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Denseness for norm attaining operator-valued functions

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Abstract

In this note we offer a short, constructive proof for Hilbert spaces of Lindenstrauss' famous result on the denseness of norm attaining operators. Specifically, we show given any $A \in \mathscr{L}(\mathscr{H})$ there is a sequence of rank-1 operators K_n such that $A + K_n$ is norm attaining for each n and K_n converges in norm to zero. We then apply our construction to establish denseness results for norm attaining operator-valued functions. © 2001 Elsevier Science Inc. All rights reserved.

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Let $\mathscr{H}$ be a separable Hilbert space and let $\mathscr{L}(\mathscr{H})$ denote the bounded linear operators on $\mathscr{H}$. Recall that $A \in \mathscr{L}(\mathscr{H})$ is called *norm attaining* if there exists a unit vector x such that $\|Ax\| = \|A\|$. We denote by $\mathscr{A}$ the collection of norm attaining operators in $\mathscr{L}(\mathscr{H})$.

Norm attaining operators have many applications, some of which are discussed in [1–3]. The denseness $\mathscr{A}$ in $\mathscr{L}(\mathscr{H})$ was established in [6] and, in a more constructive way in [4]. We begin this note with our own constructive proof of the denseness of $\mathscr{A}$. This proof may not be new but we were unable to find it in the literature and we need the specific details of this proof for our study of denseness for norm attaining operator-valued functions.

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Lemma 1. *Let B be a non-negative bounded linear operator on $\mathscr{H}$. For every $\epsilon > 0$ there exists a rank-1 non-negative operator, C, such that $\|C\| \leqslant \epsilon$ and*

$$(B + C)x = \|B + C\|x$$

for some unit vector $x \in (\mathrm{Ker}(B))^{\perp}$.

Proof. Let $\epsilon > 0$. Since

$$\|B\| = \sup_{\|x\|=1} \langle Bx, x\rangle = \sup\left\{\langle Bx, x\rangle\colon\ \|x\| = 1,\ x \in (\mathrm{Ker}(B))^{\perp}\right\},$$

there exists a unit vector $y \in (\mathrm{Ker}(B))^{\perp}$ such that $\|B\| - \epsilon/2 \leqslant \langle By, y\rangle$. Define

$$C = \epsilon\langle \cdot\,, y\rangle y.$$

Then C is a non-negative rank-1 operator of norm ϵ. Moreover,

$$\|B + C\| = \sup_{\|x\|=1} \langle (B + C)x, x\rangle \geqslant \langle (B + C)y, y\rangle \geqslant \|B\| + \frac{\epsilon}{2}.$$

Now $B + C$ is non-negative and so $\|B + C\|$ lies in the spectrum, $\sigma(B + C)$, of $B + C$. Since C is compact, Weyl's theorem implies $\sigma_{\mathrm{ess}}(B + C) = \sigma_{\mathrm{ess}}(B)$. But the spectrum of B is bounded by $\|B\|$ and hence $\|B + C\|$ must lie in the discrete spectrum of $B + C$. In other words, there exists a unit vector x such that

$$(B + C)x = \|B + C\|x.$$

Finally, we can assume without loss of generality that $x \in (\mathrm{Ker}(B))^{\perp}$. Indeed, $\mathscr{H}$ decomposes as the orthogonal direct sum of $\mathrm{Ker}(B) \oplus (\mathrm{Ker}(B))^{\perp}$. Hence if we express $x = x^1 + x^2$ in this decomposition, we see $(B + C)x^1 = \langle x^1, y\rangle y = 0$. □

To apply Lemma 1 for general operator, we will need to recall the properties of polar decompositions. Let $A \in \mathscr{L}(\mathscr{H})$ and let $A = U|A|$ denote the polar decomposition of A (cf. [7,8]). Recall that $|A| = \sqrt{A^*A}$ is a non-negative operator and U is a partial isometry. More precisely, $\mathrm{Ker}(U) = \mathrm{Ker}(|A|) = \mathrm{Ker}(A)$ and U is an isometry on $(\mathrm{Ker}(U))^{\perp}$.

Theorem 1. *Let $A \in \mathscr{L}(\mathscr{H})$. There exists a sequence of rank-1 operator K_n converging to zero in operator norm such that $A + K_n$ is norm attaining for each n. In particular, the norm attaining operators are norm dense in $\mathscr{L}(\mathscr{H})$.*

Proof. Let $A = U|A|$ be the polar decomposition of A. By Lemma 1, we know for each n there exists a non-negative rank-1 operator C_n with $\|C_n\| < 1/n$ such that for some unit vector $x_n \in (\mathrm{Ker}(|A|))^{\perp}$,

$$(|A| + C_n)x_n = \||A| + C_n\|x_n.$$

If we premultiply this expression by U and define $K_n = UC_n$, we get

$$(A + K_n)x_n = \||A| + C_n\|Ux_n.$$

But $x_n \in (\mathrm{Ker}(|A|))^{\perp} = (\mathrm{Ker}(U))^{\perp}$ and U is an isometry on this set. Thus,

$$\|(A + K_n)x_n\| = \||A| + C_n\|.$$

Now, $\|(A + K_n)\| = \|U(|A| + C_n)\| \leqslant \|(|A| + C_n)\|$. Hence the previous equation actually shows that $\|(A + K_n)\| = \|(|A| + C_n)\|$ and this norm is attained at the vector x_n. This establishes the theorem. □

This completes our constructive proof of the Lindenstrauss denseness theorem and we are ready to apply the above theorem to our study of operator-valued functions. Let $\mathscr{U}$ be an open subset of $\mathbb{C}$ and let $f : \mathscr{U} \to \mathscr{L}(\mathscr{H})$ be an operator-valued function. Recall that f is *continuous* if for each $z \in \mathscr{U}$ and $\epsilon > 0$, there exists a $\delta > 0$ such that $|z - w| < \delta$ implies $\|f(z) - f(w)\| < \epsilon$. The following lemma is an analog of Lemma 1 for continuous operator-valued functions with values in the non-negative operators.

Lemma 2. *Let $\mathscr{U}$ be an open subset of $\mathbb{C}$ and let K be a compact subset of $\mathscr{U}$. Suppose f is a continuous function on $\mathscr{U}$ with values in the non-negative linear operator on $\mathscr{H}$. For every $\epsilon > 0$ there exists finite-rank non-negative operator P of norm ϵ such that $f(z) + P$ is norm attaining for all $z \in K$.*

Proof. Let $\epsilon > 0$ and let K be a compact subset of $\mathscr{U}$. Since f is uniformly continuous on K we can fix a $\delta > 0$ such that for every $z, w \in K$, if $|z - w| < \delta$, then $\|f(z) - f(w)\| < \epsilon/8$. Compactness implies that the collection of open balls $\{B(z, \delta)\}_{z \in K}$ admits a finite subcover $\{B(z_1, \delta), \ldots, B(z_n, \delta)\}$ of K.

Now each $f(z)$ is a non-negative and so for each $z \in K$, there exists a unit vector ψ_z such that $\langle f(z)\psi_z, \psi_z\rangle > \|f(z)\| - \epsilon/2$. Let $\{\phi_1, \ldots, \phi_m\}$ denote the Gram–Schmidt ortho-normalization of the vectors $\{\psi_{z_1}, \ldots, \psi_{z_n}\}$ chosen for our finite sub cover. Define

$$P = \epsilon \sum_{i=1}^{m} \langle \cdot\,, \phi_i\rangle \phi_i.$$

Clearly, P is a non-negative finite-rank operator of norm ϵ. We claim that $f(w) + P$ is norm attaining for each $w \in K$. Let $w \in B(z_i, \delta)$. Then

$$\begin{aligned}
\|f(w) + P\| &\geqslant \langle (f(w) + P)\psi_{z_i}, \psi_{z_i}\rangle \\
&= \langle (f(w) - f(z_i))\psi_{z_i}, \psi_{z_i}\rangle + \langle (f(z_i) + P)\psi_{z_i}, \psi_{z_i}\rangle \\
&> \langle (f(w) - f(z_i))\psi_{z_i}, \psi_{z_i}\rangle + \|f(z_i)\| - \frac{\epsilon}{2} + \epsilon \\
&> -\frac{\epsilon}{8} + \|f(z_i)\| + \frac{\epsilon}{2} \\
&> \|f(w)\| - \frac{\epsilon}{8} + \frac{3\epsilon}{8} \\
&> \|f(w)\|.
\end{aligned}$$

As in Lemma 1, this implies that $f(w) + P$ is norm attaining. □

The previous lemma extends with exactly the same proof to hold for any bounded open domain $\mathcal{U}$.

Remark 1. Let $\mathcal{U}$ be a bounded open subset of $\mathbb{C}$ and suppose f is a continuous function on $\overline{\mathcal{U}}$, the closure of $\mathcal{U}$. Assume that f takes values on $\mathcal{U}$ in the non-negative linear operators on $\mathcal{H}$. Then for every $\epsilon > 0$ there exists finite-rank non-negative operator P of norm ϵ such that $f(z) + P$ is norm attaining for all $z \in \mathcal{U}$.

Proof. By assumption, the set $\overline{\mathcal{U}}$ is compact and hence, f is uniformly continuous on $\overline{\mathcal{U}}$. The calculation in the previous lemma goes through unchanged with the closure, $\overline{\mathcal{U}}$, of $\mathcal{U}$ playing the role of K. □

As in Theorem 1, the previous result extends to a result about operator-valued functions. Throughout the rest of this note, for any operator- valued function f on $\mathcal{U}$, we define the functions U and $|f|$ on $\mathcal{U}$ so that for each z in $\mathcal{U}$, $f(z) = U(z)|f|(z)$ is the polar decomposition of $f(z)$. It is easy to check that $|f|$ inherits continuity from f. However, in general, the function U is not even weakly continuous. The problem is that there can be arbitrarily close points z and w such that $\mathrm{Ker}(f(z)) \neq \mathrm{Ker}(f(w))$. But this says $\mathrm{Ker}(U(z)) \neq \mathrm{Ker}(U(w))$ and so there exists a unit vector $\psi \in \mathrm{Ker}(U(Z)) \cap \mathrm{Ker}(U(w))^{\perp}$ such that

$$\big\|\big[U(z) - U(w)\big]\psi\big\| = \|U(w)\psi\| = \|\psi\| = 1.$$

Thus, although the natural perturbation given by Theorem 1 (i.e. $P(z) = U(z)P$ for P given by the previous lemma) is an arbitrary small perturbation taking $f(z)$ into the norm attaining operators for every z, the function $P(z)$ is generally not continuous and hence not very useful. To help avoid this pathology we recall the following definition (cf. [5]).

Definition. Let $\mathcal{U}$ be an open subset of $\mathbb{C}$ and let $f : \mathcal{U} \to \mathcal{L}(\mathcal{H})$ be an operator-valued function. We say that f is *harmonic* if for each bounded linear functional Λ on $\mathcal{L}(\mathcal{H})$ the function $\Lambda(f) : \mathcal{U} \to \mathbb{C}$ is harmonic.

We only sketch a proof of the following lemma but a detailed proof appears in [9].

Lemma 3. *Let $\mathcal{U}$ be an connected open subset of $\mathbb{C}$ and f be a harmonic function on $\mathcal{U}$ with values in non-negative operators. Then for each $\psi \in \mathcal{H}$, $\langle f(z)\psi, \psi\rangle$ is either identically zero or never zero.*

Proof (*Sketch*). Let $\psi \in \mathscr{H}$ and $\mu(z) = \langle f(z)\psi, \psi\rangle$. Let $\mathscr{U}$ be a connected, simply connected open subset of $\mathbb{C}$. Suppose for some $z_0 \in \mathscr{U}$, $\mu(z_0) = 0$. The mean value property for harmonic functions implies that for all sufficiently small r the relation

$$0 = \mu(z_0) = \frac{1}{2\pi}\int_0^{2\pi} \mu(z_0 + re^{i\theta})\,d\theta \geqslant 0$$

holds. Hence μ vanishes on some open disk $B(z_0, R)$. The simply connectedness implies that μ vanishes on all of $\mathscr{U}$, since μ has a harmonic conjugate on $\mathscr{U}$. The general case of $\mathscr{U}$ open and connected follows from the fact that any two points of $\mathscr{U}$ can be put in an open connected, simply connected subset of $\mathscr{U}$. □

We now can characterize a class of operator-valued functions which have arbitrarily small continuous finite-rank perturbations into the norm attaining operators.

Theorem 2. *Let $\mathscr{U}$ be a bounded open subset of $\mathbb{C}$ and suppose that f is a continuous operator-valued function on $\overline{\mathscr{U}}$ such that $|f|$ is harmonic on $\mathscr{U}$. Then for every $\epsilon > 0$ there exists a continuous finite-rank operator-valued function $\lambda(z)$ on $\mathscr{U}$ with $\|\lambda(z)\| \leqslant \epsilon$ for all z in $\mathscr{U}$ such that each $f(z) + \lambda(z)$ is norm attaining for all $z \in \mathscr{U}$.*

Proof. Since it suffices to establish the theorem on each component of $\mathscr{U}$, we assume without loss of generality that $\mathscr{U}$ is connected. Remark 1 applies to $|f|$ and hence there is an orthonormal collection $\{\gamma_1, \ldots, \gamma_n\}$ defining an operator $P = \epsilon\sum_{i=1}^{n}\langle\cdot\,, \gamma_i\rangle\gamma_i$ such that $|f|(z) + P \in \mathscr{A}$ for all $z \in \mathscr{U}$. We will show that $\lambda(z) = U(z)P$ establishes the theorem. Lemma 3 implies that

$$\mathrm{Ker}(|f|(z)) = \mathrm{Ker}(f(z)) = \mathrm{Ker}(U(z))$$

is independent of z. Indeed, suppose $|f|(z_0)\psi = 0$. Then $\langle|f|(z)\psi, \psi\rangle$ vanishes at z_0 and hence on all of $\mathscr{U}$. But then for every $z \in \mathscr{U}$

$$\left\|\sqrt{|f|(z)}\psi\right\|^2 = \left\langle\sqrt{|f|(z)}\psi, \sqrt{|f|(z)}\psi\right\rangle = \langle|f|(z)\psi, \psi\rangle = 0.$$

Hence $|f|(z)\psi = \sqrt{|f|(z)}\sqrt{|f|(z)}\,\psi = 0$ for all z in $\mathscr{U}$.

We can assume without loss of generality that each γ_i is in $\mathrm{Ker}(|f|)^{\perp}$ (since it suffices to construct P for $|f|$ viewed as a function into the linear mappings on the Hilbert space $\mathrm{Ker}(|f|)^{\perp}$). By construction, for each z, $|f|(z) + P$ achieves its norm at some unit vector in $\mathrm{Ker}(|f|)^{\perp}$. Hence the argument given in Theorem 1 shows that $f(z) + \lambda(z) \in \mathscr{A}$ for each $z \in \mathscr{U}$. It is evident that the norm of each $\lambda(z)$ is at most ϵ and so it remains only to show the continuity of λ.

To that end, it clearly suffices to show that any fixed summand $\epsilon\langle\cdot\,, \gamma_i\rangle U(z)\gamma_i$ of λ is continuous. In other words, we need only show that $U(z)$ is strongly continuous on $\mathrm{Ker}(|f|)^{\perp} = \overline{\mathrm{Ran}(|f|)}$. Fix $z_0 \in \mathscr{U}$ and let $|f|(z_0)\psi_1 \in \mathrm{Ran}(|f|(z_0))$. Then $U(z_0)|f|(z_0)\psi_1 = f(z_0)\psi_1$ and, for all $w \in \mathscr{U}$, $U(w)|f|(z_0)\psi_1 = U(w)|f|(w)\psi_1 + U(w)[|f|(z_0) - |f|(w)]\psi_1$.

Hence

$$\begin{aligned}&\big\|(U(z_0)-U(w))|f|(z_0)\psi_1\big\|\\&\quad=\big\|\big[f(z_0)-f(w)\big]\psi_1+U(w)\big[|f|(w)-|f|(z_0)\big]\psi_1\big\|\\&\quad\leqslant\big\|f(z_0)-f(w)\big\|\|\psi_1\|+\big\||f|(w)-|f|(z_0)\big\|\|\psi_1\|.\end{aligned}$$

The inequality extends by continuity to $\overline{\mathrm{Ran}(|f|(z_0))}$. Thus U is strongly continuous at z_0 on $\overline{\mathrm{Ran}(|f|(z_0))}=\overline{\mathrm{Ran}(|f|)}$. □

The previous proof, combined with Lemma 2, implies that Theorem 2 holds on any compact subset of an open domain $\mathscr{U}$. We intend to consider the issue of extending Theorem 2 to any open domain of $\mathbb{C}$ in a subsequent paper.

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