$E = [0,1] \times [0,1]$

$\sigma$ -- subring of rectangles of type $T_{ab} : \{ a \leq x \leq b, 0 \leq y \leq 1 \}$

$m(T_{ab}) = b - a$

\[ a \quad b \]

$1)$ Describe the Lebesgue construction of the measure: What set are we going to be measurable?

Any $A \subseteq [0,1] \times [0,1]$ is Lebesgue measurable as $A \times [0,1]$. Is that all?

\[ \begin{array}{c|c}
\hline
& \bigcup \{ B \times [0,1] : B \subseteq B \times [0,1] \} = (\text{subsets of measure zero}) \\
\hline
\end{array} \]

$B \times \emptyset$ are also measurable by subsets of measure zero.
b) Define $T = \{(x, \frac{1}{2}) | x \in [0,1]\}$. Is it measurable?

**Def.** $E = [0,1] \times [0,1]$, $A \subset E$

The inner measure, $\mu_*(A) = 1 - \mu^*(E \setminus A)$

$(\mu^*(E) = 1)$

a) $\mu_*(A) \leq \mu^*(A)$

b) $A \subset M_\mu \iff \mu_*(A) = \mu^*(A)$

measurable sets on $E$

If the approximation from the inside equals the approximation from the outside, then the inner set is measurable.

Let $A \subset [0,1]$ in the decimal decompositon $x \in A$ if the decimal decomposition of $x$ you need 2 before you need 3.

$x = 0.243 \in A$ If no 2 nor 3, $x \notin A$

$x = 0.35 \notin A$

Find the Lebesgue measure of $A$. 

\[ \mu^*(T) = 1 \] 

\[ \mu_*(T) = 0 \] only using vertical strips!
Lebesgue measure data,

Counter-type exercise

The idea: pass to the complement.

\[ m(A) = 1 - m(A^c) = 1 - 0.1 - 0.08 - 0.064 - \ldots \]

3 in 2nd dec. place
no 2 in 1st place
or 3 already dropped

\[ 0.9 - \frac{8}{100} = 0.892 \]

\[ 0.9 - 0.4 = 0.5 \]
This is theory (Really boring)

Defn: A $\sigma$-additive measure $m$ defined on

$\sigma$-algebra $\mathcal{M}$ is called $\sigma$-finite if $X = \bigcup_{i=1}^{\infty} B_i$, $\exists m(B_i) < \infty$

measure is called $\sigma$-finite but we are really talking about the space.

Ex. of this is usually rectangles in a plane

Not $\sigma$-finite

Ex: $\sum_{i=1}^{\infty} B_i$ you have a ton $f$ on $[0,1]$

Then for any finite subset of $[0,1]$, \{ $x_1, x_2, x_3, x_4$ \}

you define the measure of this subset

$\mu(\{x_1, x_2, x_3, x_4\}) = \sum_{i=1}^{4} f(x_i)$

Let $f(x) = 1$; $\mu$ is not $\sigma$-finite

Since $[0,1] \neq \bigcup \{\text{finite sets}\}$

$\sum_{i=1}^{\infty} B_i$ if $B_i$ is not finite

$[0,1] = \bigcup_{i=1}^{\infty} B_i$

$\Rightarrow \mu(B_i) = +\infty$

$\mu(B) \geq \mu(A)$ provided $B \supseteq A$

We will not deal (forget about it) sets and measure that are not $\sigma$-finite.
If $\sigma_m$ does not have a unit,

$$
\mu^* (A) := \inf \sum_{n=1}^{\infty} m(B_n), \quad \sum_{n=1}^{\infty} m(B_n) < \infty
$$

for any covering for the set of all Lebesgue measurable sets.

**Theorem:** $\mu^*$ is a ring

**Theorem:** $\mu := \frac{\mu^*}{m}$ is additive on $M$

**Theorem:** $\mu$ is $\sigma$-additive on $M$

**Theorem:** The $M$ is a $\sigma$-algebra with the unit.

$\Rightarrow M$ is a $\sigma$-ring,

$$A = \bigcup_{n=1}^{\infty} A_n, \quad A_n \in M$$

$A \in M$ iff $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \text{Cost} \text{ independent of } N.$

**Corollary:** $\mu_{A} := \{B \in \mathcal{M} \mid B \subseteq A\}$ is $\sigma$-algebra with the unit $A$.

Fix any $A \in M$. If $\mu$ is $\sigma$-additive and $\sigma$-finite, then $X = \bigcup_{i=1}^{\infty} B_i$, $\mu(B_i) < \infty$. Then $\mu_{B_i} := \{C \in M \mid C \subseteq B_i\}$, $A_i := A \cap B_i$, $U = \bigcup_{i=1}^{\infty} U_i$ consider $U_i := \{A \subseteq X \mid A \cap B_i \in \mu_{B_i}\}$. 


\[ A \in \mathcal{U} \Rightarrow A = \bigcup_{i=1}^{\infty} A_i, \quad \forall A_i \in \mathcal{M}_0. \]

\[ \tilde{\mu}(A) := \sum_{i=1}^{\infty} \mu(A \cap B_i), \quad A \cap B_i \in \mathcal{M}_0. \]

\[ \tilde{\mu}(A) < \infty \]

9/9/08 Measurable funs:

"generalization of continuity"

\[(X, \sigma_X) \quad \text{set} \quad \text{collection of subsets} \quad (Y, \sigma_Y) \quad \text{set} \quad \text{collection of subsets} \]

**Definition** \( f \) is \((\sigma_X, \sigma_Y) \)-measurable if \( \forall A \in \sigma_Y, f^{-1}(A) \in \sigma_X \)

**Note** the defn. Has Nothing to do with measure. (Too Abstract)

**Ex:** \( X = Y = \mathbb{R}, \sigma_X = \sigma_Y = \text{open sets} \). Then measurability = continuity.

**Defn:** IR. You consider a collection \( \mathcal{U} \) of all open sets on IR. \( \exists \) a minimal \( \sigma \)-algebra containing \( \mathcal{U} \), Lebesgue measurable sets. Then \( \sigma \)-algebra is or Borel measurable sets called a collection of Borel sets.
\{ \text{Borel sets} \} \subseteq \{ \text{Lebesgue measurable sets} \}

\text{every subset of a Cantor set is Lebesgue measurable}

\text{Every } \{ \text{Lebesgue measurable sets} \} = \{ \text{Borel sets} \} \cup \{ \text{set of measure 0} \}

\text{Defn. } f \text{ is } \mu \text{-measurable if } \forall \text{ Borel set } A,

f^{-1}(A) \in \mathcal{M}_\mu \quad \text{set of all Lebesgue measurable sets (with respect to } \mu)\

f: \mathbb{R} \to \mathbb{R} \quad \text{are Borel functions.}

\forall \text{ Borel set } A, \ f^{-1}(A) \text{ is also Borel.}

\text{For } f \circ g, \ \forall \ f, \ g \in \{ \text{Borel functions} \} \quad f \circ g \in \{ \text{Borel sets} \}

\text{but } f \circ g, \ \forall \ f, \ g \in \{ \text{Lebesgue measurable functions} \}

f \circ g \text{ may not be a Lebesgue measurable function.}

\text{Consider } (f_n(x))_{n=1} \to \infty \text{ continuous}

\lim_{n \to \infty} f_n(x) = f(x) \text{ is continuous, NOT measurable}

\text{By: } f_n(x) = \begin{cases} 0, & x \in [0,1) \\ 1, & x = 1 \end{cases}

\text{Also, } f_n(x) = x^n \text{ as } x \to 0^+ 

f(x) = \begin{cases} 0, & x \in \mathbb{R} \setminus \{1\} \\ 1, & x = 1 \end{cases}

x \in [0,1]
Theorem. Given \((X, \mathcal{S}_X) \rightarrow (Y, \mathcal{S}_Y) \rightarrow (Z, \mathcal{S}_Z)\)

If \(f\) is \((\mathcal{S}_X, \mathcal{S}_Y)\) measurable, and \(g\) is \((\mathcal{S}_Y, \mathcal{S}_Z)\) measurable, then \(g \circ f\) is \((\mathcal{S}_X, \mathcal{S}_Z)\) measurable.

Proof: Let \(A \in \mathcal{S}_Z \Rightarrow g^{-1}(A) \in \mathcal{S}_Y \Rightarrow f^{-1}(g^{-1}(A)) \in \mathcal{S}_X \Rightarrow (g \circ f)^{-1}(A) \in \mathcal{S}_X\)

\(g \circ f\) is \((\mathcal{S}_X, \mathcal{S}_Z)\) measurable.

Corollary. Given a Borel fun. of \(\mu\)-measurable fun. (Lebesgue measurable) is measurable.

\((X, \mathcal{M}_X) \rightarrow \mathbb{R}^2 \rightarrow \mathbb{R}\)

Not True: Leb. meas. of Borel fun. is Leb. meas.

Construction:

Borel fun. \( \rightarrow \) Lebesgue measurable.

\(\mathbb{R} \rightarrow \mathbb{R} \rightarrow \mathbb{R}\)

Leb. meas. set.

How to check if an abstract fun. is continuous? Very difficult.

Is an abstract fun. measurable? Almost impossible.

Theorem. \(f\) is measurable if \(\forall c \in \mathbb{R}\), the set \(\{x \mid f(x) = c\}\) is measurable; i.e., \(\{x \mid f(x) \in (-\infty, c]\} = f^{-1}(\leq c)\).
Pf: "It is enough to check only preimages of rays."

\[ f \text{ is measurable} \implies f^{-1}(-\infty, c) \text{ is measurable. Since } (-\infty, c) \text{ is a Borel set.} \]

\[ f^{-1}(-\infty, c) \subseteq f^{-1}(A) \subseteq \sigma \text{-algebra, generated by } \{ f^{-1}(-\infty, c) \} \cup \{ x \in \mathbb{R} \} \]

\[ f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B) \]

\[ f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B) \]

\[ \Rightarrow \{ f^{-1}(\text{open sets}) \} \text{ is a } \sigma \text{-algebra.} \]

Ex: \( f \) is continuous. Does this imply \( f \) is measurable.

Ex: \( f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases} \)

Check: \[ c < 0 \implies f^{-1}(-\infty, c) = \emptyset \]

\[ c = 0 \implies f^{-1}(\emptyset) = \emptyset \text{ (meas.)} \]

\[ 0 < c < 1 \implies f^{-1}(\mathbb{Q}) = \mathbb{Q} \text{ (meas.)} \]

\[ c = 1 \implies f^{-1}(\mathbb{R}) = \mathbb{R} \text{ (meas.)} \]

\[ c > 1 \implies f^{-1}(\mathbb{R}) = \mathbb{R} \text{ (meas.)} \]
If $\lim_{n \to \infty} f_n(x)$ is continuous, then each $f_n(x)$ is

unif. cont. from pointwise cont.

from almost everywhere cont.

Bad guy: $\sup_n f_n(x) = f(x)$

Prove some arithmetic properties.

Theorem. Let $f, g$ be measurable.

Then $f + g, f/g, f^g, g^f, a f, a g$ are also measurable.

Proof. $\forall a, k \in \mathbb{R}, \ kf, \ kg, \ af, \ ag$ are measurable.

Claim $f + g$ is measurable.

Observe that $\{ x \mid f(x) > g(x) \} = \bigcup_{k=1}^{\infty} \{ x \mid f(x) > r_k \}$

$\{ r_k \}_{k=1}^{\infty}$ is a collection of rationals (countable)

\[
\Rightarrow \{ x \mid f(x) > g(x) \}
\]

So, $\{ x \mid f(x) + g(x) \} = \{ x \mid f(x) > a - g(x) \}$ is measurable.

Claim $f g$ is measurable.

If $f$ is measurable, $f^2$ is measurable.

\[
\begin{cases}
F(x) < C & \text{mean. measurable} \\
F(x) < \sqrt{C} & \text{mean. measurable}
\end{cases}
\]

\[
f g = \sqrt{f(g^2 - (g - g)^2)}
\]
Claim: \( \frac{1}{f} \) is measurable. \( f(x) \neq 0 \)

Case 1: \( c > 0 \)
\[
\{ x \mid \frac{1}{f(x)} < c \} = \left\{ x \mid f(x) > \frac{1}{c} \right\} \cup \left\{ x \mid f(x) < 0 \right\}
\]

Case 2: \( c < 0 \)
\[
\{ x \mid \frac{1}{f(x)} < c \} = \left\{ x \mid 0 < f(x) < -\frac{1}{c} \right\}
\]

Case 3: \( c = 0 \)
\[
\{ x \mid \frac{1}{f(x)} < c \} = \left\{ x \mid f(x) > 0 \right\}
\]
Consider $f : \mathbb{R} \to \mathbb{R}$ (could be $f : \mathbb{C} \to \mathbb{C}$ but this is real analysis, which means hard) think

If $f^{-1}(B)$ is a Borel set, then this is a Borel set.

A is not Borel set, but it is a Lebesgue measurable set.

$1_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \not\in A \end{cases}$ is not a Borel function.

But it is a Lebesgue function.

Not measurable. Take a set $C$, not measurable,

$1_C(x)$ is not measurable.
Theorem: \((f_n) \to f\) that is, \(m \to x.)

Then \(\lim_{n \to \infty} f_n = f(x)\) is also measurable pointwise convergence.

Aside: for a sequence of continuous functions, the limit may not be continuous. You need uniform continuity to ensure continuity.

Pf: The trick: \(\forall c \in \mathbb{R}\)

\[\{x : f(x) < c\} = \bigcup_{k \in \mathbb{N}} \bigcup_{n \in \mathbb{N}} \{x : f_n(x) < c - \frac{1}{k}\}\]

Important part: how to write down the limit. Check if \(\exists k \in \mathbb{N}\) so large that \(\forall m > n, f_m(x) < c - \frac{1}{k}\)

\(x \in \text{RHS}, \exists k \in \mathbb{N}, \exists m > n, f_m(x) < c - \frac{1}{k}\)

\(\Rightarrow f_m(x) \leq c \frac{1}{k} \Rightarrow f(x) < c\)

\(\Rightarrow x \in \text{LHS}\)

\(\Rightarrow \text{LHS} = \text{RHS}\)
Idea: Cut the tail

Let $E : m(E) < \infty$ and let $f_n(x) \xrightarrow{a.e.} f(x)$. Then, for $\delta > 0$, $\exists E_{\delta} \subseteq E$ s.t.

1. $m(E \setminus E_{\delta}) < \delta$
2. $f_n \xrightarrow{E_{\delta}} f$ uniformly continuous

For $\varepsilon > 0$, $\exists N = N(\varepsilon) \Rightarrow \forall i > N$, $\sup_{x \in E_{\delta}} |f(x_i) - f(x)| < \varepsilon$

Egorov's idea: We introduce two independent indices $m,n$ (m is large) and

$E_m^n : = \bigcap_{i \geq n} \{ x : |f_i(x) - f(x)| < \frac{1}{m} \}$

$m$ is free, $m > 0$

$E_m = \bigcup_{n=1}^{\infty} E_m^n \subseteq E$

Observation: $\forall m \quad m(E \setminus E_m) = 0$

$E_{\delta} = [0, 1-\delta]$
Since \( E_1^m \subseteq E_2^m \subseteq E_3^m \cdots \subseteq E_n^m \subseteq \cdots \)

\[ E^m = \bigcup_{n=1}^{\infty} E_n^m \]
which is approximately \( E_{n_0(m)}^m \)

\[ \text{very large} \]

so

\[ \mu(E^m \setminus E_{n_0(m)}^m) < \frac{\delta}{2^n} \]

by continuity of measure

Consider \( E_o^m = \bigcap_{n=1}^{\infty} E_n^m \):

\[ \bigcap_{n=1}^{\infty} \left( \bigcap_{i \in E_{n_0(m)}} \{ x : |f_i(x) - f_{i_0}(x)| < \frac{1}{n} \} \right) \]

Claim 1 \( f_i \xrightarrow{E_o^m} f \) \( \forall x \in E_o^m \)

uniformly (by \( x \) is "out of the game"

Claim 2 \( \mu(E \setminus E_o^m) \leq \epsilon \)

\[ \mu(E \setminus E_o^m) = \mu\left( (E \setminus E_m^m) \cup (E_{n_0(m)}^m \setminus E_o^m) \right) \leq \mu(E \setminus E_m^m) + \mu(E_{n_0(m)}^m \setminus E_o^m) \]

\[ = \mu\left( \bigcup_{n=1}^{\infty} E_n^m \right) = \mu\left( \bigcup_{n=1}^{\infty} (E_n^m \setminus E_{n_0(m)}^m) \right) \leq \sum_{n=1}^{\infty} \mu(E_n^m \setminus E_{n_0(m)}^m) < \sum_{n=1}^{\infty} \frac{\delta}{2^n} = \delta \]

He traded \( \bigcup_{n=1}^{\infty} E_n^m \) for \( \bigcup_{n=1}^{\infty} E_n^m \)