4 LU Factorization Applied to GPS Computations

When walking or driving in an unfamiliar place, we have all wondered where we are. This question easily can be answered with the aid of the Global Positioning System (GPS). A GPS system consists of GPS satellites orbiting the Earth and a GPS receiver on the Earth's surface. There are receivers in your cell phone and maybe in your car. The receiver obtains signals from the satellites and calculates its position by using the information received. Each GPS satellite continually transmits a signal at the speed of light containing the time the signal was sent and the location of the sending satellite. In order for a GPS receiver to be able to compute its position, it must *triangulate* its location from the satellites. This requires that the GPS receiver computes the distance from each satellite taking into account the travel time of the radio signals received. Since space has three dimensions, this would seem to imply that three satellites are required to compute the GPS receiver's location. However, this is only true if the GPS receiver knows exactly when the signals were sent and received, and this would require a very accurate (within nanoseconds) synchronized clock system between all the GPS satellites and the GPS receiver. It not practically realizable to achive so high accuracy for non-laboratory clocks, making a system with only three satellites impractical. However, when using four satellites instead of three, the GPS receiver can solve for time as well as position, eliminating the need for the GPS receiver to have a very accurate synchronized clock with the GPS satellites. The use of four satellites does not eliminate the need of a very accurate synchronized clock system between the satellites. Therefore, each GPS satellite is equipped with an atomic clock, which is synchronized with all other GPS satellites and universal time coordinated. Atomic clocks are the most accurate timing instrument known, incurring an error of at most 1 second every $30,000$ to $1,000,000$ years; they are accurate to within 10^{-9} seconds. The atomic clocks on the GPS satellites are periodically updated and synchronized from a control center on Earth. GPS time was set to zero at 0h 6-Jan-1980. At 1:02 p.m of September 16, 2008, the GPS time was 234,134s.

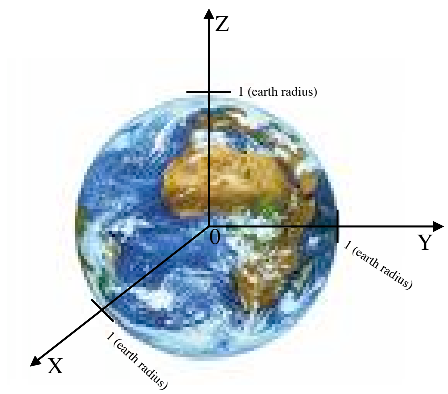
Since four GPS satellites are required in order for a GPS receiver to accurately calculate its position, the Satellite Navigation system is set up in such way that from any point on Earth at least four satellites are always "visible" by a GPS receiver.

Because the GPS receiver does not have a synchronized time clock with the satellites, the 4^{th} unknown variable in the system of equations is time, t. Ignoring atmospeheric conditions, the radio signals from the satellites travel to the receivers at the speed of light, c. Therefore, the distance to each satillites can be computed using the standard distance formula $d = c \cdot (t - t_{sent})$. Using the four satellites and the standard distance formula, we can set up a system of four equations with four unknowns. Solving this system for the unknowns yields the desired location of the GPS receiver.

[0] Version September 9, 2013

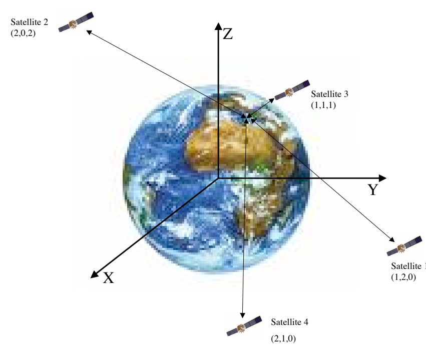
4.1 Determine the location of a GPS receiver on a boat at sea.

In order to simplify this GPS problem, we use the Cartesian xyz-coordinate system with the Earth centered at the origin, the positive z axis pointing through the north pole, and the unit of measure being the radius of the Earth. That is, each tick mark on an axis is equal to the radius of the Earth. We will also assume that any point at sea level satisfies $x^2 + y^2 + z^2 = 1$. Time will be measured in milliseconds (10^{-3} second) and the speed of light constant c is given by 0.047 Earth radii per millisecond.



The GPS receiver is on a boat and receives simultaneously from 4 satellites their position in relation to the xyz-coordinate system (units radius of Earth) and the time the signal was sent in milliseconds. The data is given in the table below. The numbers are fabricated so that they are easy to manage, but they not completely unrealistic.

Satellite	xyz-Position	Time signal was sent
1	(1,2,0)	19.9
2	(2,0,2)	2.4
3	(1,1,1)	32.6
4	(2,1,0)	19.9



Let (x, y, z) be the position of the GPS receiver and let t be the time the signal arrives. For Satellite 1, we have

$$d = 0.047 \cdot (t - 19.9),$$

and using the Euclidean distance formula for points in space gives

$$d = \sqrt{(x-1)^2 + (y-2)^2 + (z-0)^2}.$$

Combine the two equations above to get

$$\sqrt{(x-1)^2 + (y-2)^2 + (z-0)^2} = 0.047 \cdot (t - 19.9).$$

Squaring both the left-hand and right-hand sides gives

$$(x-1)^2 + (y-2)^2 + (z-0)^2 = 0.047^2 \cdot (t-19.9)^2,$$

and expanding all squares and rearranging the terms so that only linear variables are on the left-hand side yields

$$\text{(E1)} \quad 2x + 4y + 0z - 2(0.047)^2(19.9)t = 1^2 + 2^2 - 0.047^2(19.9)^2 + x^2 + y^2 + z^2 - 0.047^2t^2.$$

Exercise 4.1

Derive three equations (E2), (E3), (E4) that are analogous to (E1), for the other three satellites. Write all equations in the same format, i.e., all equations should only be linear in the left-hand side of the equal sign simialrly as in equation (E1). □

Exercise 4.2

Notice that all four equations have the same quadratic term on the right-hand side, $x^2 + y^2 + z^2 - 0.047^2t^2$. Use equation (E1) to eliminate the quadratic terms, hence creating a linear system of three equations (L1), (L2), and (L3). This can be done by subtracting (E1) from each equation, i.e., (L1) = (E2) - (E1), (L2) = (E3) - (E1), and (L3) = (E4) - (E1). □

Exercise 4.3

Write the linear equations (L1), (L2), and (L3) in augmented matrix form and solve for x, y, z, t. These are three equations with four unknowns. Let t be the free variable. □

Exercise 4.4

Substitute the variables x, y, z from Exercise 4.3 into (E1) and solve for t. You should end up with a quadratic term in t and have two time solutions. Determine which time solution is correct and report your chosen answer and the location of the GPS receiver in terms of the Cartesian coordinates (x, y, z). □

Your answer in Exercise 4.4 for the location of the GPS receiver is given with respect to the Cartesian xyz-coordinate system and would need to be converted into latitude and longitude in order to be useful. The xyz-coordinate system can be converted into ellipsoidal coordinates (ϕ, λ, h), where ϕ corresponds to the latitude, λ to longitude, and h to the ellipsoidal height. We will ignore h since we already know that the GPS receiver is at sea level. An ellipsoidal coordinates system is used since the Earth is not a perfect sphere. The size and shape of an ellipsoid can be defined by its semi-major axis a and its semi-minor axis b. The semi-major axis a is the distance from the center of the Earth to the equator and the semi-minor axis b is the distance from the center of the

Earth to the North pole. For our calculations we will use the Airy 1830 ellipsoid to represent the Earth. This gives $a = 6,377,563.396$m and $b = 6,356,256.910$m.

The longitude conversion is easy (remember the location of the quadrant of the angle),

$$\lambda = \arctan(\frac{y}{x})$$

However, the latitude conversion requires an iterative procedure,

$$\begin{aligned}
\phi_0 &= \arctan(\tfrac{z}{p(1-e)}), \\
v_0 &= \tfrac{a}{\sqrt{1-e\sin^2(\phi_0)}}, \\
\phi_i &= \arctan(\tfrac{z+ev_{i-1}\sin(\phi_{i-1})}{p}), \quad i = 1,2,3,\ldots \\
v_i &= \tfrac{a}{\sqrt{1-e\sin^2(\phi_i)}}, \quad i = 1,2,3,\ldots
\end{aligned}$$

where $e = \frac{a^2-b^2}{a^2} = 6.6705397616 \cdot 10^{-3}$, and $p = \sqrt{x^2+y^2}$.

Exercise 4.5

Convert your xyz-coordinate answer first into meters (using the fact that the radius of Earth is $6,377,563.396$ meters) and then into latitude and longitude. Use 3 iterations for the latitude value. □

These calculations give you some insight into the foundation of a GPS system. However, this is not exactly how a GPS system works. There are many missing pieces. For instance, the speed of the radio signal is not constant due to atmospheric conditions, the satellite positions are only known to an accuracy of 1 to 3 meters, and the times the signals are sent have errors. If an atomic clock has an error of only 10 nanoseconds (10^{-8} seconds), then this will create a positioning error of about 3 meters. Current GPS receivers use more accurate methamatical models than the one we described to reduce the errors in the computed positions. More information on this and related topics can be found in the book “Linear Algebra, Geodesy, and GPS” by G. Strang and K. Borre, Wellesley-Cambridge Press, Wellesley, 1997.