the numerical results, the simulated asset prices are only shown for \( j = 1, \ldots, 5 \) and \( j = 95, \ldots, 100 \). Note that in this example there is only one time step \( (N = 1) \) because we only need to simulate asset prices at the maturity date of the option.

Firstly, the constants: \( \Delta t (dt), \nu \Delta t (nudt), \sigma \sqrt{\Delta t} (sigstdt) \) and \( \ln(S)(\ln S) \) are precomputed:

\[
\Delta t = \frac{T}{N} = \frac{1}{1} = 1
\]

\[
\nu \Delta t = (\nu - \delta - \frac{1}{2}\sigma^2) \Delta t = (0.06 - 0.03 - 0.5 \times 0.2^2) \times 1 = 0.01
\]

\[
\sigma \sqrt{\Delta t} = 0.2 \sqrt{1} = 0.2
\]

\[
\ln S = \ln(4.6052)
\]

Then for each simulation \( j = 1 \) to \( M \), where \( M = 100 \), \( \ln(S_{1,i}) \) and \( \ln(S_{2,i}) \) are initialised to \( \ln(S) = 4.6052 \). Then \( \ln(S_{1,i}) \) and \( \ln(S_{2,i}) \) are simulated, for example for \( j = 1 \) and \( i = 1 \):

\[
\ln(S_{1,1}) = 4.6052 + 0.010 + 0.2 \times (-0.8265) = 4.4499
\]

\[
\ln(S_{2,1}) = 4.6052 + 0.010 + 0.2 \times (0.8265) = 4.7805
\]

\[
S_{1,T} = \exp(4.4499) = 85.62
\]

\[
S_{2,T} = \exp(4.7805) = 119.16
\]

Computing the pay-off at maturity gives:

\[
C_T = 0.5 \times (\max(0, 85.62 - 100) + \max(0, 119.16 - 100)) = 9.5807
\]

The sum of the values of \( C_T \) and the squares of the values of \( C_T \) are accumulated:

\[
\sum_{j=1}^{M} C_{T,j} = 1140.37 \quad \text{and} \quad \sum_{j=1}^{M} (C_{T,j})^2 = 20790.8
\]

The estimate of the option value \( \hat{C}_0 \) (call.value) is then given by

\[
\hat{C}_0 = \frac{1140.366}{100} \times \exp(-0.06 \times 1) = 10.7396
\]

This technique can be easily applied to virtually any Monte Carlo simulation to improve the efficiency. In the next section we describe more advanced variance reduction methods based on the hedging analogy.

### 4.4 CONTROL VARIATES AND HEDGING

The general approach of using hedges as control variates was first described by Clewlow and Carverhill (1994).

Consider the case of writing a European call option. Figure 4.4 illustrated the pay-off and its probability distribution. The distribution of this pay-off has a large standard deviation, and so if we try and estimate the call value as the mean of a number of Monte Carlo simulations then the standard error of the mean will be large.
FIGURE 4.7 Probability Distribution of the Pay-off Written Call Option after Delta Hedging

Consider the effect of delta hedging the call option. Figure 4.7 illustrates the probability distribution of the pay-off after delta hedging.

The pay-off of the hedged portfolio has a much smaller standard deviation, this is of course the whole point of the delta hedge. Let us consider the mechanics of a discretely rebalanced delta hedge in detail. The delta hedge consists of a holding of \( \partial C / \partial S \) in the asset which is rebalanced at discrete intervals, \( t_i, i = 0, \ldots, N \). The changes in the value of the hedge as the asset price changes randomly offset the changes in the option value. Because the hedge is rebalanced at discrete time intervals it is not perfect, but for reasonably frequent rebalancing we expect it to be very good. The hedging procedure consists of selling the option, putting the premium in the bank and rebalancing the holding in the asset at discrete intervals with resultant cash flows into and out of the bank account. At the maturity date the hedge, consisting of the cash account plus the asset, closely replicates the pay-off of the option. We can express this mathematically as follows:

\[
C_T e^{r(T-t)} = \left[ \sum_{i=0}^{N} \left( \frac{\partial C_{t_i}}{\partial S} - \frac{\partial C_{t_{i-1}}}{\partial S} \right) S_{t_i} e^{r(T-t_i)} \right] = C_T + \eta
\]

(4.15)

where \( \partial C_{t-1} / \partial S = 0 \). The first term in equation (4.15) is the premium received for writing the option, inflated at the riskless rate to the maturity date, the second term represents the cash flows from rebalancing the hedge at each date \( t_i \) and the third term is the pay-off of the option \( C_T \) and the hedging error \( \eta \). The expression in square brackets is the delta.
hedge. Expanding the summation term in the square brackets in equation (4.15) gives
\[
\frac{\partial C_{t_0}}{\partial S} S_{t_0} e^{r(T-t_0)} + \frac{\partial C_{t_1}}{\partial S} S_{t_1} e^{r(T-t_1)} + \ldots + \frac{\partial C_{t_{N-1}}}{\partial S} S_{t_{N-1}} e^{r(T-t_{N-1})} + \frac{\partial C_{t_N}}{\partial S} S_{t_N}
\]

\[- \frac{\partial C_{t_0}}{\partial S} S_{t_1} e^{r(T-t_1)} - \frac{\partial C_{t_1}}{\partial S} S_{t_2} e^{r(T-t_2)} - \ldots - \frac{\partial C_{t_{N-1}}}{\partial S} S_{t_N} \]  

(4.16).

Rewriting equation (4.16) grouping terms with \( \partial C_i / \partial S \) at the same time step:
\[
- \frac{\partial C_{t_0}}{\partial S} (S_{t_1} - S_{t_0} e^{r\Delta t}) e^{r(T-t_1)} - \frac{\partial C_{t_1}}{\partial S} (S_{t_2} - S_{t_1} e^{r\Delta t}) e^{r(T-t_2)} \ldots
\]

\[- \frac{\partial C_{t_{N-1}}}{\partial S} (S_{t_N} - S_{t_{N-1}} e^{r\Delta t}) + \frac{\partial C_{t_N}}{\partial S} S_{t_N} \]  

(4.17).

If we assume that the final term in (4.17) is zero, which corresponds to not buying the final delta amount of the asset, but simply liquidating the holding from the previous rebalancing date into cash, then the hedged portfolio becomes
\[
C_{t_0} e^{r(T-t_0)} + \sum_{i=0}^{N-1} \frac{\partial C_{t_i}}{\partial S} (S_{t_{i+1}} - S_{t_i} e^{r\Delta t}) e^{r(T-t_{i+1})} = C_T + \eta \]  

(4.18).

The expression in square brackets, which is the delta hedge, we call a delta-based martingale control variate (cv1). This can be seen by writing it as follows:
\[
\text{cv1} = \sum_{i=0}^{N-1} \frac{\partial C_{t_i}}{\partial S} (S_{t_{i+1}} - S_{t_i} e^{r\Delta t}) e^{r(T-t_{i+1})}
\]  

(4.19).

Thus, the expectation or mean of cv1 will be zero. Rearranging equation (4.18) we have
\[
C_{t_0} e^{r(T-t_0)} = C_T - \left[ \sum_{i=0}^{N-1} \frac{\partial C_{t_i}}{\partial S} (S_{t_{i+1}} - E[S_{t_{i+1}}]) e^{r(T-t_{i+1})} \right] + \eta \]  

(4.20).

and we can interpret equation (4.20) as saying that the expectation of the pay-off plus the hedge is equal to the initial premium inflated to the maturity date at the riskless rate of interest. Therefore if we simulate the pay-off and the hedge and compute the mean of these we will obtain an estimate of the option value but with a much smaller variance. This can be more easily visualised by plotting the pay-off against the control variate cv1 originating from a Monte Carlo simulation. This is done in Figure 4.8 for European call option with parameter values: \( K = 100, T = 1.0, S = 100, \sigma = 0.2, r = 0.06, \delta = 0.03 \). The Monte Carlo simulation has been repeated for a number of different rebalancing intervals (or Monte Carlo time steps) corresponding to monthly, weekly and daily time steps \( (N = 12, N = 52, N = 250) \) with \( M = 1000 \).

Figure 4.8 shows that the combination of the initial premium inflated at the riskless rate of interest plus the control variate cv1 which is the cash accumulated by the delta hedging process is approximately equal to the pay-off of the option. The hedge gets better as the time step is decreased or equivalently as the hedge is rebalanced more often. Therefore, with this method the key to obtaining accurate prices is to have small time steps rather than a large number of simulations. In the terminology of Monte Carlo simulation, cv1 is
FIGURE 4.8 Black–Scholes Monte Carlo Simulation with Delta-based Control Variate

$N = 12$

$N = 50$

$N = 250$
a control variate, a random variable, whose expected value we know, which is correlated with the variable we are trying to estimate (in our case the option value). In this case the known mean of \( cv_1 \) is zero. In the same way as for \( cv_1 \) we can construct other control variates equivalent to other hedges. For example, a \textit{gamma} hedge

\[
\begin{align*}
    cv_2 &= \sum_{i=0}^{N-1} \frac{\partial^2 C}{\partial S^2} ((\Delta S_i)^2) - E[(\Delta S_i)^2] e^{(T-t_0)} \\
    &\quad \text{where } E[(\Delta S_i)^2] = \frac{S^2_0}{\tau} (e^{(2\tau + \tau^2)} - 2e^{\tau} + 1).
\end{align*}
\]  

For the general case of a European option paying off \( C_T \) at time \( T \), setting \( t_0 = 0 \) and with \( m \) control variates, equation (4.20) becomes

\[
C_0 e^{rT} = C_T - \sum_{k=1}^{m} \beta_k cv_k + \eta
\]  

where the \( \beta \) factors are included to account for the sign of the hedge, for errors in the hedges due to the discrete rebalancing and only having approximate hedge sensitivities (i.e. \textit{delta}, \textit{gamma}, etc.). This is important for the practical implementation of this method. In reality we will be using Monte Carlo to value an option for which we do not have an analytical expression. Therefore we will not have analytical expressions for the hedge sensitivities; however, we are quite likely to have an analytical formula for a similar option. For example we might be valuing a path-dependent option which is similar to a lookback option where we have analytical expressions for continuously fixed lookback options under the Black–Scholes assumptions. Therefore we can use the hedge sensitivities from the analytical lookback formula in the control variates to value the more complex option. We describe this example in detail in the next section.

We rewrite equation (4.22) as follows:

\[
C_T = \beta_0 + \sum_{k=1}^{m} \beta_k cv_k + \eta
\]  

where \( \beta_0 = C_0 e^{rT} \) is the forward price of the option. We can interpret equation (4.23) as a linear equation relating the pay-off of the option to the control variates via the \( \beta \) coefficients. If we perform \( M \) simulations we can regard the pay-offs and control variates \( (C_T, cv_{1,j}, \ldots, cv_{m,j}; j = 1, \ldots, M) \) as samples from this linear relationship with noise. The noise comes from the discrete rebalancing and the imperfect sensitivities. We can then obtain an estimate of the "true" relationship by least-squares regression. The least-squares estimate of the \( \beta \) is

\[
\beta = (X'X)^{-1}X'Y
\]  

where \( \beta = (\beta_0, \beta_1, \ldots, \beta_m) \), \( X \) is a matrix whose rows correspond to each simulation and are \( (1, cv_{1,j}, \ldots, cv_{m,j}) \) and \( Y \) is the vector of simulated pay-offs (the "dash" denotes transpose). The matrices \( X'X \) and \( X'Y \) can be accumulated as the simulation proceeds as follows:

\[
\begin{align*}
    (X'X)_{k+1,j} &= (X'X)_{k,j} + cv_{k+1,j}cv_{k+1,j} \\
    (X'Y)_{k+1,j} &= (X'Y)_{k,j} + cv_{k+1,j}C_{T,j+1}
\end{align*}
\]
where $k$ and $l$ index the rows and columns of the matrix and $j$ is the time step as usual. It is important to note that since the pay-offs and control variates are not jointly normally distributed then the estimate of $\beta$ will be biased. This is particularly important for the forward value of the option $\beta_0$, as we do not want biased estimates of the option value. This problem is easily overcome by precomputing the $\beta_k; k = 1, \ldots, m$ by the least-squares regression method or fixing them at some appropriate value for the type of hedge. All options can then be priced, keeping the $\beta$ fixed, by simply taking the mean of the hedged portfolio under a different set of simulated paths. This is our recommended method for implementing this technique.

There is one other subtle but important aspect of the hedge control variate idea. If we form a control variate delta hedge for a variable which in the analytical model is stochastic and which in the simulation is following the same process as in the model, then the control variate hedge is simply replicating the model option. In this case it is much more efficient to form a static hedge portfolio which is long the option we want to price and short the analytical model option. We then value the difference between the two options using the Monte Carlo simulation which has much smaller variance than the option we want to price; we use this idea in section 4.9 to price an Asian option. However, a control variate hedge for a variable which is a constant parameter in the analytical model, but which is stochastic in the simulation, cannot be simplified. The analytical model does not price any possible pay-offs due to randomness of this variable. However, using the sensitivity from the analytical model in the simulation will approximately hedge the risk and therefore help reduce the variability of the Monte Carlo estimate. We use this method in section 4.10 to price a lookback option under stochastic volatility.

### 4.5 Monte Carlo Simulation with Control Variates

In this section we illustrate the use of control variates with a series of examples based on a European call option. For our first example we consider the Monte Carlo valuation of a European call option with a delta-based control variate. Figure 4.9 gives the pseudo-code algorithm.

The code which has been added from the simple Monte Carlo example in Figure 4.2 is highlighted in bold. Notice also that the method of simulating the asset price is slightly different. Since we need the asset price at each time step we simulate this directly rather than its natural logarithm. The variable `erddt` allows us to compute $E(S_t)$ in equation (4.19) efficiently. We set $\text{beta} = -1$ which is the appropriate value for this example where we have the exact delta. The `delta` variable is computed as the Black–Scholes delta, by the function `Black.Scholes.delta` ($S_t, t, K, T, \sigma, r, \delta$), at the start of the time step, i.e. before the asset price has been evolved and the control variate is then accumulated after the asset price has been evolved. The pay-off of the hedged portfolio is computed after the end of the time step loop and at the end the mean and standard error of this are computed. Since the mean of the control variate is zero, the mean gives us an estimate of the option price, but the standard error is greatly reduced by the control variate hedge. Figure 4.10 gives a numerical example.
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... and $j$ is the time step as all control variates are not jointly d. This is particularly important if biased estimates of the option is the $\beta_k; k = 1, \ldots, m$ by the appropriate value for the type of xed, by simply taking the mean paths. This is our recommended

hedge control variate idea. If we the analytical model is stochastic this case is much more efficient ion we want to price and short the two operations using variance than the option we want sian option. However, a control factor in the analytical model, but fixed. The analytical model does not variable. However, using the will approximately hedge the risk into estimate. We use this method stic volatility.

CONTROLS

ith a series of examples based on the Monte Carlo valuation of a Figure 4.9 gives the pseudo-code

Monte Carlo example in Figure 4.2 of simulating the asset price is each time step we simulate this rddt allows us to compute $E[S_t]$ which is the appropriate value for theta variable is computed as the ta $(S_t, t, K, T, \sigma, \delta)$, at the start and the control variate is then paid. The payoff of the hedged portfolio s: the end the mean and standard variate is zero, the mean gives is greatly reduced by the control

FIGURE 4.9 Pseudo-code for Monte Carlo Valuation of a European Call Option in a Black–Scholes World with a Delta-based Control Variate

```plaintext
initialise parameters (K, T, S, sig, r, div, N, M)

{ precompute constants }

dt = T/N
nudt = (r-div-0.5*sig^2)*dt
sigadt = sig*sqrt(dt)
erddt = exp((r-div)*dt)

beta1 = -1

sum_CT = 0
sum_CT2 = 0

for j = 1 to M do { for each simulation }

St = S

for i = 1 to N do { for each time step }

t = (i-1)*dt

delta = BlackScholes.delta(St, t, K, T, sig, r, div)
s = standard.normal.sample
Stn = St*exp( nudt + sigadt*sqrt(dt) )
cv = cv + delta*(Stn-St*erddt)
St = Stn
next i

CT = max( 0, St - K ) + beta1*cv

sum_CT = sum_CT + CT
sum_CT2 = sum_CT2 + CT*CT

next j

call.value = sum_CT/M*exp(-r*T)
SD = sqrt( 1/sum_CT2 - sum_CT^2/sum_CT/M )*exp(-2*r*T)/(N-1)
SE = SD/sqrt(M)
```

Example: Pricing a European Call Option by Monte Carlo Simulation with a Delta-based Control Variate

We price a one-year maturity, at-the-money European call option with the current asset price at 100 and volatility of 20 per cent. The continuously compounded interest rate is assumed to be 6 per cent per annum, the asset pays a continuous dividend yield of 3 per cent per annum. The simulation has 10 time steps and 100 simulations; $K = 100, T = 1$
year, $S = 100$, $\sigma = 0.2$, $r = 0.06$, $\delta = 0.03$, $N = 10$, $M = 100$. Figure 4.10 illustrates the numerical results for the simulation of the path for $j = 100$.

Firstly, the constants; $\Delta t$ ($dt$), $\nu\Delta t$ ($nu dt$), $\sigma\sqrt{\Delta t}$ ($sigstdt$), $\exp((r - \delta)\Delta t)$ ($erddt$) and $\beta_1$ ($beta1$) are precomputed:

$$\Delta t = \frac{T}{N} = \frac{1}{10} = 0.1$$

$$nu dt = (r - \delta - \frac{1}{2}\sigma^2) \Delta t = (0.06 - 0.03 - 0.5 \times 0.2^2) \times 0.1 = 0.001$$

$$sigstdt = \sigma\sqrt{\Delta t} = 0.2\sqrt{0.1} = 0.0632$$

$$erddt = \exp(-(r - \delta)\Delta t) = \exp(-(0.06 - 0.03) \times 0.1) = 1.0030$$

$$lnS = \ln(S) = 4.6052$$

$$\beta_1 = -1$$

Then for each simulation $j = 1$ to $M$, where $M = 100$, $S_t$ is initialised to $S = 100$ and the control variate $cv = 0$. Then for each time step $i = 1$ to $N$, where $N = 10$, delta is computed, $S_i$ is simulated and the control variate $cv$ is accumulated. For example for $j = 100$ and $i = 0$, $delta = 0.58101$, this is the delta which is used at $i = 1$ for accumulating the control variate. At $i = 1$ we have:

$$S_t = S \times \exp(nu dt + sigstdt \times \varepsilon)$$

$$= 100 \times \exp(0.0010 + 0.06325 \times 0.37656) = 102.513$$

$$cv = cv + delta \times (S_t - S \times erddt) = 0 + 0.5810 \times (102.513 - 100 \times 1.0030) = 1.2853$$

After the $i$ loop we have

$$C_T = \max(0, S_T - K) + \beta_1 \times cv$$

$$= \max(0, 100.49 - 100) + (-1) \times (-6.5133) = 7.0029$$

The sum of the values of $C_T$ and the squares of the values of $C_T$ are accumulated in sum_CT and sum_CT2, giving sum_CT = 963.128 and sum_CT2 = 9670.3. The estimate of the option value is then given by

$$\hat{C}_0 = \text{sum}_\text{CT}/M \times \exp(-r \times T) = 963.128/100 \times \exp(-0.06 \times 1) = 9.0704$$

It is straightforward to combine the antithetic and control variate methods, we simply accumulate control variates for the standard and antithetic asset paths. Figure 4.11 illustrates the pseudo-code for combining antithetics with delta-based control variates for a European call option. The lines which have been added from Figure 4.9 are highlighted in bold. Note that in the calculation of the pay-off of the hedged portfolio (CT) both cv1 and cv2 are multiplied by beta1 and then added together. Therefore we do not need separate variables for the standard and antithetic control variates, they could both be accumulated in cv1. This method is used in the final example.

**Example : Pricing a European Call Option by Monte Carlo Simulation with Antithetic and Delta-based Control Variates**

We price a one-year maturity, at-the-money European call option with the current asset price at 100 and volatility of 20 per cent. The continuously compounded interest rate is
FIGURE 4.11  Pseudo-code for Monte Carlo Valuation of a European Call Option in a Black–Scholes World with Antithetic and Delta-based Control Variates

initialise parameters \[ K, T, S, \sigma, r, \text{div}, N, M \]

\[
\begin{align*}
\text{precompute constants } & \\
\Delta t & = T/N \\

\nu & = (r-\text{div}-0.5\sigma^2) \Delta t \\

\sigma \Delta t & = \sigma \sqrt{\Delta t} \\

e^{\sigma \Delta t} & = \exp((r-\text{div}) \Delta t) \\

\beta & = -1 \\

\text{sum.CT} & = 0 \\

\text{sum.CT}^2 & = 0 \\

\text{for } j = 1 \text{ to } M \text{ do } \{ \text{for each simulation} \} \\

S_1 & = S \\
S_2 & = S \\

C_{v1} & = 0 \\

C_{v2} & = 0 \\

\text{for } i = 1 \text{ to } N \text{ do } \{ \text{for each time step} \} \\

\tau & = (i-1) \Delta t \\

\Delta \tau_1 & = \text{Black}\_\text{Scholes}\_\text{delta}(S_1, t; K, T, \sigma, r, \text{div}) \\

\Delta \tau_2 & = \text{Black}\_\text{Scholes}\_\text{delta}(S_2, t; K, T, \sigma, r, \text{div}) \\

\epsilon & = \text{standard}\_\text{normal}\_\text{sample} \\

St_1 & = S_1 \exp(\nu \Delta t + \sigma \Delta t \epsilon) \\

St_2 & = S_2 \exp(\nu \Delta t + \sigma \Delta t (-\epsilon)) \\

C_{v1} & = C_{v1} + \Delta \tau_1 (St_1 - St_1 \epsilon) \\

C_{v2} & = C_{v2} + \Delta \tau_2 (St_2 - St_2 \epsilon) \\

St_1 & = St_1 \\

St_2 & = St_2 \\

\text{next } i \\

\text{CT} & = 0.5 \times (\max(0, St_1 - K) + \beta \cdot C_{v1} + \\

& \max(0, St_2 - K) + \beta \cdot C_{v2}) \\

\text{sum.CT} & = \text{sum.CT} + \text{CT} \\

\text{sum.CT}^2 & = \text{sum.CT}^2 + \text{CT} \cdot \text{CT} \\

\text{next } j \\

\text{call.Value} & = \text{sum.CT}/M \exp(-r T) \\

\text{SD} & = \sqrt{1 / (\text{sum.CT}^2 - \text{sum.CT} \cdot \text{sum.CT}/M) \times \exp(-2 r T)/(M-1)} \\

\text{SE} & = \text{SD}/\sqrt{M}
\end{align*}
\]

assumed to be 6 per cent per annum, the asset pays a continuous dividend yield of 3 per cent per annum. The simulation has 10 time steps and 100 simulations: \[ K = 100, T = 1 \text{ year, } S = 100, \sigma = 0.2, r = 0.06, \delta = 0.03, N = 10, M = 100. \] Figure 4.12 illustrates the numerical results for the simulation of the path for \[ j = 100. \]
FIGURE 4.12 Monte Carlo Valuation of a European Call Option in a Black–Scholes World with Antithetic and Delta-based Control Variates

<table>
<thead>
<tr>
<th>K</th>
<th>T</th>
<th>S</th>
<th>sig</th>
<th>r</th>
<th>div</th>
<th>N</th>
<th>M</th>
<th>sig</th>
<th>r</th>
<th>div</th>
<th>N</th>
<th>M</th>
<th>sum_CT</th>
<th>sum_CT2</th>
<th>SD</th>
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</thead>
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<tr>
<td>100</td>
<td>1</td>
<td>100</td>
<td>0.2</td>
<td>0.06</td>
<td>0.03</td>
<td>10</td>
<td>100</td>
<td>0.2</td>
<td>0.06</td>
<td>0.03</td>
<td>10</td>
<td>100</td>
<td>962.75</td>
<td>9597.3</td>
<td>1.7153</td>
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</tbody>
</table>

<table>
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<tr>
<th>dt</th>
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<th>slsgdt</th>
<th>erddt</th>
<th>beta1</th>
<th>call_value</th>
<th>SE</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.0010</td>
<td>0.0632</td>
<td>1.0000</td>
<td>-1</td>
<td>9.0669</td>
<td>0.1715</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>j = 100</th>
</tr>
</thead>
<tbody>
<tr>
<td>i 0 1 2 3 4 5 5 7 8 9 10</td>
</tr>
<tr>
<td>e 0 0.7987 0.0516 0.4174 -0.2495 0.2289 -0.2336 -0.2831 -1.5925 -0.5560 0.0549</td>
</tr>
<tr>
<td>St1 100 105.29 105.74 108.67 107.08 108.75 106.92 105.13 95.21 92.01 92.43</td>
</tr>
<tr>
<td>St2 100 95.17 94.95 92.57 94.14 92.86 94.66 96.46 106.72 110.65 110.38</td>
</tr>
<tr>
<td>delta1 0.581012 0.6760 0.6871 0.7442 0.7242 0.7677 0.7450 0.7171 0.7921 0.1077 0.0000</td>
</tr>
<tr>
<td>delta2 0.581012 0.4774 0.4825 0.3922 0.4147 0.3504 0.3866 0.4201 0.7946 0.9506 1.0000</td>
</tr>
</tbody>
</table>

European Call Option in a Black–Scholes World with Antithetic and Delta-based Control Variates
The calculations are identical for \{delta1, Stn1, cv1\} and \{delta2, Stn2, cv2\} as for \{delta, Stn, cv\} in Figure 4.10. The only other difference is the computation of the pay-off:

\[
C_T = 0.5 \times (\max(0, S_{1,T} - K) + \beta_1 \times cv_1 + \max(0, S_{2,T} - K) + \beta_1 \times cv_2)
\]

The final example using a European call option combines antithetic, delta- and gamma-based control variates, the pseudo-code appearing in Figure 4.13.

**FIGURE 4.13** Pseudo-code for Monte Carlo Valuation of a European Call Option in a Black–Scholes World with Antithetic, Delta- and Gamma-based Control Variates

```plaintext
initialise parameters [ K, T, S, r, div, N, M ]

| precompute constants |

nu = (r - div - 0.5*sig^2)*dt
sig = sig*sqrt(dt)
err = exp((r - div)*dt)
egamma = exp((2*(r - div) + sig^2)*dt) - 2*err + 1
beta1 = -1
beta2 = -0.5

sum_cT = 0
sum_cT2 = 0

for j = 1 to M do { for each simulation }

    St1 = S
    St2 = S
    cv1 = 0
    cv2 = 0

    for i = 1 to N do { for each time step }

        | compute hedge sensitivities |
        t = (i-1)*dt
        delta1 = BlackScholes.delta(St1, t, K, r, div)
        delta2 = BlackScholes.delta(St2, t, K, r, div)
        gamma1 = BlackScholes.gamma(St1, t, K, r, div)
        gamma2 = BlackScholes.gamma(St2, t, K, r, div)

        | evolves asset prices |
        s = standard.normal.sample
        Stn1 = St1*exp( nu*t + sig*sqrt(t)*s )
        Stn2 = St2*exp( nu*t + sig*sqrt(t)*(-s) )

        | accumulate control variates |
        cv1 = cv1 + delta1*(Stn1-St1*err) + delta2*(Stn2-St2*err)
        cv2 = cv2 + gamma1*( (Stn1-St1)^2 - St1^2*egamma ) + gamma2*( (Stn2-St2)^2 - St2^2*egamma )

        St1 = Stn1
        St2 = Stn2

next i
```
Example: Pricing a European Call Option by Monte Carlo Simulation with Antithetic, Delta- and Gamma-based Control Variates

We price a one-year maturity, at-the-money European call option with the current asset price at 100 and volatility of 20 per cent. The continuously compounded interest rate is assumed to be 6 per cent per annum, the asset pays a continuous dividend yield of 3 per cent per annum. The simulation has 10 time steps and 100 simulations; $K = 100$, $T = 1$ year, $S = 100$, $\sigma = 0.2$, $r = 0.06$, $\delta = 0.03$, $N = 10$, $M = 100$. Figure 4.14 illustrates the numerical results for the simulation of the path for $j = 100$. The calculations are very similar to the previous examples (compare also the pseudo-code implementations).

Table 4.1 illustrates the typical standard errors and computation times which can be achieved for example by the application of antithetic, delta- and gamma-based control variates to the valuation of a standard European call option in a Black-Scholes world.

| TABLE 4.1 Typical Standard Errors and Relative Computation Times for the Monte Carlo Valuation of a European Call Option in a Black-Scholes World with Antithetic, Delta- and Gamma-based Control Variates |
|---------------------------------|-----------------|-----------------|
|                                | Standard error  | Relative computation time |
| Strike price                   | 100             |                               |
| Time to maturity               | 1 year          |                               |
| Initial asset price            | 100             |                               |
| Volatility                     | 20%             |                               |
| Riskless interest rate         | 6%              |                               |
| Continuous dividend yield      | 3%              |                               |
| Number of time steps           | 52              |                               |
| Number of simulations          | 1000            |                               |
| Standard European call value   | 9.1352          |                               |
| Simple estimate                | 0.4348          | 1.00                          |
| With antithetic variate        | 0.2353          | 1.29                          |
| With control variates          | 0.0072          | 3.64                          |
| Combined variates              | 0.0048          | 6.43                          |
### FIGURE 4.14 Monte Carlo Valuation of a European Call Option in a Black–Scholes World with Delta- and Gamma-based Control Variates

<table>
<thead>
<tr>
<th>( K )</th>
<th>( T )</th>
<th>( S )</th>
<th>( \sigma )</th>
<th>( r )</th>
<th>( \text{div} )</th>
<th>( N )</th>
<th>( M )</th>
<th>\text{sum}_{\text{CT}}</th>
<th>\text{sum}_{\text{CT}2}</th>
<th>\text{SD}</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>1</td>
<td>100</td>
<td>0.2</td>
<td>0.06</td>
<td>0.03</td>
<td>10</td>
<td>100</td>
<td>981.87</td>
<td>9662.2</td>
<td>0.4390</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( d_t )</th>
<th>( n_{d_t} )</th>
<th>( \sigma_{d_t} )</th>
<th>( \text{er}_{d_t} )</th>
<th>( \text{egamma} )</th>
<th>( \beta_1 )</th>
<th>( \beta_2 )</th>
<th>\text{call_value}</th>
<th>\text{SE}</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.0010</td>
<td>0.063246</td>
<td>1.0030</td>
<td>0.004041</td>
<td>-1</td>
<td>-0.5</td>
<td>9.2499</td>
<td>0.043896</td>
</tr>
</tbody>
</table>

\( j = 100 \)

<table>
<thead>
<tr>
<th>( i )</th>
<th>( j )</th>
<th>( \varepsilon )</th>
<th>( \text{St}_1 )</th>
<th>( \text{St}_2 )</th>
<th>( \text{delta}_1 )</th>
<th>( \text{delta}_2 )</th>
<th>( \text{cv}_1 )</th>
<th>( \text{gamma}_1 )</th>
<th>( \text{gamma}_2 )</th>
<th>( \text{cv}_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>-0.0344</td>
<td>-1.4005</td>
<td>-1.0658</td>
<td>0.4191</td>
<td>-0.9075</td>
<td>2.9757</td>
<td>0.0462</td>
<td>-0.5343</td>
<td>-1.3023</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0.1</td>
<td>0.2</td>
<td>0.3</td>
<td>0.4</td>
<td>0.5</td>
<td>0.6</td>
<td>0.7</td>
<td>0.8</td>
<td>0.9</td>
</tr>
<tr>
<td>( \varepsilon )</td>
<td>0</td>
<td>-0.0344</td>
<td>-1.4005</td>
<td>-1.0658</td>
<td>0.4191</td>
<td>-0.9075</td>
<td>2.9757</td>
<td>0.0462</td>
<td>-0.5343</td>
<td>-1.3023</td>
</tr>
<tr>
<td>( \text{St}_1 )</td>
<td>100</td>
<td>99.50</td>
<td>91.16</td>
<td>85.30</td>
<td>87.68</td>
<td>82.97</td>
<td>100.14</td>
<td>100.53</td>
<td>97.29</td>
<td>89.68</td>
</tr>
<tr>
<td>( \text{St}_2 )</td>
<td>100</td>
<td>100.70</td>
<td>110.14</td>
<td>117.93</td>
<td>114.96</td>
<td>121.88</td>
<td>101.07</td>
<td>100.88</td>
<td>194.45</td>
<td>113.53</td>
</tr>
<tr>
<td>( \text{delta}_1 )</td>
<td>0.591012</td>
<td>0.5690</td>
<td>0.3754</td>
<td>0.2246</td>
<td>0.2517</td>
<td>0.1229</td>
<td>0.5603</td>
<td>0.5683</td>
<td>0.4198</td>
<td>0.0501</td>
</tr>
<tr>
<td>( \text{delta}_2 )</td>
<td>0.591012</td>
<td>0.5917</td>
<td>0.7599</td>
<td>0.8656</td>
<td>0.8476</td>
<td>0.9284</td>
<td>0.5866</td>
<td>0.5805</td>
<td>0.7208</td>
<td>0.9785</td>
</tr>
<tr>
<td>( \text{cv}_1 )</td>
<td>0</td>
<td>-0.2308</td>
<td>0.2653</td>
<td>3.6309</td>
<td>1.2304</td>
<td>5.5206</td>
<td>-12.0463</td>
<td>-12.2374</td>
<td>-12.4048</td>
<td>-9.3998</td>
</tr>
<tr>
<td>( \text{gamma}_1 )</td>
<td>0.018762</td>
<td>0.0201</td>
<td>0.0229</td>
<td>0.0208</td>
<td>0.0233</td>
<td>0.0173</td>
<td>0.0307</td>
<td>0.0353</td>
<td>0.0447</td>
<td>0.0182</td>
</tr>
<tr>
<td>( \text{gamma}_2 )</td>
<td>0.018762</td>
<td>0.0196</td>
<td>0.0148</td>
<td>0.0097</td>
<td>0.0121</td>
<td>0.0066</td>
<td>0.0299</td>
<td>0.0349</td>
<td>0.0355</td>
<td>0.0063</td>
</tr>
<tr>
<td>( \text{cv}_2 )</td>
<td>0</td>
<td>-1.5026</td>
<td>0.0338</td>
<td>0.2245</td>
<td>-0.7289</td>
<td>-0.9820</td>
<td>6.1436</td>
<td>3.6703</td>
<td>1.6087</td>
<td>3.8441</td>
</tr>
</tbody>
</table>
In this example the total standard error is reduced by a factor of 90. To achieve this order of variance reduction in the simple Monte Carlo method would require increasing the number of simulations by a factor of 8100, that is, 8.1 million simulations with a computation time of approximately 3.15 hours. However, this is a slightly unrealistic example because we have the delta and gamma analytically, and so the hedge works perfectly in the limit as the time step is decreased to zero. In following sections we describe more realistic examples.

4.6 COMPUTING HEDGE SENSITIVITIES

The standard hedge sensitivities, delta, gamma, vega, theta and rho can be computed by approximating them by finite difference ratios:

\[
\delta\text{elta} = \frac{\partial C}{\partial S} = \frac{C(S + \Delta S) - C(S - \Delta S)}{2\Delta S}
\]  

\[
\delta\text{amma} = \frac{\partial^2 C}{\partial S^2} = \frac{C(S + \Delta S) - 2C(S) + C(S - \Delta S)}{\Delta S^2}
\]  

\[
\delta\text{ega} = \frac{\partial C}{\partial \sigma} = \frac{C(\sigma + \Delta \sigma) - C(\sigma - \Delta \sigma)}{2\Delta \sigma}
\]  

\[
\delta\text{theta} = \frac{\partial U}{\partial t} = \frac{C(t + \Delta t) - C(t)}{\Delta t}
\]  

\[
\delta\text{rho} = \frac{\partial C}{\partial r} = \frac{C(r + \Delta r) - C(r - \Delta r)}{2\Delta r}
\]

where \(C(S + \Delta S)\) is the Monte Carlo estimate using an initial asset price of \(S + \Delta S\), and \(S\) is a small fraction of \(S\), e.g. \(\Delta S = 0.001S\) and the other \(C(.)\)'s are defined similarly, one that every price \(C(.)\) in equations (4.27)–(4.31) should be computed using the same \(t\) of random numbers. If this is not done then the random error in the prices from the Monte Carlo simulation can be a large proportion of the price differences in the numerator of the finite difference ratios leading to very large errors in the sensitivity estimates. By the same random numbers the pricing errors will tend to cancel out.

A more efficient way to compute delta and from this gamma is by applying the discounted expectations approach. We can express the standard European call delta as follows:

\[
\delta\text{elta} = \frac{\partial C}{\partial S} = \frac{\partial}{\partial S} \left( e^{-rT} E \left[ (S_T - K) 1_{S_T > K} \right] \right)
\]

where \(S_T = S \exp(\nu T + \sigma T)\) and \(1_{S_T > K}\) is the indicator function which is one if \(S_T > K\) and zero otherwise. Substituting \(S_T\) in equation (4.32) and differentiating we obtain

\[
\delta\text{elta} = e^{-rT} E[\exp(\nu T + \sigma T) 1_{S_T > K}]
\]

to compute delta by Monte Carlo simulation we simulate the asset price as usual compute the discounted expectation of an instrument which pays off \(\exp(\nu T + \sigma T)\) if \(T > K\) and zero otherwise. Figure 4.15 gives a pseudo-code implementation of this.