ON CROFT, FALCONER AND GUY QUESTIONS OF UNIQUENESS FOR BODIES OF REVOLUTION

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Abstract. We prove that if a convex body of revolution $K \subset \mathbb{R}^4$ has the property that its floating body and its body of centers are concentric Euclidean balls, then K is a Euclidean ball.

1. Introduction

In [24, Problem 19], Ulam proposed the following problem: *If a convex body $K \subset \mathbb{R}^3$ made of material of uniform density $\delta \in (0,1)$ floats in equilibrium in any orientation in a liquid of density 1, must K be a Euclidean ball?*

The *surface of centers* $\mathcal{S}_\delta(K)$ of a convex body K of density $\delta \in (0,1)$ is defined as the geometric locus of all the centers of mass of the submerged part of K in each orientation, and the *body of centers* is the body whose boundary is the surface of centers (see Section 2 for a precise definition). It is known that the condition of floating in equilibrium in all orientations is equivalent to the fact that the surface of centers of K is a sphere centered at the center of mass of K (see, for example [12], [20], [21], [9, Chapter XXIV]).

Given $\delta \in (0,1)$, for each orientation of K, there is a unique water surface (the plane cutting off the portion of K of volume $\delta \mathrm{vol}_3(K)$) corresponding to the submerged body in that orientation. If we assume that K is fixed in one position, while the water surfaces are rotated, the *Dupin floating body* $K_{[\delta]}$ of K is defined as the envelope of all the water surfaces [8]. It is an open question whether the Euclidean ball is the only convex body whose Dupin floating body is a Euclidean ball.

In [7, p. 19], Croft, Falconer and Guy posed a series of problems of Ulam's type; see also the related classical questions of Busemann and Petty [6]. Consider a body K in $\mathbb{R}^3$ and a collection of affine planes $\{H(\xi)\}$ with normal ξ, for every unit vector ξ in $\mathbb{R}^3$. Assume that the body K is such that one (or more) of the following conditions are satisfied:

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(V) The volumes cut off from K by each plane $H(\xi)$ in the collection are equal in every direction ξ.

(I) All the sections $K \cap H(\xi)$ have constant equal principal moments of inertia $I_{K\cap H(\xi)}(l) = \int\limits_{K\cap H(\xi)} \text{dist}^2(l,v)dv$ with respect to the lines l passing through the center of mass of the sections.

(A) All the sections $K \cap H(\xi)$ have equal area.

(P) All the sections $K \cap H(\xi)$ have equal perimeter.

(S) The surface areas cut off from the boundary of K by each plane $H(\xi)$ are equal in every direction ξ.

Croft, Falconer and Guy ask if any two of these constraints imply that the body must be a ball. They point out that Ulam's problem is equivalent to Problem (V, I) in their formulation (see [21, Thm 1] for a proof of this fact). Of course, the questions can be posed in any dimension $d \geq 3$ and additional conditions can be added to the list, for example:

(H) The distance from $H(\xi)$ to the origin is positive and independent of ξ.

The answer to Ulam's problem (V, I) when $d \geq 3$ and $\delta = 1/2$ has been shown to be negative in [20], but it was known to be positive when $K = -K$ [10, 22] (see also [15, 11]). Problem (V, A) also has a negative answer when $\delta = 1/2$, and (A, I), (A, P) have a positive answer in the class of bodies of revolution for all densities [1].

For the two-dimensional version of Ulam's problem, Auerbach [2] found counterexamples in the case $\delta = 1/2$. For some specific densities less than 1/2, Bracho, Montejano and Oliveros [5] proved that the answer is affirmative, while Wegner [25, 26] has found counterexamples for other densities. In general, the question is open.

What happens if we impose three conditions rather than two? In this paper we consider the (V, I, H) and the (V, A, H) problems:

Problem 1.1. Let K be a convex body in $\mathbb{R}^d$, $d \geq 3$, with density $\delta \in (0, 1)$. If both the body of centers and the Dupin floating body $K_{[\delta]}$ of K are Euclidean balls, does it follow that K is a Euclidean ball?

Problem 1.2. Let K be a convex body in $\mathbb{R}^d$, $d \geq 3$, with density $\delta \in (0, 1)$. If the Dupin floating body $K_{[\delta]}$ of K is a Euclidean ball, and the areas of the sections $K \cap H(\xi)$ are constants independent of ξ, does it follow that K is a Euclidean ball?

In dimension 2, the answer to both problems is known to be affirmative, in fact conditions (V, H) and (A, H) are enough to conclude that K is a disc [4, 13]. In dimensions 3 and higher, both problems are open. Kurusa and Ódor also proved that the answer to Problem 1.1 is affirmative under an additional normalization condition: Denoting the unit Euclidean ball by B, they show that if $\text{vol}_d(K) = \text{vol}_d(B)$, $K_{[\delta]} = B_{[\delta]}$ and $S_\delta(K) = S_\delta(B)$, then $K = B$. We improve their result by showing that the last hypothesis is not needed.

Theorem 1.3. *Let K be a convex body in $\mathbb{R}^d$, $d \geq 3$ with density $\delta \in (0, 1/2)$. If $\mathrm{vol}_d(K) = \mathrm{vol}_d(B)$ and $K_{[\delta]} = B_{[\delta]}$, then $K = B$.*

We also give an affirmative answer to Problem 1.1 in dimension 4 in the class of bodies of revolution, for any density $\delta \in (0, 1/2)$.

Theorem 1.4. *Let $K \subset \mathbb{R}^4$ be a convex body of revolution with C^1 boundary and density $\delta \in (0, 1/2)$, such that its body of centers $\mathcal{S}_\delta$ and its Dupin floating body are concentric Euclidean balls, with center at the center of mass of K. Then K is a Euclidean ball.*

Similarly, one can show that Problem 1.2 also has an affirmative answer in the same class of bodies.

Theorem 1.5. *Let K be a convex body of revolution in $\mathbb{R}^4$ with C^1 boundary, center of mass at the origin and containing the Euclidean ball of radius t centered at the origin in its interior. Let $\{H(\xi)\}_{\xi \in S^3}$ be the collection of hyperplanes tangent to this ball, such that K satisfies conditions (V, A) for this collection. Then K is a Euclidean ball.*

In the proofs of Theorems 1.4 and 1.5 we rewrite the hypotheses as a system of nonlinear ODEs with a Cauchy type condition at infinity. To obtain the affirmative answer, we show the asymptotical instability of the system. The proof of Theorem 1.5 is almost identical to the one of Theorem 1.4.

2. Notation and auxiliary results

A convex body $K \subset \mathbb{R}^d$, $d \geq 2$, is a convex compact set with non-empty interior int$K$. Given $\xi \in S^{d-1}$ we denote by $\xi^\perp = \{p \in \mathbb{R}^d : p \cdot \xi = 0\}$ the the subspace orthogonal to ξ, where $p \cdot \xi = p_1\xi_1 + \cdots + p_d\xi_d$ is the usual inner product in $\mathbb{R}^d$. The *center of mass* of a convex body $L \subset \mathbb{R}^d$ will be denoted by $\mathcal{C}(L)$,

$$\mathcal{C}(L) = \frac{1}{\mathrm{vol}_d(L)} \int\limits_L x dx.$$

We say that a hyperplane H is the supporting hyperplane of a convex body L if $L \cap H \neq \emptyset$, but int $L \cap H = \emptyset$.

For $d \geq 2$ we denote by S^{d-1} the unit sphere in $\mathbb{R}^d$ centered at the origin, and by $B_2^d(r)$ the Euclidean ball in $\mathbb{R}^d$ with center at the origin and radius r. We will write B for $B_2^d(1)$ when the dimension is clear. We will denote by κ_d the volume of $B_2^d(1)$ and by σ_{d-1} the surface area of S^{d-1}.

Let $K \subset \mathbb{R}^d$ be a convex body and let $\delta \in (0, 1)$ be fixed. Given a direction $\xi \in S^{d-1}$ and $t = t(\xi) \in \mathbb{R}$, we call a hyperplane

$$H(\xi) = H_t(\xi) = \{p \in \mathbb{R}^d : p \cdot \xi = t\}, \tag{2.1}$$

the *cutting hyperplane* of K (or the *water surface*) in the direction ξ, if

$$\frac{\mathrm{vol}_d(K \cap H^-(\xi))}{\mathrm{vol}_d(K)} = \delta, \qquad H^-(\xi) = \{p \in \mathbb{R}^d : p \cdot \xi \leq t(\xi)\}. \tag{2.2}$$

We recall several facts and definitions from fluid mechanics; see [20], [21], [9, Ch. XXIV] or [27].

Definition 2.1. Let $\delta \in (0,1)$, let $\xi \in S^{d-1}$, and let $\mathcal{C}(\xi) = \mathcal{C}_\delta(\xi)$ be the center of mass of the submerged part $K \cap H^-(\xi)$ satisfying (2.2). A convex body K *floats in equilibrium in every orientation* $\xi \in S^{d-1}$ *at the level* δ if (2.2) holds and the line $\ell(\xi)$ connecting $\mathcal{C}(K)$ with $\mathcal{C}_\delta(\xi)$ is orthogonal to the "free water surface" $H(\xi)$, i.e., the line $\ell(\xi)$ is "vertical" (parallel to ξ).

Definition 2.2. The geometric locus of $\{\mathcal{C}_\delta(\xi) : \xi \in S^{d-1}\}$ is called the *surface of centers* $\mathcal{S} = \mathcal{S}_\delta$, or the *surface of buoyancy*. The *body of centers* $\mathcal{K}_\delta$ is the body whose surface is S_δ.

In order to obtain analytic expressions equivalent to conditions (V, I, A), we need to introduce the concepts of characteristic points of a family of hyperplanes and of moments of inertia.

Definition 2.3. For $d \geq 2$, let $\mathcal{Q}$ be a set of hyperplanes in $\mathbb{R}^d$, so that for each $\xi \in S^{d-1}$, there is a unique corresponding hyperplane $H = H(\xi) \in \mathcal{Q}$ orthogonal to ξ. Let $H \in \mathcal{Q}$ and let Γ be a $(d-2)$-dimensional subspace parallel to H. Assume also that for any sequence $\{H_k\}_{k=1}^\infty \subseteq \mathcal{Q}$ parallel to Γ and converging to H, the limit $\Pi_\Gamma(H) = \lim_{k\to\infty} H \cap H_k$ exists. A point $e \in H$ is termed the *characteristic point* of $\mathcal{Q}$ relative to H if, for any Γ and $\{H_k\}_{k=1}^\infty$ as above, e belongs to $\Pi_\Gamma(H)$.

We remark that by writing that a sequence of hyperplanes $\{H_k\}_{k=1}^\infty$, $H_k = H_{t_k}(\xi_k)$, converges to $H(\xi) = H_t(\xi)$ as $k \to \infty$ if $\xi_k \to \xi$, $t_k \to t$.

Olovjanischnikoff [19] proved the equivalence between the constant volume condition (V) and the fact that the characteristic points of the collection of cutting hyperplanes are exactly the centers of mass of the sections of K by these hyperplanes. A proof of this result can be found in [20, Theorem 3].

Theorem 2.4. *Let $d \geq 3$, let $K \subset \mathbb{R}^d$ be a convex body, and let $\delta \in (0,1)$. The characteristic points of the family of cutting hyperplanes $\{H(\xi) : \xi \in S^{d-1}\}$ for which equation* (2.2) *holds are the centers of mass of the sections $\{K \cap H(\xi) : \xi \in S^{d-1}\}$.*

Conversely, if the characteristic points of the family of hyperplanes $\{H(\xi) : \xi \in S^{d-1}\}$ intersecting the interior of K and corresponding to the sections $\{K \cap H(\xi) : \xi \in S^{d-1}\}$ coincide with the centers of mass of these sections, then the function $\xi \mapsto \frac{\mathrm{vol}_d(K\cap H^-(\xi))}{\mathrm{vol}_d(K)}$ is constant on S^{d-1} and the constant is equal to some $\delta \in (0,1)$.

Next, we recall the notion of moment of inertia, [27, p. 553]. Let $d \geq 3$, let $\delta \in (0,1)$, and let $\xi \in S^{d-1}$ be any direction. Consider a convex body K and an affine $(d-1)$-dimensional subspace $H(\xi)$ defined by (2.1) such that (2.2) holds. Choose any $(d-2)$-dimensional affine subspace $l \subset H(\xi)$

passing through the center of mass $\mathcal{C}(K \cap H(\xi))$, and let $\eta_1, \dots, \eta_{d-2}, \eta_{d-1}$ be an orthonormal basis of $\xi^\perp = \{p \in \mathbb{R}^d : p \cdot \xi = 0\}$ such that

$$l = \mathcal{C}(K \cap H(\xi)) + \operatorname{span}(\eta_1, \dots, \eta_{d-2}), \quad H(\xi) = \mathcal{C}(K \cap H(\xi)) + \xi^\perp. \tag{2.3}$$

Definition 2.5. The moment of inertia $I_{K\cap H(\xi)}(l)$ of $K \cap H(\xi)$ with respect to l is defined as

$$I_{K\cap H(\xi)}(l) = \int\limits_{K\cap H(\xi)} \operatorname{dist}^2(l, v)dv = \int\limits_{K\cap H(\xi) - \mathcal{C}(K\cap H(\xi))} (u \cdot \eta_{d-1})^2 \, du, \tag{2.4}$$

where $\operatorname{dist}(l, v) = \min\limits_{\{u \in l\}} |u - v|$.

Ulam's problem can now be restated as problem (V, I). A proof of this fact can be found in [21, Thm. 1].

Theorem 2.6. *Let $d \geq 3$, let K be a convex body and let $\delta \in (0, 1)$. If K floats in equilibrium at the level δ in every orientation, then $\forall \xi \in S^{d-1}$ the cutting sections $K \cap H(\xi)$ have equal moments of inertia $I_{K\cap H(\xi)}(l)$ for all $(d-2)$-dimensional affine subspace $l \subset H(\xi)$ passing through the center of mass $\mathcal{C}(K \cap H(\xi))$.*

Conversely, if $\mathcal{C}(\mathcal{S}) = \mathcal{C}(K)$ and for every cutting hyperplane $H(\xi)$, $\xi \in S^{d-1}$, the cutting section $K \cap H(\xi)$ have equal principal moments of inertia, then a C^1-smooth body K floats in equilibrium in every orientation at the level δ.

Finally, we recall the notion of *Dupin floating body*. The floating body $K_{[\delta]}$ of K was introduced by C. Dupin in 1822 [8].

Definition 2.7. A non-empty convex set $K_{[\delta]}$ is the Dupin floating body of K if each supporting hyperplane $H(\xi)$ of $K_{[\delta]}$, $\xi \in S^{d-1}$, cuts off a set $K \cap H^-(\xi)$ of fixed volume satisfying (2.2).

We remark that $K_{[\delta]}$ does not necessarily exist for every convex K, (see [14] or [16], Chapter 5), but if K has a sufficiently smooth boundary and $\delta > 0$ is small enough (as is the case under our assumptions), then $K_{[\delta]}$ exists [14, Satz 2]. We refer the reader to [3] and [23] for further information about convex floating bodies.

3. Proof of Theorem 1.3

We assume that $(B)_\delta = K_\delta$ and $\operatorname{vol}_d(K) = \operatorname{vol}_d(B)$. Then, for each $\xi \in S^{d-1}$, we have

$$\delta = \frac{\operatorname{vol}_d(K \cap \{x \cdot \xi \geq t\})}{\operatorname{vol}_d(K)} = \frac{\operatorname{vol}_d(B \cap \{x \cdot \xi \geq t\})}{\operatorname{vol}_d(B)}, \tag{3.1}$$

where $t \in (0, +\infty)$ is independent of ξ. We also assume that μ is the uniform probability measure on S^{d-1}. As in [13] we write

$$\int_{S^{d-1}} \mathrm{vol}_d(K \cap \{x \cdot \xi \geq t\}) d\mu(\xi) = \int_{K \setminus tB} dx \int_{\{\xi \in S^{d-1} \colon \xi \cdot \frac{x}{|x|} \geq \frac{t}{|x|}\}} d\mu(\xi) =$$

$$\sigma_{d-2} \int_{K \setminus tB} dx \int_{\frac{t}{|x|}}^{1} (1-y^2)^{\frac{d-3}{2}} dy = \tag{3.2}$$

$$\sigma_{d-2} \int_{S^{d-1}} d\mu(\theta) \int_{t}^{\rho_K(\theta)} r^{d-1} dr \int_{\frac{t}{r}}^{1} (1-y^2)^{\frac{d-3}{2}} dy.$$

Here, in the last equality, we passed to polar coordinates in the outer integral. Now we consider the auxiliary function $f(x)$ defined for $x \geq t^d$ as

$$f(x) = \sigma_{d-2} \int_{t}^{x^{1/d}} r^{d-1} dr \int_{\frac{t}{r}}^{1} (1-y^2)^{\frac{d-3}{2}} dy.$$

Observe that f is *convex*, i.e., $f''(x) \geq 0$. Indeed,

$$f'(x) = \frac{\sigma_{d-2}}{d} x^{\frac{1}{d}-1} x^{\frac{d-1}{d}} \int_{\frac{t}{x^{1/d}}}^{1} (1-y^2)^{\frac{d-3}{2}} dy = \frac{\sigma_{d-2}}{d} \int_{\frac{t}{x^{1/d}}}^{1} (1-y^2)^{\frac{d-3}{2}} dy$$

and

$$f''(x) = t \frac{\sigma_{d-2}}{d^2} x^{-\frac{1}{d}-1} \left(1 - \left(\frac{t^2}{x^{2/d}}\right)\right)^{\frac{d-3}{2}} \geq 0.$$

We now write equation (3.2) in terms of f and the radial function ρ_K of K as a function on the unit sphere, and apply Jensen's inequality,

$$\int_{S^{d-1}} \mathrm{vol}_d(K \cap \{x \cdot \xi \geq t\}) d\mu(\xi) = \int_{S^{d-1}} f(\rho_K^d(\theta)) d\mu(\theta)$$

$$\geq f\left(\int_{S^{d-1}} \rho_K^d(\theta) d\mu(\theta) \right).$$

Since $\mathrm{vol}_d(K) = \mathrm{vol}_d(B)$, by (3.1) we have

$$f(1) = \int_{S^{d-1}} f(\rho_B^d(\theta)) d\mu(\theta) = \int_{S^{d-1}} f(\rho_K^d(\theta)) d\mu(\theta) \geq$$

$$f\left(\int_{S^{d-1}} \rho_K^d(\theta)d\mu(\theta)\right) = f(1).$$

The equality in Jensen's inequality implies that ρ_K is constant or f is linear. But f is not a linear function, hence $\rho_K = const$, as we wanted to show. □

4. Bodies of Revolution

Let $d \geq 3$. We follow the notation from [17], [18]. We will consider bodies of revolution

$$K_f = \{x \in \mathbb{R}^d : x_2^2 + x_3^2 + \cdots + x_d^2 \leq f^2(x_1)\}$$

obtained by rotation of a smooth concave function supported on $[-R_1, R_2]$ about the x_1-axis.

Let $L_s(\xi) = L(s, h, \xi) = s\xi + h$ be a linear function with slope $s > 0$, and let

$$H(L_s) = \{x \in \mathbb{R}^d : x_d = L_s(x_1)\}$$

be the corresponding hyperplane. Let $a(s) = -h'(s)$ be the x_1-coordinate of the point of tangency of $H(L_s)$ with the floating body of K_f which is $B_2^d(r)$. We will assume that $a(0) = 0$.

We remark that for each given $s > 0$, there are two parallel hyperplanes $H(L_s)$ tangent to $B_2^d(r)$, see Figure 1. In this paper, we will mostly refer to the upper hyperplane. The considerations related to the lower one will be very similar.

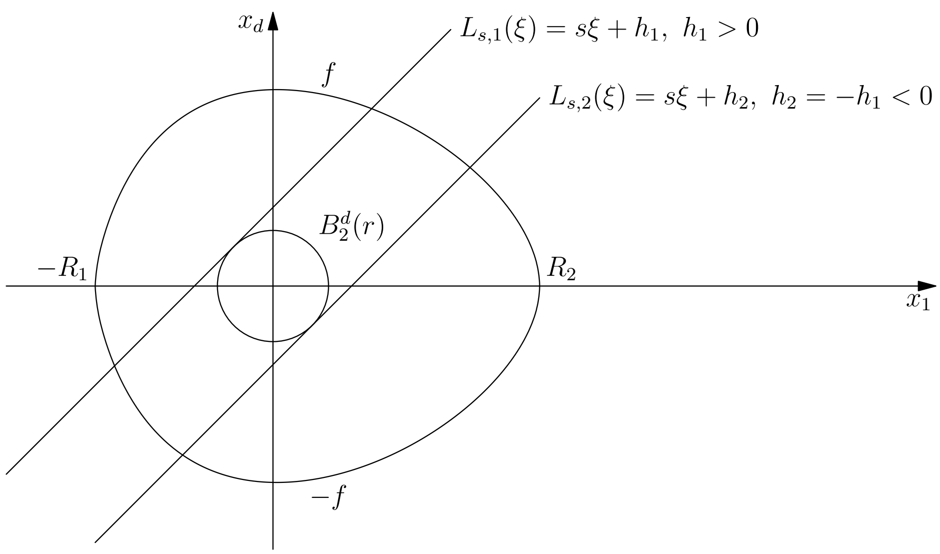

Figure 1. The two parallel hyperplanes

Denote by l_j the $(d-2)$-dimensional planes, $j = 2, \ldots, d-1$,

$$l_1 = \{x \in \mathbb{R}^d : x \in K \cap L_s,\, x_1 = a(s)\}, \quad l_j = \{x \in \mathbb{R}^d : x \in K \cap L_s,\, x_j = 0\}.$$

Let $-x = -x(s)$ and $y = y(s)$ be the first coordinates of the points of intersection of $\pm f$ and L_s, i.e.,

$$f(y(s)) = sy(s) + h(s), \quad -f(-x(s)) = -sx(s) + h(s); \tag{4.1}$$

see Figure 2. We remark that, although the notation and Figure 2 seem to suggest that $x(s), y(s)$ are positive functions, in fact it is possible for them to be negative for certain values of s.

It is not hard to compute that the center of mass of the section $C_{K\cap H(L_s)}$, where $H(L_s)$ satisfies (2.2), has coordinates given by

$$C_{K\cap H(L_s)} = \{(-h'(s), 0, \dots, 0, L_s(-h'(s))) \in \mathbb{R}^d : s \in [0, \infty)\}, \tag{4.2}$$

see [20, Lemma 1]. Therefore, using Theorems 2.4 and 2.6, conditions (V, A, I) can be restated for bodies of revolution by means of the next two Lemmas.

Lemma 4.1. [20, Lemma 2] *The condition* $\mathcal{C}(K\cap H(L_s)) = (a, 0, \dots, 0, L_s(a))$ *reads as*

$$\int_{-x(s)}^{y(s)} (\xi - a)(f^2(\xi) - L_s^2(\xi))^{\frac{d-2}{2}} d\xi = 0. \tag{4.3}$$

The conditions that the second moments of inertia I_j of $K \cap H(L_s)$ with respect to l_j are constant read as

$$I_1 = \kappa_{d-2}(1 + s^2)^{\frac{3}{2}} \int_{-x(s)}^{y(s)} (\xi - a)^2(f^2(\xi) - L_s^2(\xi))^{\frac{d-2}{2}} d\xi = c, \tag{4.4}$$

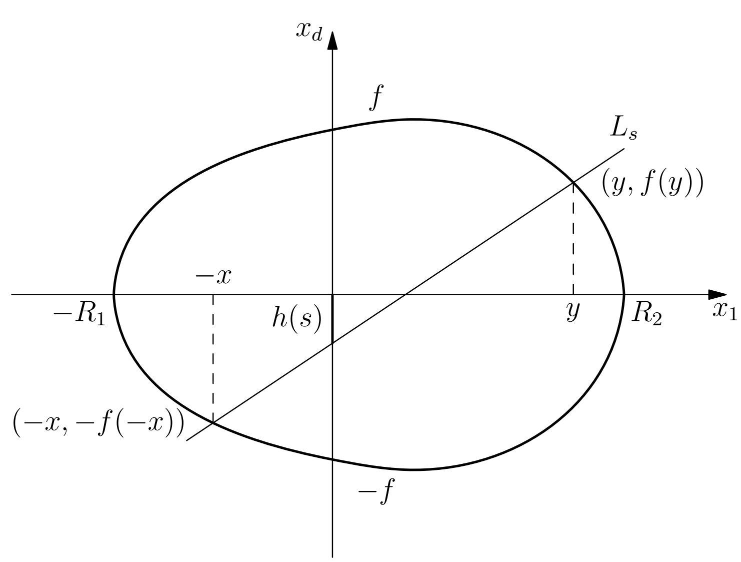

FIGURE 2. The body of revolution.

$$I_j = \sqrt{1+s^2}\,\gamma_{d-2} \int\limits_{-x(s)}^{y(s)} (f^2(\xi) - L_s^2(\xi))^{\frac{d}{2}} d\xi = c, \tag{4.5}$$

where

$$\gamma_{d-2} = \int\limits_{B_2^{d-2}(1)} y_j^2 dy, \quad j = 2, \dots, d-1.$$

We remark that

$$\gamma_{d-2} = \frac{1}{d-2} \int\limits_{B_2^{d-2}(1)} |y|^2 dy = \frac{1}{d-2} \int\limits_{S^{d-3}} d\sigma \int\limits_0^1 r^{2+d-3} dr = \frac{\sigma_{d-3}}{d(d-2)}.$$

In particular, if $d = 4$, then $\gamma_2 = \frac{\pi}{4}$.

The analytic form of condition (A) for bodies of revolution is as follows (see [18]):

Lemma 4.2. *The $(d-1)$-dimensional volume of the intersection $K_f \cap H(L_s)$ is constant for all s if and only if*

$$\int_{x(s)}^{y(s)} (f^2(\xi) - L_s^2(\xi))^{\frac{d-2}{2}} d\xi = \frac{\widetilde{c}}{\sqrt{1+s^2}}, \tag{4.6}$$

where $\widetilde{c}$ is an absolute constant.

Proof. Fix $s > 0$. Let us define the $(d-1)$-dimensional hyperplanes $H^\xi = \{x \in \mathbb{R}^d : x_1 = \xi\}$, where $\xi \in (-x(s), y(s))$. The slice $(K_f \cap H(L_s)) \cap H^\xi$ for each $t \in (-x(s), y(s))$ is a $(d-2)$-dimensional Euclidean ball with radius $r = \sqrt{f^2(\xi) - L_s^2(\xi)}$. Therefore,

$$\mathrm{vol}_{d-1}(K_f \cap H(L_s)) = \kappa_{d-2}\sqrt{1+s^2} \int_{-x(s)}^{y(s)} (f^2(\xi) - L_s^2(\xi))^{(d-2)/2} d\xi. \tag{4.7}$$

Since the latter does not depend on s, (4.6) follows. $\square$

5. Proof of Theorem 1.4

5.1. **The system of 3 dependent equations.** Using Theorem 2.6 and Lemma 4.1, we see that if K floats in equilibrium at the level $\delta \in (0, \frac{1}{2})$, then f, x, y, a and h must satisfy the following system of 3 integral equations,

$$\begin{cases} \int\limits_{-x(s)}^{y(s)} (\xi + h'(s))(f^2(\xi) - L^2(\xi)) d\xi = 0, \\ \pi(1+s^2)^{\frac{3}{2}} \int\limits_{-x(s)}^{y(s)} (\xi + h'(s))^2 (f^2(\xi) - L^2(\xi)) d\xi = c, \\ \frac{\pi}{4}\sqrt{1+s^2} \int\limits_{-x(s)}^{y(s)} (f^2(\xi) - L^2(\xi))^2 d\xi = c, \end{cases} \tag{5.1}$$

where $h(0) = h_0 \geq 0$ is given, and from now on we will write L instead of L_s. The constant c is the same for the second and third equations and its value is $\frac{4\pi}{15}(R^2 - r^2)^{\frac{5}{2}}$ (see Appendix, Section 7.1).

It is clear that Euclidean balls float in equilibrium in every direction. If $K = B_2^4(R)$ and $K_\delta = B_2^4(r)$, $r \in (0, R)$, then the corresponding functions

$$f(\xi) = f_o(\xi) \equiv \sqrt{R^2 - \xi^2}, \tag{5.2}$$

$$h(s) = h_o(s) \equiv r\sqrt{1+s^2}, \qquad a(s) = a_o(s) \equiv -\frac{rs}{\sqrt{1+s^2}}, \tag{5.3}$$

and

$$x(s) = x_o(s) \equiv \frac{sr + \sqrt{R^2 - r^2}}{\sqrt{1+s^2}}, \quad y(s) = y_o(s) \equiv \frac{-sr + \sqrt{R^2 - r^2}}{\sqrt{1+s^2}}, \tag{5.4}$$

satisfy (5.1).

In [20, Lemmas 2-4], it was shown that the third equation in system (5.1) depends on the first two. Since we are assuming that the floating body of K is a Euclidean ball of radius $r > 0$, we have that $h(s) = r\sqrt{1+s^2}$. Thus, we have a system of two equations with the unknown variables f, x and y. However, these three variables are not independent, since they are related by (4.1). The system reads as

$$\begin{cases} \int\limits_{-x(s)}^{y(s)} (\xi + h'(s))(f^2(\xi) - L^2(\xi))d\xi = 0, \\ \pi(1+s^2)^{\frac{3}{2}} \int\limits_{-x(s)}^{y(s)} (\xi + h'(s))^2(f^2(\xi) - L^2(\xi))d\xi = c. \end{cases} \tag{5.5}$$

In the Appendix, sections 7.2 and 7.3, we show that this system can be reduced to a system of first order ODEs,

$$\begin{cases} -h''y'\,(y+h')^2(sy+h) - h''x'\,(-x+h')^2(-sx+h) = P(s,x,y) \\ y'\,(y+h')^3(sy+h) + x'(-x+h')^3(-sx+h) = Q(s,x,y). \end{cases} \tag{5.6}$$

In sections 7.2 and 7.3, we also compute the exact expressions for P and Q, (7.9) and (7.12). Here, we write them with the additional assumption that $r = 1$, omitting the argument s in $x(s), y(s)$ and $a(s)$.

$$P(s,x,y) = \tag{5.7}$$

$$\frac{3}{2(1+s^2)^{\frac{7}{2}}}\left((y-a)^2 - (x+a)^2\right) + \frac{2s}{(1+s^2)^3}\left((y-a)^3 + (x+a)^2\right)$$
$$+\frac{1+4s^2}{4(1+s^2)^{\frac{5}{2}}}\left((y-a)^4 - (x+a)^4\right),$$

$$Q(s,x,y) = \tag{5.8}$$

$$3a'\left(\frac{s}{4}\left((y-a)^4 - (x+a)^4\right) + \frac{1}{3(1+s^2)^{\frac{1}{2}}}\left((y-a)^3 + (x+a)^3\right)\right) -$$

$$-\frac{1}{5}\left((y-a)^5+(x+a)^5\right)+\frac{3c}{2\pi}\frac{1-4s^2}{(1+s^2)^{\frac{7}{2}}},$$

where the value of c is

$$c=\frac{4\pi}{15}(R^2-1)^{\frac{5}{2}}, \tag{5.9}$$

see (7.2). In short, system (5.6) has the form

$$\begin{cases} y'A+x'B=P \\ y'C+x'D=Q. \end{cases}$$

Since $h''(s)=\frac{1}{(1+s^2)^{3/2}}\neq 0$ for all $s>0$, the determinant

$$\mathcal{D}=AD-BC= \tag{5.10}$$

$$(y+h')^2(sy+h)(-x+h')^2(-sx+h)h''((-x+h'-(y+h'))$$
$$=-(y+h')^2(sy+h)(-x+h')^2(-sx+h)h''(x+y)$$

will be zero only when one of the factors of

$$(y+h')(sy+h)(-x+h')(-sx+h)(x+y)$$

is zero. Let us consider $s>0$ such that

$$y+h'\neq 0,\quad sy+h\neq 0,\quad -x+h'\neq 0,\quad -sx+h\neq 0,\quad x+y\neq 0. \tag{5.11}$$

Observe that for finite s, the terms $y+h'$, $-x+h'$ and $x+y$ in (5.11) are nonzero. Indeed, only when the hyperplane $H(L_s)$ is perpendicular to the axis of revolution x_1 it is possible to have $-x(s)=y(s)=a(s)$, (recall that $-h'(s)=a(s)$ is the x_1 coordinate of the point of tangency of L_s with the floating body). The term $sy+h$ is zero only for a negative value s_{**}, but it is never zero for positive s if we consider the upper hyperplane in Figure 3. On the other hand, the term $-sx(s)+h$ can be zero for a single finite value $s_*>0$.

Lemma 5.1. *We introduce new variables*

$$\alpha=\sqrt{1+s^2}(y-a),\qquad \beta=\sqrt{1+s^2}(x+a), \tag{5.12}$$

provided $y\geq a$ and $-x\leq a$ (see Figure 4 for the geometric meaning of α,β) Then system (5.6) has the following form

$$\begin{cases} \alpha^2(s\alpha+1)\alpha'+\beta^2(-s\beta+1)\beta'= \\ -\frac{1}{1+s^2}\left(\frac{\alpha^2-\beta^2}{2}+\frac{\alpha^4-\beta^4}{4}\right) \\ \alpha^3(s\alpha+1)\alpha'-\beta^3(-s\beta+1)\beta'= \\ \frac{1}{1+s^2}\left((4s^2-1)\left(\frac{\alpha^5+\beta^5-2(R^2-1)^{\frac{5}{2}}}{5}\right)+\frac{5s}{4}(\alpha^4-\beta^4)\right). \end{cases} \tag{5.13}$$

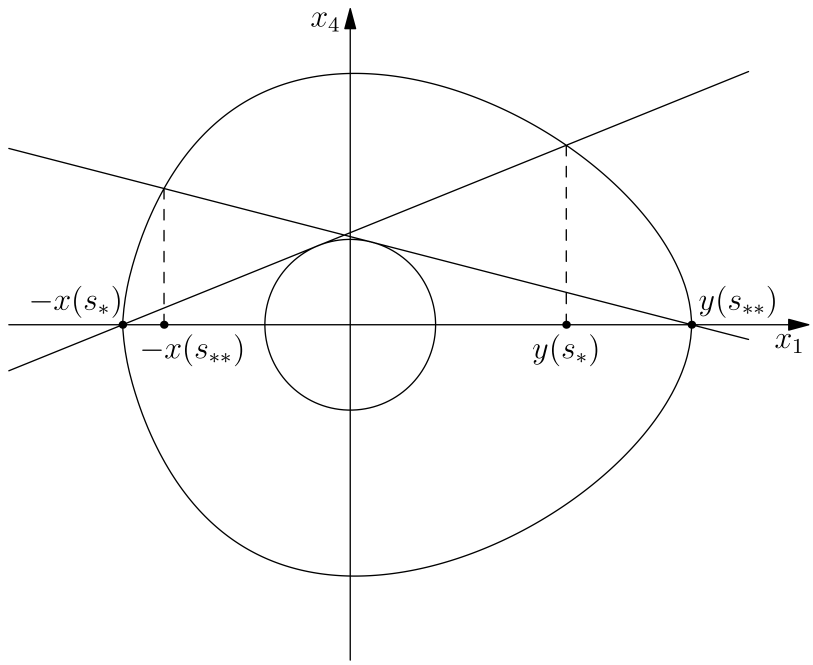

FIGURE 3. Slopes s_* and s_{**}

Proof. We start with the first equation in our system (5.6). It reads as

$$a'\left(y'(y-a)^2(sy+h)+x'(x+a)^2(-sx+h)\right)=P(s,x,y),$$

where $P(s,x,y)$ is defined in (7.10), and for convenience we are setting $r=1$ in the expression for P. Using (5.12) we have

$$y-a=\frac{\alpha}{\sqrt{1+s^2}},\quad x+a=\frac{\beta}{\sqrt{1+s^2}},$$

$$sy+h=s(y-a)+sa+h=\frac{s\alpha}{\sqrt{1+s^2}}+sa+h=$$

$$\frac{s\alpha}{\sqrt{1+s^2}}+\frac{1}{\sqrt{1+s^2}}, \tag{5.14}$$

$$-sx+h=-s(x+a)+sa+h=-\frac{s\beta}{\sqrt{1+s^2}}+sa+h=$$

$$-\frac{s\beta}{\sqrt{1+s^2}}+\frac{1}{\sqrt{1+s^2}}. \tag{5.15}$$

After these substitutions, our first equation in (5.6) is

$$a'\left(y'\left(\frac{\alpha^2}{1+s^2}\right)\left(\frac{s\alpha+1}{\sqrt{1+s^2}}\right)+x'\left(\frac{\beta^2}{1+s^2}\right)\left(\frac{-s\beta+1}{\sqrt{1+s^2}}\right)\right)=$$

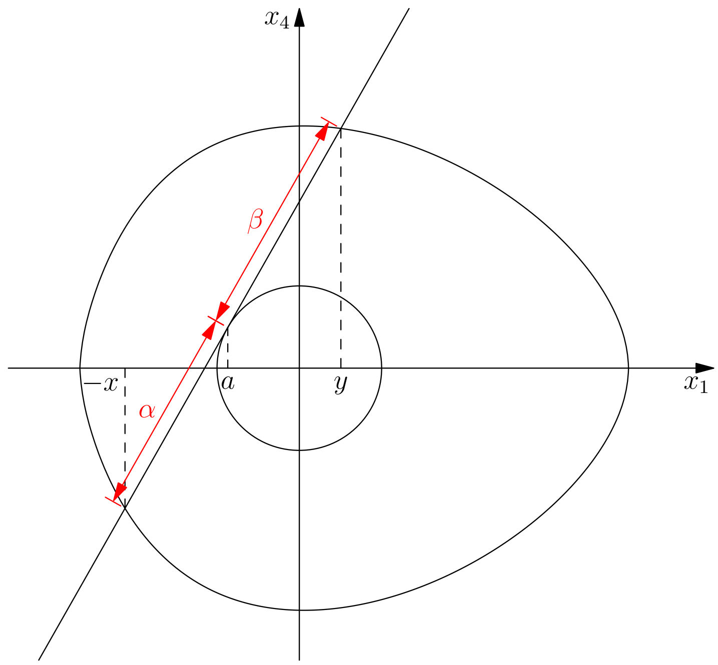

FIGURE 4. Geometric meaning of α, β

$$\frac{3}{2(1+s^2)^{\frac{7}{2}}}\left(\frac{\alpha^2}{1+s^2}-\frac{\beta^2}{1+s^2}\right)+\frac{2s}{(1+s^2)^3}\left(\frac{\alpha^3}{(1+s^2)^{\frac{3}{2}}}+\frac{\beta^3}{(1+s^2)^{\frac{3}{2}}}\right)+$$
$$\frac{1+4s^2}{4(1+s^2)^{\frac{5}{2}}}\left(\frac{\alpha^4}{(1+s^2)^2}-\frac{\beta^4}{(1+s^2)^2}\right).$$

Since

$$a' = -\frac{1}{(1+s^2)^{\frac{3}{2}}},$$

we can cancel $(1+s^2)^3$ in both parts of our equation to get

$$-\big(y'\alpha^2(s\alpha+1)+x'\beta^2(-s\beta+1)\big) = \tag{5.16}$$
$$\frac{3}{2(1+s^2)^{\frac{3}{2}}}(\alpha^2-\beta^2)+\frac{2s}{(1+s^2)^{\frac{3}{2}}}(\alpha^3+\beta^3)+\frac{1+4s^2}{4(1+s^2)^{\frac{3}{2}}}(\alpha^4-\beta^4).$$

To finish the computations related to the first equation, it remains to express y', x' via α' and β'. We have

$$\alpha' = \frac{s}{\sqrt{1+s^2}}(y-a)+\sqrt{1+s^2}\,y'-a'\sqrt{1+s^2}.$$

Hence,

$$y' = \frac{\alpha'+a'\sqrt{1+s^2}-\frac{s(y-a)}{\sqrt{1+s^2}}}{\sqrt{1+s^2}} = \frac{\alpha'-\frac{1}{1+s^2}-\frac{s\alpha}{1+s^2}}{\sqrt{1+s^2}} = \frac{(1+s^2)\alpha'-(s\alpha+1)}{(1+s^2)^{\frac{3}{2}}}.$$

Similarly,

$$\beta' = \frac{s}{\sqrt{1+s^2}}(x+a) + \sqrt{1+s^2}\,x' + a'\sqrt{1+s^2} =$$
$$\frac{s}{1+s^2}\beta + \sqrt{1+s^2}x' - \frac{1}{1+s^2},$$

i.e.,

$$x' = \frac{\beta' + \frac{1}{1+s^2} - \frac{s\beta}{1+s^2}}{\sqrt{1+s^2}} = \frac{(1+s^2)\beta' + (-s\beta+1)}{(1+s^2)^{\frac{3}{2}}}.$$

Substituting into (5.16), we have

$$-\left((1+s^2)\alpha' - (s\alpha+1)\right)\alpha^2(s\alpha+1) - \left((1+s^2)\beta' + (-s\beta+1)\right)\beta^2(-s\beta+1)$$
$$= \frac{3}{2}(\alpha^2-\beta^2) + 2s(\alpha^3+\beta^3) + \frac{1+4s^2}{4}(\alpha^4-\beta^4),$$

or, in other words,

$$(5.17) \qquad (1+s^2)\alpha^2(s\alpha+1)\,\alpha' + (1+s^2)\beta^2(-s\beta+1)\,\beta'$$
$$= \alpha^2(s\alpha+1)^2 - \beta^2(-s\beta+1)^2 - \frac{3}{2}(\alpha^2-\beta^2) - 2s(\alpha^3+\beta^3) - \frac{1+4s^2}{4}(\alpha^4-\beta^4).$$

We can simplify (5.17) even further, by writing the first two terms on the right hand as

$$\alpha^2(s\alpha+1)^2 - \beta^2(-s\beta+1)^2 = \alpha^2(s^2\alpha^2+2s\alpha+1) - \beta^2(s^2\beta^2-2s\beta+1)$$
$$= s^2(\alpha^4-\beta^4) + 2s(\alpha^3+\beta^3) + \alpha^2 - \beta^2.$$

Thus, our first equation in (5.13) is

$$(1+s^2)\alpha^2(s\alpha+1)\,\alpha' + (1+s^2)\beta^2(-s\beta+1)\,\beta' = -\frac{\alpha^2-\beta^2}{2} - \frac{\alpha^4-\beta^4}{4}.$$

Now we consider the second equation in (5.13), where the right hand side Q is defined in (5.6). Using our change of variables (5.12), we have

$$Q(s,x,y) = -\frac{3}{(1+s^2)^{\frac{3}{2}}}\left(\frac{s}{4}\cdot\frac{\alpha^4-\beta^4}{(1+s^2)^2} + \frac{1}{3(1+s^2)^{\frac{1}{2}}}\cdot\frac{\alpha^3+\beta^3}{(1+s^2)^{\frac{3}{2}}}\right) -$$
$$-\frac{1}{5}\cdot\frac{\alpha^5+\beta^5}{(1+s^2)^{\frac{5}{2}}} + \frac{3c}{2\pi}\frac{1-4s^2}{(1+s^2)^{\frac{7}{2}}}.$$

For the left hand side, using (5.14) and (5.15) we have

$$(y-a)^3(sy+h)y' - (x+a)^3(-sx+h)x'$$
$$= \frac{\alpha^3}{(1+s^2)^{\frac{3}{2}}}\cdot\frac{s\alpha+1}{\sqrt{1+s^2}}\,y' - \frac{\beta^3}{(1+s^2)^{\frac{3}{2}}}\cdot\frac{-s\beta+1}{\sqrt{1+s^2}}\,x'$$
$$= \frac{\alpha^3}{(1+s^2)^2}\cdot(s\alpha+1)\cdot\frac{(1+s^2)\alpha' - (s\alpha+1)}{(1+s^2)^{\frac{3}{2}}} -$$
$$-\frac{\beta^3}{(1+s^2)^2}\cdot(-s\beta+1)\cdot\frac{(1+s^2)\beta' + (-s\beta+1)}{(1+s^2)^{\frac{3}{2}}}.$$

We multiply both parts of the resulting equation by $(1+s^2)^{\frac{7}{2}}$, obtaining

$$\alpha^3(s\alpha+1)\left((1+s^2)\alpha' - (s\alpha+1)\right) - \beta^3(-s\beta+1)\left((1+s^2)\beta' + (-s\beta+1)\right) =$$

$$-\frac{3s}{4}(\alpha^4-\beta^4) - (\alpha^3+\beta^3) - \frac{1+s^2}{5}\cdot(\alpha^5+\ \beta^5) + \frac{3c}{2\pi}(1-4s^2), \tag{5.18}$$

where c is defined in (5.9). We further simplify the left hand side of (5.18),

$$\alpha^3(s\alpha+1)(1+s^2)\alpha' - \beta^3(-s\beta+1)(1+s^2)\beta' - \alpha^3(s\alpha+1)^2 - \beta^3(-s\beta+1)^2 =$$

$$\alpha^3(s\alpha+1)(1+s^2)\alpha' - \beta^3(-s\beta+1)(1+s^2)\beta' - s^2(\alpha^5+\beta^5) - 2s(\alpha^4-\beta^4) - (\alpha^3+\beta^3).$$

This yields

$$\alpha^3(s\alpha+1)(1+s^2)\alpha' - \beta^3(-s\beta+1)(1+s^2)\beta' =$$

$$\frac{4s^2-1}{5}(\alpha^5+\beta^5) + \frac{5s}{4}(\alpha^4-\beta^4) + \frac{3c}{2\pi}(1-4s^2).$$

This is the second equation in (5.13). □

Our goal is to prove that system (5.13) has a unique solution

$$\alpha(s) = \beta(s) = \sqrt{R^2-1} \tag{5.19}$$

for all $s \in \mathbb{R}$. We will show at first that (5.19) holds in the neighborhood of infinity, i.e., there exists $s_o > 0$ such that (5.19) holds for $s \geq s_o$. This will be a consequence of several lemmas proved below.

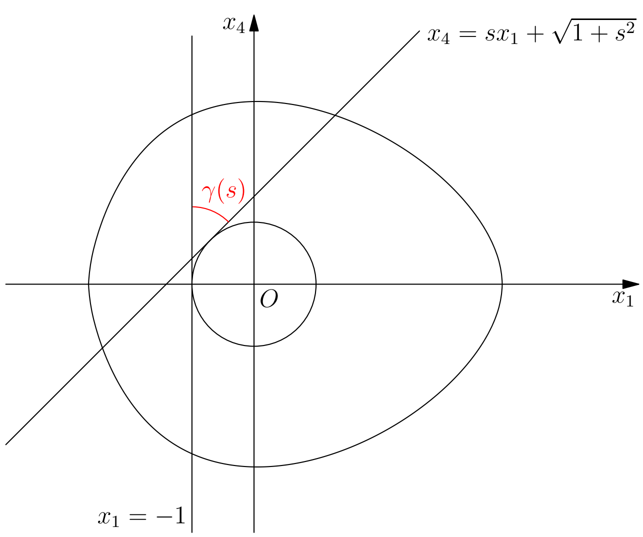

FIGURE 5. Angle between the lines $x_1 = -1$ and $x_4 = sx_1 + \sqrt{1+s^2}$

Lemma 5.2. *Denote*

$$u(s)=\alpha(s)^5+\beta(s)^5-2(R^2-1)^{\frac{5}{2}},\quad v(s)=\alpha(s)^2-\beta(s)^2. \tag{5.20}$$

Then, there exists $s_o>0$ *and constants* $c,C>0$ *such that*

$$|u(s)|\le C\quad \forall s\ge s_o. \tag{5.21}$$

and

$$|v(s)|\le \frac{c}{s}\quad \forall s\ge s_o. \tag{5.22}$$

Proof. By (5.12) both functions α and β are bounded. Hence, (5.21) holds. To show (5.22), we let $\gamma(s)=\frac{\pi}{2}-\arctan s$ be the angle between the lines $x_1=-1$ and $x_4=sx_1+\sqrt{1+s^2}$, see Figure 5. Then (5.22) follows if we put $\tilde v(\gamma(s))=v(s)$ and use $\tilde v(0)=0$ to see that $|\tilde v(\gamma)|\le c\gamma$ in a neighborhood of $\gamma=0$. □

Lemma 5.3. *Let* u *and* v *be defined by* (5.20)*. Assume that there exists* $s_o>0$ *and positive constants* c_1*,* c_2 *and* c_3 *such that*

$$u'(s)\mathrm{sign}(u(s))\ge c_1\left(\frac{|u(s)|}{s}-\frac{|v(s)|}{s^2}\right)\qquad \forall s\ge s_o, \tag{5.23}$$

$$|v'(s)|\le c_2\,\frac{|u(s)|+|v(s)|}{s^2}\qquad \forall s\ge s_o, \tag{5.24}$$

and

$$|u(s)|\le \frac{c_3}{s}|v(s)|\qquad \forall s\ge s_o. \tag{5.25}$$

Then $\alpha(s)=\beta(s)=\sqrt{R^2-1}$ *holds for all* $s\ge s_o$*.*

Proof. We claim at first that $\alpha\equiv\beta$ or $v\equiv 0$ for all $s\ge s_o$. Indeed, if this is not the case, then there is $s_1\ge s_o$ where $v(s_1)\ne 0$. We take a maximal interval (s_2,s_3) containing s_1 where $v(s)$ is not zero. Then, for this interval, (5.25) and (5.24) yield

$$|v'(s)|\le \frac{c}{s^2}|v(s)|.$$

Hence,

$$(\ln|v|)'\ge -\frac{c}{s^2},$$

and integrating this inequality from s_2 to s, with $s<s_3$, we see that

$$\ln|v(s)|-\ln|v(s_2)|\ge \frac{1}{s}-\frac{1}{s_2}. \tag{5.26}$$

Since $v(s)$ tends to zero when s tends to s_3, the left hand side of the previous inequality tends to $-\infty$, while the right hand side is bounded. Hence, if v is not identically zero, then $v(s)\ne 0$ for every $s>s_o$. Now, using the same argument and inequality (5.26) with s_o instead of s_2, we obtain that $v(s)$ is separated from zero for $s>s_o$, which contradicts (5.22). Thus, we conclude that $v(s)=0$ for all $s\ge s_o$.

We substitute $|v(s)| = 0$ into (5.23) to have

$$u' \operatorname{sign}(u) \geq \frac{c|u|}{s}.$$

Therefore, u' and u have the same sign, and so if u is non-zero at s_0, then it is non-zero for all $s \geq s_0$. Thus we can divide both sides of the inequality above by u, then integrate to obtain $\ln|u| \geq c \ln s + C$, which shows that $|u|$ is unbounded. This contradicts (5.21), so $|u(s)| = 0$ for $s \geq s_o$, and $\alpha(s) = \beta(s) = \sqrt{R^2 - 1}$ for these values of s. □

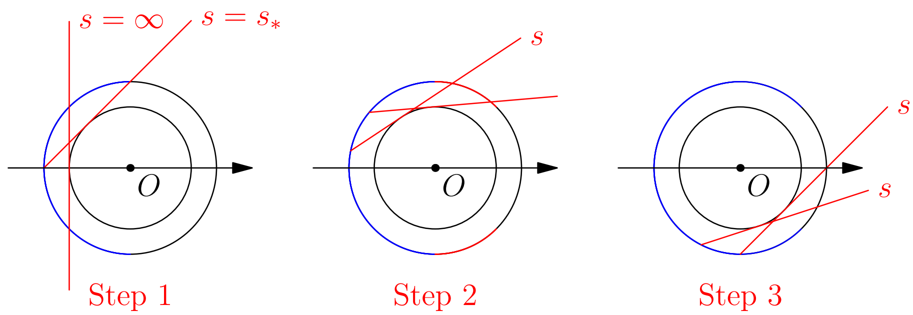

Figure 6. Extending the solution to the whole sphere.

Observe that we have reduced the problem of finding the solution of system (5.13) with Cauchy-type condition at infinity to a usual Cauchy problem for this system. Indeed, since (5.19) holds for $s \geq s_0$, we can take any $s_1 \geq s_o$ and look for a solution of (5.13) in the neighborhood of s_1. This process can be continued until we reach the chord with slope s_*, corresponding to the line such that $-s_* x(s_*) + h = 0$. At this point, the determinant (5.10) of system (5.6) becomes zero. Since α and β satisfying (5.19) are the only solution of (5.13) for all $s > s_*$, the boundary of K_f for the corresponding values of s is a sphere, see Figure 6, Step 1.

In order to define the rest of the boundary, we now consider a chord with positive slope $s < s_*$, and solve the system using that the left end of this chord lies on the sphere, which means that $\alpha(s) = \sqrt{R^2 - 1}$ for this s (see Figure 6, Step 2). Plugging $\alpha = \sqrt{R^2 - 1}$ in the first equation of system (5.13), we obtain the following first order differential equation for β,

$$\beta^2(-s\beta + 1)\beta' = -\frac{1}{1+s^2}\left(\frac{\alpha^2 - \beta^2}{2} + \frac{\alpha^4 - \beta^4}{4}\right).$$

Using equality (5.15) we see that the coefficient $(-s\beta + 1)$ of β' is not zero, since we are taking a value of $s < s_*$, for which $sx(s) + h \neq 0$. Hence, by the classical theory of ordinary differential equations, this equation has a unique solution, which must be the constant solution $\beta(s) = \sqrt{R^2 - 1}$. Thus, we have extended the range of s where $\alpha(s) = \beta(s) = \sqrt{R^2 - 1}$ are the only solutions beyond the value s_*.

By the symmetry of bodies of revolution, we can now repeat this process using the bottom hyperplanes (as in Figure 1 with $h_2 < 0$) instead of the top ones. This allows us, in a finite number of steps, to finish defining the boundary at all points, see Figure 6, Step 3. Since all chords tangent to the unit disc centered at the origin and inscribed in $K_f \cap \{x \in \mathbb{R}^4 : x_2 = x_3 = 0\}$ have the same length, by the results of [4] and [13] we obtain that $f(t) = \sqrt{R^2 - t^2}$ for all $t \in [-R, R]$. Thus, K is a Euclidean ball.

It remains to prove (5.23), (5.24) and (5.25). This will be done in the next two lemmas.

Lemma 5.4. *Let u and v be defined by (5.20). Then there exists $s_o > 0$ such that (5.23) and (5.24) hold.*

Proof. We write our system (5.13) in the form

$$\begin{cases} \alpha^2(s\alpha+1)\alpha' - \beta^2(s\beta-1)\beta' = -\frac{1}{s^2}F_1 v \\ \alpha^3(s\alpha+1)\alpha' + \beta^3(s\beta-1)\beta' = F_2 u + \frac{1}{s}F_3 v, \end{cases}$$

where $F_j = F_j(u, v, s)$, $j = 1, 2, 3$, are positive functions, bounded from above and below by positive constants, for large s.

Now we solve the system with respect to $\alpha'(s\alpha+1)$ and $\beta'(s\beta-1)$. By Cramer's rule, we obtain

$$\begin{cases} \alpha'(s\alpha+1) = \frac{1}{\alpha^2\beta^2(\alpha+\beta)}\left(-\beta^3\frac{1}{s^2}F_1 v + \beta^2\left(F_2 u + \frac{1}{s}F_3 v\right)\right) \\ \beta'(s\beta-1) = \frac{1}{\alpha^2\beta^2(\alpha+\beta)}\left(\alpha^3\frac{1}{s^2}F_1 v + \alpha^2\left(F_2 u + \frac{1}{s}F_3 v\right)\right). \end{cases}$$

Then,

$$u' = 5\alpha^4\alpha' + 5\beta^4\beta' = \frac{W_1 u}{s} + \frac{W_2 v}{s^2},$$

where W_1 is positive and bounded away from zero and infinity for large s, and W_2 is bounded (may be negative). We also have

$$v' = 2\alpha\alpha' - 2\beta\beta'.$$

Then

$$v' = 2\left(\frac{\alpha\beta^2}{s\alpha+1} - \frac{\beta\alpha^2}{s\beta-1}\right)F_2 u.$$

We can replace $s\alpha + 1$ by αs in the above expression, resulting in an extra term of order $\frac{c}{s^2}$. Similarly, replace $s\beta - 1$ by βs. Hence,

$$v' = \frac{\Phi_1 u}{s^2} - 2\frac{v}{s}F_2 u + \frac{\Phi_2 v}{s^2},$$

where Φ_1 and Φ_2 are bounded functions. Thus, our system is written in a simpler form

$$\begin{cases} u'(s) = \frac{W_1(s)u(s)}{s} + \frac{W_2(s)v(s)}{s^2} \\ v'(s) = \frac{\Phi_1(s)u(s)}{s^2} - 2\frac{v}{s}F_2(s)u(s) + \frac{\Phi_2(s)v(s)}{s^2}. \end{cases} \tag{5.27}$$

Then, taking into account estimate (5.22) we obtain (5.23) and (5.24) as direct consequences of the first and the second equations in (5.27) and the fact that W_2, Φ_1, Φ_2 and F_2 are bounded functions, and W_1 is positive and bounded from below. ☐

Lemma 5.5. *Let u and v be defined by* (5.20)*. Then there exists $s_o > 0$ such that* (5.25) *holds, i.e.*

$$|u(s)| \leq \frac{c_3}{s}|v(s)| \qquad \forall s \geq s_o.$$

Proof. We will show that the statement of the lemma holds with $c_3 = c_2/c_1$, where c_1 are c_2 are the constants from (5.23) and (5.24). We will argue by contradiction. Assume that for every $s_0 > 0$ there exists $s > s_0$ such that

$$|u(s)| > \frac{c_3}{s}|v(s)|.$$

Fix any number s_1 satisfying the previous inequality. Denote $M = \frac{c_3}{s_1}$. Then for all $s_2 > s_1$ there is $s_3 > s_2$ such that

$$|u(s_3)| \geq M|v(s_3)|. \tag{5.28}$$

We will further assume that

$$s_2 > s_1 + \frac{2s_1(s_1+1)}{c_3}. \tag{5.29}$$

Take any s_3 satisfying (5.28). By continuity of u and v there is $s_4 > s_3$ such that

$$|u(s)| \geq \frac{M}{2}|v(s)|$$

for all $s \in [s_3, s_4]$.

We use (5.23) and (5.24) to obtain

$$\begin{cases} u'(s)\text{sign}(u(s)) \geq \frac{c_1|u(s)|}{s} - \frac{c_1|v(s)|}{s^2} \geq c_1\left(\frac{1}{s} - \frac{2s_1}{c_3 s^2}\right)|u(s)| \\ |v'(s)| \leq c_2\left(\frac{1}{s^2} + \frac{2s_1}{c_3 s^2}\right)|u(s)|, \end{cases} \tag{5.30}$$

for all $s \in [s_3, s_4]$. This implies that u' and u have the same sign and

$$|v'(s)| \leq \frac{c_2\left(\frac{1}{s^2} + \frac{2s_1}{c_3 s^2}\right)}{c_1\left(\frac{1}{s} - \frac{2s_1}{c_3 s^2}\right)}|u'(s)|,$$

for all $s \in [s_3, s_4]$.

Observe that by (5.29) we have

$$\frac{c_2\left(\frac{1}{s^2} + \frac{2s_1}{c_3 s^2}\right)}{c_1\left(\frac{1}{s} - \frac{2s_1}{c_3 s^2}\right)} = c_3\frac{1 + \frac{2s_1}{c_3}}{s - \frac{2s_1}{c_3}} \leq c_3\frac{1 + \frac{2s_1}{c_3}}{s_1 + \frac{2s_1^2}{c_3}} = \frac{c_3}{s_1} = M,$$

since $s > s_2$.

Thus,

$$|v'(s)| \leq M|u'(s)|,$$

for all $s \in [s_3, s_4]$. Integrating the inequality

$$-M|u'(s)| \le v'(s) \le M|u'(s)|,$$

from s_3 to s, and using (5.28), we obtain

$$|v(s)| \le M \left| \int_{s_3}^{s} |u'(t)|dt + |u(s_3)| \right|.$$

As u' and u have the same sign, this yields

$$|v(s)| \le M|u(s)| \tag{5.31}$$

for all $s \in [s_3, s_4]$.

Since now $|v(s_4)| \le M|u(s_4)|$, we repeat this process to obtain the inequality (5.31) for a larger interval $[s_3, s_5]$, where $s_5 > s_4$. Eventually we obtain that (5.31) holds for all $s \ge s_3$. However this is impossible due to the boundedness of u. Indeed, from (5.23) we have

$$|u(s)|' \ge \frac{c_1}{s}\left(1 - \frac{M}{s}\right)|u(s)| \ge \frac{c_1}{2s}|u(s)|,$$

for s large enough. Since u' and u have the same sign, if u is non-zero at s_3, then it is also non-zero for all $s \ge s_3$. Thus we have

$$\frac{|u(s)|'}{|u(s)|} \ge \frac{c_1}{2s},$$

and therefore

$$\ln|u(s)| \ge \frac{c_1}{2}\ln(s) + C,$$

which means that u is not bounded. This contradicts (5.21). Thus we have proven (5.25). □

6. Proof of Theorem 1.5

6.1. **The system of 2 equations.** Using (4.3) and (4.6) from Lemmas 4.1 and 4.2, we see that if K satisfies conditions (V, A), then f, x, y, a and h must satisfy the following system of 2 integral equations,

$$\begin{cases} \int\limits_{-x(s)}^{y(s)} (\xi + h'(s))(f^2(\xi) - L_s^2(\xi))d\xi = 0, \\ \int\limits_{-x(s)}^{y(s)} (f^2(\xi) - L_s^2(\xi))d\xi = \frac{\tilde{c}}{\sqrt{1+s^2}}, \end{cases} \tag{6.1}$$

where $h(0) = h_0 \ge 0$ is given. Since the Euclidean ball $K = B_2^d(R)$ satisfies conditions (V, A, H), the functions f, h, x, y given by (5.2), (5.3), (5.4) satisfy (6.1). Using (4.7) we see that and the value of the constant $\tilde{c}$ for the ball is $\frac{4}{3}\left(R^2 - r^2\right)^{3/2}$.

Without loss of generality we can put $r = 1$. We claim that differentiating the equations in the system above with respect to s we obtain the following system

$$\begin{cases} -h''y'\,(y-a)^2(sy+h) - h''x'\,(x+a)^2(-sx+h) = P(s,x,y) \\ y'\Big(s(y-a)^2 + \frac{y-a}{\sqrt{1+s^2}}\Big) + x'\Big(s(x+a)^2 - \frac{x+a}{\sqrt{1+s^2}}\Big) = \tilde{Q}(s,x,y), \end{cases} \tag{6.2}$$

where

$$\tilde{Q}(s,x,y) = a'\Big(s(y-a)^2 + \frac{y-a}{\sqrt{1+s^2}}\Big) - a'\Big(s(x+a)^2 - \frac{x+a}{\sqrt{1+s^2}}\Big) \tag{6.3}$$

$$-\Big(\frac{(y-a)^3}{3} + \frac{(x+a)^3}{3}\Big) + \frac{s}{(1+s^2)^{\frac{3}{2}}}\Big(\frac{(y-a)^2}{2} - \frac{(x+a)^2}{2}\Big) + \frac{\tilde{c}}{2}\frac{1-2s^2}{(1+s^2)^{\frac{5}{2}}}.$$

Observe that system (6.2) is very similar to system (5.6). In fact, the first equations of the systems are exactly the same with P defined by (5.7). The second equation is easily obtained by taking two derivatives of both sides of the second equation in (6.1) and using (4.1) (see Appendix, Section 7.4 for computations).

The rest of the proof follows almost verbatum the lines of the proof of Theorem 1.4. As in Lemma 5.1 we introduce new variables (5.12), provided $y \geq a$ and $-x \leq a$. Then we use (5.14) and (5.15) to see that system (6.2) has the following form

$$\begin{cases} \alpha^2(s\alpha+1)\alpha' + \beta^2(-s\beta+1)\beta' = \\ -\frac{1}{1+s^2}\left(\frac{\alpha^2-\beta^2}{2} + \frac{\alpha^4-\beta^4}{4}\right) \\ \alpha(s\alpha+1)\alpha' - \beta^3(-s\beta+1)\beta' = \\ \frac{1}{1+s^2}\left(\frac{(2s^2-1)}{3}(\alpha^3+\beta^3-2(R^2-1)^{\frac{3}{2}}) + \frac{3}{2}s(\alpha^2-\beta^2)\right). \end{cases} \tag{6.4}$$

Finally, to show that the only solution of our system is $\alpha(s) = \beta(s) = \sqrt{R^2-1}$, we use the change of variables

$$\tilde{u}(s) = \alpha(s)^3 + \beta(s)^3 - 2(R^2-1)^{\frac{3}{2}}, \quad \tilde{v}(s) = \alpha(s)^2 - \beta(s)^2, \tag{6.5}$$

similar to (5.20), and proceed as in Lemmas 5.2, 5.3, 5.4 and 5.5.

7. Appendix

7.1. **Computation of the constant on the right-hand side of (5.1).** We compute the constant on the right-hand side of the last two equations in the case of $K = B_2^4(R)$ and $K_\delta = B_2^4(r)$, $0 < r < R$. In this situation, the functions f, h, a, x, and y are given by (5.2)-(5.4). In particular, $x+a = y-a$ and

$$x + a = \frac{\sqrt{R^2-r^2}}{\sqrt{1+s^2}}, \quad y + x = 2\frac{\sqrt{R^2-r^2}}{\sqrt{1+s^2}}. \tag{7.1}$$

Since $f^2(\xi) - L^2(\xi) = R^2 - \xi^2 - (s\xi + h)^2$, we obtain

$$J_2 = \int\limits_{-x(s)}^{y(s)} (\xi - a)^2 (f^2(\xi) - L^2(\xi)) d\xi = \int\limits_{-x(s)}^{y(s)} (\xi - a)^2 (R^2 - \xi^2 - (s\xi + h)^2) d\xi.$$

Integrating by parts twice, we have that

$$\begin{aligned} J_2 &= \frac{1}{3}\left((\xi - a)^3 (R^2 - \xi^2 - (s\xi + h)^2)|_{-x}^{y} + 2 \int\limits_{-x(s)}^{y(s)} (\xi - a)^3 ((1 + s^2)\xi + sh) d\xi \right) \\ &= \frac{1}{6}\left((\xi - a)^4 ((1 + s^2)\xi + sh)|_{-x}^{y} - \int\limits_{-x(s)}^{y(s)} (\xi - a)^4 (1 + s^2) d\xi \right) \\ &= \frac{1}{6}\Big((y - a)^4 ((1 + s^2) y + sh) - (x + a)^4 (-(1 + s^2) x + sh) \\ &\quad - \frac{1}{5}(1 + s^2)((y - a)^5 + (x + a)^5) \Big). \end{aligned}$$

Since $x + a = y - a$, this equals

$$\begin{aligned} &\frac{1}{6}(1 + s^2)\left(y(y - a)^4 + x(x + a)^4 - \frac{1}{5}((y - a)^5 + (x + a)^5) \right) \\ &\qquad = \frac{1}{6}(1 + s^2)\left((x + a)^4 (y + x) - \frac{2}{5}(x + a)^5 \right) \\ &\qquad = \frac{1}{30}(1 + s^2)(x + a)^4 \left(5(y + x) - 2(x + a) \right). \end{aligned}$$

Finally, using (7.1) we obtain

$$J_2 = \frac{8}{30(1 + s^2)^{\frac{3}{2}}} (R^2 - r^2)^{\frac{5}{2}}.$$

It follows that the desired constant in the right-hand side of the second equation in (5.1) is

$$c = \frac{4\pi}{15}(R^2 - r^2)^{\frac{5}{2}}. \tag{7.2}$$

Similarly, using the fact that

$$R^2 - \xi^2 - (s\xi + h)^2 = -(1 + s^2)(\xi + x)(\xi - y),$$

we have

$$J_3 = \int\limits_{-x(s)}^{y(s)} (f^2(\xi) - L^2(\xi))^2 d\xi = (1 + s^2)^2 \int\limits_{-x(s)}^{y(s)} (\xi + x)^2 (\xi - y)^2 d\xi.$$

Making the substitution $\xi = \eta(y+x) - x$ we obtain

$$J_3 = \frac{1}{30}(1+s^2)^2(y+x)^5 = \frac{32}{30}\frac{(R^2-r^2)^{\frac{5}{2}}}{(1+s^2)^{\frac{1}{2}}}.$$

Thus, the constant on the right-hand side of the third equation in (5.1) also has the value given by (7.2).

7.2. **The first equation in (5.5).** Recall that $a = -h'$. We differentiate the first equation with respect to s, obtaining

$$\int\limits_{-x(s)}^{y(s)} (f^2(\xi) - L^2(\xi))d\xi = -\frac{2}{a'}\int\limits_{-x(s)}^{y(s)} (\xi - a)^2(s\xi + h(s))d\xi.$$

Differentiating once again, we have

$$(a')^2 \int\limits_{-x(s)}^{y(s)} (\xi - a)(s\xi + h(s))d\xi =$$

$$a'\left(-2a'I_{12} + I_{13} + y'(y-a)^2(sy+h) + x'(x+a)^2(-sx+h)\right) - a''I_{11},$$

where

$$I_{11} = \int\limits_{-x(s)}^{y(s)} (\xi - a)^2(s\xi + h(s))d\xi, \quad I_{12} = \int\limits_{-x(s)}^{y(s)} (\xi - a)(s\xi + h(s))d\xi,$$

$$I_{13} = \int\limits_{-x}^{y} (\xi - a)^3 d\xi.$$

This yields

$$3(a')^2 I_{12} + a''I_{11} - a'I_{13} = a'\left(y'(y-a)^2(sy+h) + x'(x+a)^2(-sx+h))\right). \tag{7.3}$$

We see that our first equation reads as

$$a'\left(y'(y-a)^2(sy+h) + x'(x+a)^2(-sx+h))\right) = P(s,x,y,h), \tag{7.4}$$

where

$$P(s,x,y,h) = 3(a')^2 I_{12} + a''I_{11} - a'I_{13} \tag{7.5}$$

We compute the above integrals using integration by parts,

$$I_{11} = \frac{(\xi - a(s))^3}{3}(s\xi + h(s))|_{-x(s)}^{y(s)} - \frac{s}{3}\int\limits_{-x(s)}^{y(s)} (\xi - a)^3 d\xi$$

$$= \frac{(y(s) - a(s))^3}{3}(sy(s) + h(s)) + \frac{(x(s) + a(s))^3}{3}(-sx(s) + h(s)) - \frac{s}{12}((y(s) - a(s))^4 - (x(s) + a(s))^4). \tag{7.6}$$

Observe that in the case of Euclidean balls, we have

$$I_{11} = \frac{1}{12}(y - a)^3(3s(y - x) + 8h + 2sa)$$
$$= \frac{2}{3}(y - a)^3 r\left(\sqrt{1 + s^2} - \frac{s^2}{\sqrt{1 + s^2}}\right) = \frac{2r}{3}\frac{(R^2 - r^2)^{\frac{3}{2}}}{(1 + s^2)^2} > 0.$$

We also compute

$$I_{12} = \frac{(\xi - a)^2}{2}(s\xi + h(s))\Big|_{-x}^{y} - s\frac{(\xi - a)^3}{6}\Big|_{-x}^{y},$$

or

$$I_{12} = \frac{1}{2}\left((y - a)^2(sy + h) - (x + a)^2(-sx + h)\right) - \frac{s}{6}\left((y - a)^3 + (x + a)^3\right). \tag{7.7}$$

Finally,

$$I_{13} = \int_{-x}^{y} (\xi - a)^3\, d\xi = \frac{1}{4}\left((y - a)^4 - (x + a)^4\right). \tag{7.8}$$

Now we substitute (7.6), (7.7) and (7.8) into (7.5), grouping terms with the same powers of $(y - a), (x + a)$. Thus,

$$\begin{aligned} P &= 3(a')^2 I_{12} + a'' I_{11} - a' I_{13} \\ &= \frac{3}{2}(a')^2\left[(y - a)^2(sy + h) - (x + a)^2(-sx + h)\right] \\ &+ (y - a)^3\left[(a'')\frac{sy + h}{3} - 3(a')^2\frac{s}{6}\right] + (x + a)^3\left[(a'')\frac{-sx + h}{3} - 3(a')^2\frac{s}{6}\right] \\ &+ \left((y - a)^4 - (x + a)^4\right)\left[(a'')(-\frac{s}{12}) - \frac{a'}{4}\right]. \end{aligned} \tag{7.9}$$

Since $a' = \frac{-r}{(1+s^2)^{3/2}}$ and $a'' = \frac{3rs}{(1+s^2)^{5/2}}$, the coefficient of the last term is

$$(a'')(-\frac{s}{12}) - \frac{a'}{4} = \frac{r}{4(1 + s^2)^{5/2}}.$$

To simplify the first summand in (7.9), we rewrite

$$(y - a)^2(sy + h) = (y - a)^2(s(y - a) + as + h),$$

$$(x+a)^2(-sx+h) = (x+a)^2(-s(x+a)+as+h).$$

Then

$$\begin{aligned}P = \frac{3}{2}(a')^2 &\left[s(y-a)^3 + (y-a)^2(as+h) + s(x+a)^3 - (x+a)^2(sa+h)\right] \\ &+ \frac{sa''}{3}(y-a)^4 + (y-a)^3\left[\frac{a''(as+h)}{3} - 3(a')^2\frac{s}{6}\right] \\ &- \frac{sa''}{3}(x+a)^4 + (x+a)^3\left[\frac{a''(as+h)}{3} - 3(a')^2\frac{s}{6}\right] \\ &+ \frac{r}{4(1+s^2)^{5/2}}\left((y-a)^4 - (x+a)^4\right).\end{aligned}$$

Grouping similar powers again, we have

$$\begin{aligned}P = &\frac{3}{2}(a')^2(as+h)\left((y-a)^2 - (x+a)^2\right) \\ &+ \left(\frac{3s}{2}(a')^2 + \frac{a''(as+h)}{3} - 3(a')^2\frac{s}{6}\right)\left((y-a)^3 + (x+a)^3\right) \\ &+ \left(\frac{r}{4(1+s^2)^{5/2}} + \frac{sa''}{3}\right)\left((y-a)^4 - (x+a)^4\right).\end{aligned}$$

Now we simplify the coefficients:

$$\frac{3}{2}(a')^2(as+h) = \frac{3r^3}{2(1+s^2)^{7/2}},$$

$$s(a')^2 + \frac{a''(as+h)}{3} = \frac{2r^2 s}{(1+s^2)^3},$$

and

$$\frac{r}{4(1+s^2)^{5/2}} + \frac{sa''}{3} = \frac{r(1+4s^2)}{4(1+s^2)^{5/2}}.$$

The final expression for P is

$$P(s,x,y) = \frac{3r^3}{2(1+s^2)^{7/2}}\left((y-a)^2 - (x+a)^2\right) \tag{7.10}$$

$$+\frac{2r^2 s}{(1+s^2)^3}\left((y-a)^3 + (x+a)^3\right) + \frac{r(1+4s^2)}{4(1+s^2)^{5/2}}\left((y-a)^4 - (x+a)^4\right).$$

7.3. **The second equation in (5.5).** We write it as

$$\int_{-x(s)}^{y(s)} (\xi-a)^2(f^2(\xi) - L^2(\xi))d\xi = \frac{c}{\pi(1+s^2)^{\frac{3}{2}}}.$$

We differentiate with respect to s and use $h' = -a$ and the first equation in (5.1) to obtain

$$-2a'(s)\int\limits_{-x(s)}^{y(s)}(\xi-a)(f^2(\xi)-L^2(\xi))d\xi-2\int\limits_{-x(s)}^{y(s)}(\xi-a)^2(s\xi+h(s))(\xi+h'(s))d\xi$$

$$=-2\int\limits_{-x(s)}^{y(s)}(\xi-a)^2(s\xi+h(s))(\xi+h'(s))d\xi=\left(\frac{c}{\pi(1+s^2)^{\frac{3}{2}}}\right)',$$

i.e.,

$$\int\limits_{-x(s)}^{y(s)}(\xi-a)^3(s\xi+h(s))d\xi=-\left(\frac{c}{2\pi(1+s^2)^{\frac{3}{2}}}\right)'.$$

We differentiate again to obtain

$$(y-a)^3(sy+h)y'-(x+a)^3(-sx+h)x'=Q(s,x,y), \tag{7.11}$$

where

$$Q=3a'\int\limits_{-x}^{y}(\xi-a)^2(s\xi+h)d\xi-\int\limits_{-x}^{y}(\xi-a)^4d\xi-\left(\frac{c}{2\pi(1+s^2)^{\frac{3}{2}}}\right)''$$
$$=3a'I_{11}-\frac{1}{5}\left((y-a)^5-(x+a)^5\right)+\frac{3c}{2\pi}\left(\frac{1-4s^2}{(1+s^2)^{7/2}}\right).$$

By (7.6), this equals

$$3a'\Big[\Big(\frac{(y(s)-a(s))^3}{3}(sy(s)+h(s))+\frac{(x(s)+a(s))^3}{3}(-sx(s)+h(s))$$
$$-\frac{s}{12}((y(s)-a(s))^4-(x(s)+a(s))^4)\Big]$$
$$-\frac{1}{5}\left((y-a)^5-(x+a)^5\right)+\frac{3c}{2\pi}\left(\frac{1-4s^2}{(1+s^2)^{7/2}}\right).$$

Using the expression for I_{11} given by (7.6), and once again the identities $(y-a)^2(sy+h)=(y-a)^2(s(y-a)+as+h)$, $(x+a)^2(-sx+h)=(x+a)^2(-s(x+a)+as+h)$, we group terms with the same powers and finally obtain

$$Q=\frac{3a's}{4}\left((y-a)^4-(x+a)^4\right)+\frac{a'r}{(1+s^2)^{\frac{1}{2}}}\left((y-a)^3+(x+a)^3\right)$$
$$-\frac{1}{5}\left((y-a)^5+(x+a)^5\right)+\frac{3c}{2\pi}\frac{1-4s^2}{(1+s^2)^{\frac{7}{2}}}. \tag{7.12}$$

7.4. **The second equation in (6.2).** Differentiating the second equation in (6.1) and using the fact that $f(y(s)) = sy(s)+h(s), -f(-x(s)) = -sx(s)+h(s)$, we obtain

$$\int_x^y (s\xi - sa + sa + h)(\xi - a)d\xi$$

$$= s\int_{-x}^y (\xi - a)^2 d\xi + (sa+h)\int_{-x}^y (\xi - a)d\xi = \frac{\tilde{c}s}{2(1+s^2)^{\frac{3}{2}}}.$$

Taking the integrals in the left hand side we obtain

$$s\Big(\frac{(y-a)^3}{3} + \frac{(x+a)^3}{3}\Big) + (sa+h)\Big(\frac{(y-a)^2}{2} - \frac{(x+a)^2}{2}\Big) = \frac{\tilde{c}s}{2(1+s^2)^{\frac{3}{2}}}.$$

Differentiating both parts once again and using the fact that $h' = -a$, we have

$$\Big(\frac{(y-a)^3}{3} + \frac{(x+a)^3}{3}\Big) + s((y-a)^2(y'-a') + (x+a)^2(x'+a'))$$
$$+ sa'\Big(\frac{(y-a)^2}{2} - \frac{(x+a)^2}{2}\Big) + (sa+h)((y-a)(y'-a') - (x+a)(x'+a'))$$
$$= \frac{\tilde{c}s}{2(1+s^2)^{\frac{3}{2}}}.$$

Finally, using the fact that

$$sa + h = \frac{1}{\sqrt{1+s^2}}, \qquad a' = -\frac{1}{(1+s^2)^{\frac{3}{2}}},$$

and rearranging the terms, we obtain

$$y'\Big(s(y-a)^2 + \frac{y-a}{\sqrt{1+s^2}}\Big) + x'\Big(s(x+a)^2 - \frac{x+a}{\sqrt{1+s^2}}\Big)$$
$$= a'\Big(s(y-a)^2 + \frac{y-a}{\sqrt{1+s^2}}\Big) - a'\Big(s(x+a)^2 - \frac{x+a}{\sqrt{1+s^2}}\Big)$$
$$- \Big(\frac{(y-a)^3}{3} + \frac{(x+a)^3}{3}\Big) + \frac{s}{(1+s^2)^{\frac{3}{2}}}\Big(\frac{(y-a)^2}{2} - \frac{(x+a)^2}{2}\Big) + \frac{\tilde{c}}{2}\frac{1-2s^2}{(1+s^2)^{\frac{5}{2}}},$$

which is (6.3).

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