4.8 PATH-DEPENDENT OPTIONS

An important application of Monte Carlo simulation is in pricing complex or exotic path-dependent options. Simple analytical formulae exist for certain types of exotic options, these options being classified by the property that the path-dependent condition applies to the continuous path. For example, a popular class of exotic option is the barrier option. These are standard European options except that the option either ceases to exist or only comes into existence if the underlying asset price crosses a predetermined barrier level. If we assume that the underlying asset price is checked continuously for the crossing of the barrier, then simple analytical formulae exist for the price of these options. In contrast, with actual barrier options the underlying asset price is checked (fixed) at most once a day and often much less frequently. This significantly affects the price of the option since the price is much less likely to be observed crossing the barrier if the fixings occur infrequently, and it also complicates the pricing formulae. However, these options can be priced very easily by Monte Carlo simulation.

Consider pricing a daily fixed down-and-out call option. This is a particular type of barrier option which is a normal call option unless the underlying asset price observed once per day crosses the predetermined barrier level $H$ from above, in which case the option ceases to exist. For this option we must simulate the underlying asset price for each fixing date in order to check for the crossing of the barrier. Assuming the asset price follows GBM, the simulation of the asset price takes the usual form:

$$S_{t+\Delta t} = S_t \exp(\nu \Delta t + \sigma \sqrt{\Delta t} z)$$  \hspace{1cm} (4.45)

where we assume $\Delta t$ is one day and $z$ is a standard normal random variate as usual. The Monte Carlo simulation proceeds in exactly the same way as for a standard option, except that at each time step we check whether the asset price has crossed the barrier level $H$. If so then we terminate the simulation of that path and the pay-off for that path is zero. The pseudo-code is given in Figure 4.21. We can use the analytical formulae for continuously fixed barrier options to construct hedge sensitivity-based control variates as we described in earlier sections.

Example: Pricing a European Down and Out Call Option by Monte Carlo Simulation

We price a one-year maturity, at-the-money European down and out call option with the current asset price at 100 and volatility of 20 per cent. The continuously compounded interest rate is assumed to be 6 per cent per annum, the asset pays a continuous dividend yield of 3 per cent per annum, and the barrier is at 99. The simulation has 10 time steps and 100 simulations; $K = 100, T = 1 \text{ year}, S = 100, \sigma = 0.2, r = 0.06, \delta = 0.03, H = 99, N = 10, M = 100$. Figure 4.22 illustrates the numerical results for the simulation of the path for $j = 100$.

Firstly, the constants; $\Delta t$ ($dt$), $\nu \Delta t$ ($nu \text{d}t$), $\sigma \sqrt{\Delta t}$ ($sig \text{d}t$) are precomputed:

$$\Delta t = \frac{T}{N} = \frac{1}{10} = 0.1$$

$$nu \text{d}t = (r - \delta - \frac{1}{2} \sigma^2) \Delta t = (0.06 - 0.03 - 0.5 \times 0.2^2) \times 0.1 = 0.001$$

$$sig \text{d}t = \sigma \sqrt{\Delta t} = 0.2 \sqrt{0.1} = 0.0632$$
FIGURE 4.21 Pseudo-code for Monte Carlo Valuation of a European Down and Out Call Option in a Black–Scholes World

\[
\begin{align*}
\text{initialise parameters } & \{ K, T, S, \sigma, r, \text{div, } N, N, M \} \\
& \{ N \text{ is the number of days in the life of the option } T \} \\
& \{ \text{precompute constants } \} \\
St & = T/N \\
\text{nudt} & = (r-\text{div}-0.5*\sigma*\sigma)*dt \\
\sigma\text{sd} & = \sigma*\text{sqrt}(dt) \\
\text{sum\_CT} & = 0 \\
\text{sum\_CT2} & = 0 \\
\text{for } j = 1 \text{ to } M \text{ do } \{ \text{for each simulation} \} \\
& St = S \\
& \text{BARRIER\_CROSSED = FALSE} \\
& \text{for } i = 1 \text{ to } N \text{ do } \{ \text{for each time step} \} \\
& \epsilon = \text{standard normal sample} \\
& St = St*\exp( \text{nudt} + \sigma\text{sd}*\epsilon) \\
& \text{if } ( St \leq H ) \text{ then} \\
& \text{BARRIER\_CROSSED = TRUE} \\
& \text{exit loop} \\
& \text{next } i \\
& \text{if BARRIER\_CROSSED then } CT = 0 \\
& \text{else } CT = \max( 0, St - K ) \\
& \text{sum\_CT} = \text{sum\_CT} + CT \\
& \text{sum\_CT2} = \text{sum\_CT2} + CT^2 \\
& \text{next } j \\
& \text{call value} = \text{sum\_CT}/M*\exp(-r*T) \\
& \text{SD} = \text{sqrt}( ( \text{sum\_CT2} - (\text{sum\_CT}/M)^2 )*\exp(-2*r*T)/(M-1) ) \\
& \text{SE} = \text{SD}/\text{sqrt}(M) \\
\end{align*}
\]

Then for each simulation \( j = 1 \) to \( M \) where \( M = 100 \), \( S_t \) is initialised to \( S = 100 \) and BARRIER\_CROSSED = FALSE which indicates that the barrier has not yet been crossed. Then for each time step \( i = 1 \) to \( N \), where \( N = 10 \), \( S_t \) is simulated and the crossing of the barrier is checked. For example for \( j = 100 \) and \( i = 1 \) we have

\[
S_t = S_i \times \exp(\text{nudt} + \sigma\text{sd} \times \epsilon)
\]

\[
= 100 \times \exp(0.001 + 0.0632 \times 0.5087) = 103.37
\]

\( S_t > H \), therefore BARRIER\_CROSSED is FALSE and the loop continues. For \( i = 4 \) we have
\[ F = 0.0432 \times 0.5987 = 0.037 \]

\[ \text{FALSE and the loop continues. For } i = 4 \]

\[ \text{which indicates that the barrier has not yet } \]

\[ \text{been crossed.} \]

\[ \text{For example, for } j = 100 \text{ and } i = 1 \]

\[ \text{we have } S = 10 \text{ and } H = 2 \text{.} \]

\[ \text{FIGURE 4.22 Monte Carlo Valuation of a European Down and Out Call Option in a Black–Scholes World} \]

<table>
<thead>
<tr>
<th>K</th>
<th>T</th>
<th>S</th>
<th>a(t)</th>
<th>r</th>
<th>div</th>
<th>H</th>
<th>N</th>
<th>M</th>
<th>sum_CT</th>
<th>sum_CT2</th>
<th>SD</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0.1</td>
<td>0.2</td>
<td>0.06</td>
<td>0.03</td>
<td>99</td>
<td>10</td>
<td>100</td>
<td></td>
<td>4.1049</td>
<td>1.0657</td>
<td>10</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>dt</th>
<th>nudi</th>
<th>sigadt</th>
<th>call_value</th>
<th>SE</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.0010</td>
<td>0.0032</td>
<td>3.8659</td>
<td>1.0064</td>
</tr>
</tbody>
</table>

\[ j = 100 \]

\[ S = 100 \text{ and } H = 10 \]

\[ \text{Barrier crossed? FALSE FALSE FALSE TRUE TRUE TRUE TRUE TRUE TRUE} \]

\[ \text{TRUE} \]

\[ \text{TRUE} \]
\[ S_t = S_0 \times \exp(mudt + sigd \times \varepsilon) \]
\[ = 109.531 \times \exp(0.001 + 0.0632 \times (-1.8043)) = 97.82 \]

\( S_t < H \), therefore BARRIER\_CROSSED = TRUE and the loop terminates.
For \( j = 100 \) we have BARRIER\_CROSSED = TRUE therefore:
\( C_T = 0 \)

The sum of the values of \( C_T \) and the squares of the values of \( C_T \) are accumulated in sum\_CT and sum\_CT2, giving sum\_CT = 410.493 and sum\_CT2 = 13057.7. The estimate of the option value is then given by
\[ \text{call\_value} = \text{sum\_CT}/M \times \exp(-r \times T) = 410.493/100 \times \exp(-0.06 \times 1) = 3.8659 \]

### 4.9 An Arithmetic Asian Option with a Geometric Asian Option Control Variate

In this example we price a European arithmetic Asian (average price) call option.\(^3\) This option pays the difference, if positive, between the arithmetic average of the asset price \( A_T \) and the strike price \( K \) at the maturity date \( T \). The arithmetic average is taken on a set of observations (fixings) of the asset price \( S_i \) (which we assume follows GBM) at dates \( t_i; i = 1, \ldots, N \)

\[ A_T = \frac{1}{N} \sum_{i=1}^{N} S_i \]  
(4.46)

Thus the pay-off at the maturity date is
\[ \max(0, A_T - K) \]  
(4.47)

Figure 4.23 illustrates two typical asset price paths and the fixing dates.

There is no analytical solution for the price of an arithmetic Asian option; however, there is a simple analytical formula for the price of a geometric Asian option. A geometric Asian call option pays the difference if positive, between the geometric average of the asset price \( G_T \) and the strike price \( K \) at the maturity date \( T \). The geometric average is defined as
\[ G_T = \left( \prod_{i=1}^{N} S_i \right)^{1/N} \]  
(4.48)

Since the geometric average is essentially the product of lognormally distributed variables then it is also lognormally distributed. Therefore the price of the geometric Asian call option is given by a modified Black–Scholes formula:
\[ C_{\text{GEOMETRIC\_ASIAN}} = \exp(-rT) \left( \exp\left(\frac{1}{2} \sigma \right) \mathcal{N}(x) - K \mathcal{N}(x - \sigma) \right) \]  
(4.49)

where
\[ x = \ln(G_t) + \frac{N - m}{N} (\ln(S) + \nu(t_{m+1} - t) + \frac{1}{2} \nu(T - t_{m+1})) \]
$$532 \times (-1.8043)) = 97.82$$

UE and the loop terminates.

$$t = \text{TRUE} \text{ }$$

which of the values of \( C_T \) are accumulated in

$$93 \text{ and sum } C_T = 13057.7 \text{. The estimate}$$

$$10.493/100 \times \exp(-0.06 \times 1) = 3.8659$$

**ION WITH A CONTROL VARIATE**

ic Asian (average price) call option. This is the arithmetic average of the asset price \( T \). The arithmetic average is taken on a set (which we assume follows GBM) at dates

$$\sum_{i=1}^{N} S_{ti} \quad (4.46)$$

$$\gamma \sim (r - K) \quad (4.47)$$

paths and the fixing dates.

ce of an arithmetic Asian option; however, ce of a geometric Asian option. A geometric average, between the geometric average of the maturity date \( T \). The geometric average is

$$\left[ S_{t_{i+1}} \right]^{1/N} \quad (4.48)$$

product of lognormally distributed variables xfore the price of the geometric Asian call s formula:

$$\left( a + \frac{1}{2} b \right) N(x) - KN(x - \sqrt{b}) \quad (4.49)$$

$$\gamma = r - \delta - \frac{1}{2} \sigma^2, \quad x = \frac{a - \ln(K) + b}{\sqrt{b}}$$

where \( G_t \) is the current geometric average and \( m \) is the last known fixing. The geometric Asian option makes a good static hedge style control variate for the arithmetic Asian option. Figure 4.24 shows a pseudo-code implementation of the Monte Carlo valuation of a European Asian call option with a geometric Asian call option control variate. We simulate the difference between the arithmetic and geometric Asian options or a hedged portfolio which is long one arithmetic Asian and short one geometric Asian option. This is much faster than using the delta of the geometric Asian option to generate a delta hedge control variate because we do not have to compute the delta at every time step and it is equivalent to a continuous delta hedge. Note the bold highlighted lines where we precompute the drift and volatility constant expressions required for the simulation of the asset price between the fixing dates. This increases the efficiency of the simulation significantly because these constants are used for every time step for every simulation.

**Example: Pricing a European Asian Call Option by Monte Carlo Simulation with Geometric Asian Call Option Control Variate**

We price a one-year maturity, European Asian call option with a strike price at 100, current asset price at 100 and volatility of 20 per cent. The continuously compounded interest rate is assumed to be 6 per cent per annum, the asset pays a continuous dividend yield of 3 per
FIGURE 4.24 Pseudo-code for Monte Carlo Valuation of a European Arithmetic Asian Call Option with a Geometric Asian call Option Control Variate

\[
\text{Initialize parameters: } [\text{K}, \text{t}[i]], S, \text{sig}, r, \text{div}, N, M \\
\text{t[i] is an array containing the fixing times }
\]

Them \precompute constants \}

\text{for } i = 1 \text{ to } N \text{ do [for each fixing ]}
\text{nudt[i] = (r-div-0.5*sig*sig)*t[i]-t[i-1])}
\text{sigsd[t] = sig*sqrt(t[i]-t[i-1])}
\text{next } i

\text{sum_CT = 0}
\text{sum_CT2 = 0}
\\text{for } j = 1 \text{ to } M \text{ do [for each simulation ]}

\text{St = S}
\text{sumSt = 0}
\text{productSt = 1}
\\text{for } i = 1 \text{ to } N \text{ do [for each fixing ]}
\text{e = standard.normal.sample}
\text{St = St*exp( nudt[i]*e )}
\text{sumSt = sumSt + St}
\text{productSt = productSt * St}
\text{next } i

\text{A = sumSt/N}
\text{B = productSt*t[i]/N}
\text{CT = max( 0, A - K ) - max( 0, G - K )}
\text{sum_CT = sum_CT + CT}
\text{sum_CT2 = sum_CT2 + CT*CT}
\text{next } j

\text{portfolio.value = sum_CT/M + exp(-r*T)}
\text{SD = sqrt(1/sum_CT2 - sum_CT*sum_CT/M + exp(-2*r*T)/(M-1) )}
\text{SE = SD/sqrt(M)}
\\text{add back in control variate value }
\text{call.value = portfolio.value + geometric.Asian.call( K, t[i], S, sig, r, div, N )}

\text{cent per annum, and their are 10 equally spaced fixing dates. The simulation has 10 time}
\text{steps and 100 simulations; } K = 100, T = 1 \text{ year, } S = 100, \sigma = 0.2, r = 0.06, \delta = 0.03.
\text{N = 10, M = 100. Figure 4.25 illustrates the numerical results for the simulation of the}
\text{path for } j = 100.
By spaced fixing dates. The simulation has $J = 100$, $\sigma = 0.2$, $r = 0.06$, $S = 0.05$.

![Figure 4.25 Monte Carlo Valuation of a European Asian Call Option with a Geometric Asian Call Option Control Variate](image)

<table>
<thead>
<tr>
<th>$K$</th>
<th>$T$</th>
<th>$S$</th>
<th>sig</th>
<th>$r$</th>
<th>div</th>
<th>$N$</th>
<th>$M$</th>
<th>$\text{sum}_{\text{CT}}$</th>
<th>$\text{sum}_{\text{CT}2}$</th>
<th>SD</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>1</td>
<td>100</td>
<td>0.2</td>
<td>0.00</td>
<td>0.03</td>
<td>10</td>
<td>100</td>
<td>22.8411</td>
<td>17.9102</td>
<td>0.3373</td>
</tr>
</tbody>
</table>

$dt = 0.1$, $nudt = 0.0010$, $nusdlt = 0.003246$

<table>
<thead>
<tr>
<th>$n=100$</th>
<th>$i$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\epsilon$</td>
<td>0</td>
<td>0.1</td>
<td>0.2</td>
<td>0.3</td>
<td>0.4</td>
<td>0.5</td>
<td>0.6</td>
<td>0.7</td>
<td>0.8</td>
<td>0.9</td>
<td>$t$</td>
</tr>
<tr>
<td>$\sigma^2$</td>
<td>100</td>
<td>114.68</td>
<td>115.10</td>
<td>117.64</td>
<td>121.12</td>
<td>126.65</td>
<td>119.25</td>
<td>132.79</td>
<td>132.67</td>
<td>135.59</td>
<td>136.61</td>
</tr>
<tr>
<td>$\text{sum}_{\text{St}}$</td>
<td>0</td>
<td>114.68</td>
<td>229.78</td>
<td>347.42</td>
<td>468.54</td>
<td>595.19</td>
<td>708.44</td>
<td>841.23</td>
<td>973.90</td>
<td>1109.48</td>
<td>1245.99</td>
</tr>
<tr>
<td>$\text{product}_{\text{St}}$</td>
<td>1</td>
<td>1.1E+02</td>
<td>1.3E+04</td>
<td>1.6E+06</td>
<td>1.9E+08</td>
<td>2.3E+10</td>
<td>2.7E+12</td>
<td>3.6E+14</td>
<td>4.8E+16</td>
<td>6.6E+18</td>
<td>8.8E+20</td>
</tr>
</tbody>
</table>

Geometric Asian Call Option

<table>
<thead>
<tr>
<th>$GA_a$</th>
<th>$GA_b$</th>
<th>$GA_{ru}$</th>
<th>$GA_x$</th>
<th>$\text{call}_{\text{value}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.5107</td>
<td>0.0164</td>
<td>0.0100</td>
<td>0.1584</td>
<td>5.5577</td>
</tr>
</tbody>
</table>

$\text{geometric}_{\text{Asian}}_{\text{call}} = 5.3426$
Firstly, the constants; \( \Delta t (dt) \), \( \nu \Delta t (nudt) \), \( \sigma \sqrt{\Delta t} (sigdst) \) are precomputed:

\[
\Delta t = \frac{T}{N} = \frac{1}{10} = 0.1
\]

\[
nudt = (r - \delta - \frac{1}{2} \nu^2) \Delta t = (0.06 - 0.03 - 0.5 \times 0.2^2) \times 0.1 = 0.001
\]

\[
sigdst = \sigma \sqrt{\Delta t} = 0.2 \sqrt{0.1} = 0.0632
\]

Then for each simulation \( j = 1 \) to \( M \) where \( M = 100 \), \( S_t \) is initialised to \( S = 100 \), \( sumSt = 0 \) and \( productSt = 1 \). Then for each time step \( i = 1 \) to \( N \), where \( N = 10 \), \( S_t \) is simulated and the sum and product of the asset prices at the fixing times are accumulated. For example, for \( j = 100 \) we have, for \( i = 1 \),

\[
S_t = S_i \times \exp(nudt + sigdst \times \epsilon)
\]

\[
= 100 \times \exp(0.0010 + 0.06325 \times 2.14929) = 114.675
\]

\[
sumSt = sumSt + S_t = 0 + 114.675 = 114.675
\]

\[
productSt = productSt \times S_t = 1 \times 114.675 = 114.675
\]

For \( i = 5 \):

\[
S_t = S_i \times \exp(nudt + sigdst \times \epsilon)
\]

\[
= 121.118 \times \exp(0.0010 + 0.06325 \times (-0.0765)) = 120.654
\]

\[
sumSt = sumSt + S_t = 468.539 + 120.654 = 589.193
\]

\[
productSt = productSt \times S_t = 1.98 \times 120.654 = 2.38 \times 10
\]

After the \( i \) loop we have

\[
A_T = \frac{sumSt}{N} = 1245.99 / 10 = 124.599
\]

\[
G_T = productSt^{(1/N)} = (8.8E + 20)^{1/10} = 124.326
\]

\[
C_T = \max(0, A - K) - \max(0, G - K)
\]

\[
= \max(0, 124.599 - 100) - \max(0, 124.326 - 100)
\]

\[
= 24.599 - 24.326 = 0.27303
\]

The sum of the values of \( C_T \) and the squares of the values of \( CT \) are accumulated in \( sum_{CT} \) and \( sum_{CT2} \) giving \( sum_{CT} = 22.8411 \) and \( sum_{CT2} = 17.9192 \). The estimate of the portfolio value is then given by

\[
portfolio.value = sum_{CT}/M \times \exp(-r \times T)
\]

\[
= 22.8411/100 \times \exp(-0.06 \times 1) = 0.21511
\]

Finally the estimate of the option value is given by

\[
call.value = portfolio.value + geometric.Asian.call(K, \{t_1, \ldots, t_N\}, S, \sigma, r, \delta, N)
\]

\[
= 0.21511 + 5.3426 = 5.5577
\]

Table 4.2 gives the prices, standard errors and relative computation times with no variance reduction, an antithetic control variate, a geometric Asian control variate and
\( \Delta t = (0.06 - 0.03 - 0.5 \times 0.2^2) \times 0.1 = 0.001 \)
\[ \sqrt{\Delta t} = 0.0632 \]

\( \Delta t = 1 \) to \( M \) where \( M = 100 \), \( S_i \) is initialised to \( S = 100 \).

Then for each time step \( i = 1 \) to \( N \), where \( N = 10 \), \( S_i \) is updated by:
\( \Delta S_i = \sigma \sqrt{\Delta t} \) \( \times \) \( \n \)
\( n(0.0010 + 0.06325 \times 2.14929) = 114.675 \)
\( \Delta S_i = 0 + 114.657 = 114.675 \)
\( \times S_{i-1} = 1 \times 114.675 = 114.675 \)
\( + \sigma \sqrt{\Delta t} \times e \)
\( \theta(0.0010 + 0.06325 \times (-0.0765)) = 120.654 \)
\( e^{\theta(0.0010 + 0.06325 \times (-0.0765))} = 589.193 \)
\( 1.98 \times 120.654 = 2.3 \times 10 \)
\( 5.99/10 = 124.599 \)
\( = (8.8 \times 20)^{(1/10)} = 124.326 \)
\( \max(0, G - K) \)
\( -100 \) and \( \max(0, 124.326 - 100) \)
\( = 0.27363 \)

The square of the value of \( C_T \) are accumulated in
\( 22.8411 \) and sum \( C_T^2 = 17.9192 \). The estimate
\( I \times \exp(-r \times T) \)
\( 0.0 \times \exp(-0.06 \times 1) = 0.21511 \)

is given by
\( e^{\text{Asian call}(K, \{t_i \}, \ldots, t_N), S_i, \sigma, r, \delta, N) \)

and relative computation times with no rate, a geometric Asian control variate and

| Table 4.2 Results from Pricing a European Arithmetic Asian Option by Monte Carlo Simulation |
|-----------------------------------------------|----------------|-----------------|
| Simple Monte Carlo                           | 5.038019       | 0.248236        | 1.00            |
| Antithetic                                   | 5.156263       | 0.135463        | 1.23            |
| Control variate                              | 5.207977       | 0.010366        | 1.05            |
| Antithetic and control variate               | 5.216232       | 0.006596        | 1.32            |

Finally both an antithetic control variate and a geometric Asian control variate. The addition of the antithetic and geometric Asian control variate increase the computation time by approximately 30 per cent, but reduces the standard error by approximately 37 times.

To achieve this reduction with the simple Monte Carlo would require increasing the number of simulations by \( 37 \times 37 = 1369 \) times with a roughly equivalent increase in the computation time.

### 4.10 A Lookback Call Option Under Stochastic Volatility with Delta, Gamma and Vega Control Variates

In this example we price a European fixed strike lookback call option. This option pays the difference, if positive, between the maximum of a set of observations (fixings) of the asset price \( S_t \) at dates \( t_i; i = 1, \ldots, N \) and the strike price. Thus the pay-off at the maturity date is

\[ \max(0, \max(S_{i}; i = 1, \ldots, N) - K) \]

We will also assume that the asset price and the variance of the asset price returns \( V = \sigma^2 \) are governed by the following stochastic differential equations:

\[ dS = rS \, dt + \sigma S \, dZ_1 \]

\[ dV = \alpha(V - \bar{V}) \, dt + \frac{\rho}{\sqrt{V}} \, dZ_2 \]

and that the Wiener processes \( dZ_1 \) and \( dZ_2 \) are uncorrelated, but this is easily generalised as we saw in section 4.7. Figure 4.26 illustrates two typical asset price paths and the fixing dates.

There is no analytical solution for the price of European fixed strike lookback call option with discrete fixings and stochastic volatility. However, there is a simple analytical formula for the price of a continuous fixing fixed strike lookback call with constant volatility.

\[ C_{\text{Fixed Strike Lookback Call}} = G + Se^{-rT}(x + \sigma \sqrt{T}) - Ke^{-rT}N(x) \]

\[ - \frac{S}{E} \left( e^{-rT} \frac{E}{S} \right)^{\frac{1}{2}} N \left( x + (1 - B)\sigma \sqrt{T} \right) \]

\[ - e^{-rT} N \left( x + \sigma \sqrt{T} \right) \]