forall A \leq M. \text{ Since } f_0 \in K, \]

then \[ \int_{f_0} \, d\mu = \phi(A) \]

Consider: \[ \lambda(A) = \phi(A) - \int_{f_0} \, d\mu \]

If $\lambda \neq 0$, then perturb $f_0$ and show that \[ \int_{f_0} \, d\mu \neq M = \sup_{f \in K} \int f \, d\mu \]

\[ \lambda \text{ is a measure}, \quad \lambda \left( \bigcup_{i=1}^{\infty} A_i \right) = \phi \upharpoonright \leq \mu \]

\[ \lambda \ll \mu \]

absolutely continuous w.r.t. $\mu$

\[ \text{erased (it is obvious)} \]
Now we are using the trick.

Consider $h_i = f_0 + \frac{1}{n} 1_B$; $1_B(x) = \{1, x \in B \}$, $\{0, x \notin B\}$

the perturbation $\Rightarrow$

a) $h \in K$

b) $\int h \, d\mu > \int f_0 \, d\mu$

b) Since $h > f_0$ by defn. of $f$, $\int h \, d\mu > \int f_0 \, d\mu$

\[\int h \, d\mu \geq \int f_0 \, d\mu\]

\[\forall E \in \mathcal{M}, \int h \, d\mu = \phi(E)\]

\[\int h \, d\mu = \int f_0 \, d\mu + \frac{1}{n} \int 1_B \, d\mu = \int f_0 \, d\mu + \frac{1}{n} (E \cap B)\]

\[\text{from much earlier.}\]

\[
\int h \, d\mu \leq \int f_0 \, d\mu + \lambda (E \cap B) \\
= \int f_0 \, d\mu + \phi(E \cap B) - \int f_+ \, d\mu
\]