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Lipschitz-type bounds for the map $A \to |A|$ on $\mathscr{L}(\mathscr{H})$

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Abstract

It is well known that the absolute value map on the self-adjoint operators on an infinite dimensional Hilbert spaces is not Lipschitz continuous, although Lipschitz continuity holds on certain subsets of operators. In this note, we provide an elementary proof that the absolute value map is Lipschitz continuous on the set of all operators which are bounded below in norm by any fixed positive constant. Applications are indicated.
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Throughout this note, let $\mathscr{H}$ be an infinite dimensional Hilbert space and let $\mathscr{L}(\mathscr{H})$ denote the algebra of bounded linear operators on $\mathscr{H}$. Denote by $\mathscr{L}^{\mathrm{S.A.}}(\mathscr{H})$ and $\mathscr{L}^{\mathrm{P}}(\mathscr{H})$ the subsets of $\mathscr{L}(\mathscr{H})$ consisting of self-adjoint and non-negative operators. Recall that a map $f : \mathscr{L}(\mathscr{H}) \to \mathscr{L}(\mathscr{H})$ is called *Lipschitz continuous* if there exists a constant M such that for every $A, B \in \mathscr{L}(\mathscr{H})$, $\|f(A) - f(B)\| \leqslant M\|A - B\|$. In the text [12], it was pointed out that it was unknown whether the absolute value mapping, $A \to |A| = \sqrt{A^*A}$, is Lipschitz continuous on $\mathscr{L}(\mathscr{H})$. In his 1973 paper [8], Kato provided a counterexample showing that Lipschitz continuity of the absolute value map fails on the subset of self-adjoint operators, $\mathscr{L}^{\mathrm{S.A.}}(\mathscr{H})$. Since then other studies of the Lipschitz continuity of the absolute value map on self-adjoint operators have been offered. Notably, some interesting counter examples

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were given in [11] and the Lipschitz continuity of the absolute value map on finite dimensional spaces was proved in [7]. Many other logarithmic Lipschitz bounds have been developed for various important functions on operators (cf. [1,2,7,9]).

In [4,5], using Frechet derivatives, it is shown that the absolute value map satisfies a bound of the form $\||A| - |B|\| \leqslant a_1\|A - B\| + a_2\|A - B\|^2 + \mathcal{O}(\|A - B\|^3)$ provided A is invertible and B is sufficiently close to A. In this note, we drop the invertibility requirements and the error terms $a_2\|A - B\|^2 + \mathcal{O}(\|A - B\|^3)$ in the bound. Specifically, we show that the absolute value map is Lipschitz continuous on the set of all operators which are bounded below in norm by any fixed positive constant. We apply our construction to provide an example which shows that there are subsets of $\mathscr{L}^{\mathrm{S.A.}}(\mathscr{H})$ which properly contain $\mathscr{L}^{\mathrm{P}}(\mathscr{H})$ on which the absolute value map is norm Lipschitz continuous. In particular, this provides an example of a subset of $\mathscr{L}^{\mathrm{S.A.}}(\mathscr{H})$ containing operators of arbitrarily small norm on which the absolute value map is Lipschitz continuous. This, in turn, yields an important interpretation in terms of the logarithmic Lipschitz continuity property of the absolute value map which was established in [8]. We begin by showing that the square root function is Lipschitz continuous on positive operators whose norms are bounded below by a fixed constant.

Definition. For any constant $R > 0$, define

$$B_R = \{A \in \mathscr{L}(\mathscr{H})\colon \|A\| > R\} \quad \text{and} \quad B_R^+ = \{A \in \mathscr{L}^{\mathrm{P}}(\mathscr{H})\colon \|A\| > R\}.$$

Lemma 1. *Fix R with $0 < R < 1/16$. The square root map is Lipschitz continuous on the set B_R^+. Specifically, whenever A and B are operators in $\mathscr{L}^{\mathrm{P}}(\mathscr{H})$ with $R < \|B\| \leqslant \|A\|$,*

$$\left\|\sqrt{A} - \sqrt{B}\right\| \leqslant \frac{\sqrt{8}\,\|A - B\|}{\sqrt{R\|A\|}}. \tag{1}$$

Thus, for all $A, B \in B_R^+$,

$$\left\|\sqrt{A} - \sqrt{B}\right\| \leqslant \frac{\sqrt{8}\,\|A - B\|}{R}. \tag{2}$$

Proof. Fix R with $0 < R < 1/16$. Let $A, B \in B_R^+$ with $R < \|B\| \leqslant \|A\|$. We consider separately the cases: (i) $\sqrt{R}\|A\| > \|B\|$ and (ii) $\sqrt{R}\|A\| \leqslant \|B\|$. In case (i), $\sqrt[4]{R}\sqrt{\|A\|} > \sqrt{\|B\|}$ and so

$$\left(1 - \sqrt[4]{R}\right)\sqrt{\|A\|} < \sqrt{\|A\|} - \sqrt{\|B\|}.$$

Since $R < 1/16$ implies that $\sqrt{R} < \left(1 - \sqrt[4]{R}\right)$, this yields:

$$\left\|\sqrt{A} - \sqrt{B}\right\| \leqslant \left\|\sqrt{A}\right\| + \left\|\sqrt{B}\right\| = \frac{\|A\| - \|B\|}{\sqrt{\|A\|} - \sqrt{\|B\|}} \leqslant \frac{\|A\| - \|B\|}{\sqrt{R}\sqrt{\|A\|}} \leqslant \frac{\|A - B\|}{\sqrt{R}\sqrt{\|A\|}}.$$

Now assume $\sqrt{R}\|A\| \leqslant \|B\| \leqslant \|A\|$. That is, $\|A\|/\|B\| < 1/\sqrt{R}$.

Define $C = A/\|A\|$ and $D = B/\|A\|$ and let $\alpha = \|B\|/\|A\|$. Let $\Gamma = \{z \in \mathbb{C}: |z| = 1 + \alpha\}$ and $m = \frac{1}{2}$.

$$\begin{aligned}
\|C^m - D^m\| &= \left\| \frac{1}{2\pi i} \int_\Gamma w^m [(w - C)^{-1} - (w - D)^{-1}] \, \mathrm{d}w \right\| \\
&\leqslant \sqrt{8} \sup_{w \in \Gamma} \|(w - C)^{-1} - (w - D)^{-1}\| \\
&\leqslant \sqrt{8} \sup_{w \in \Gamma} \|(w - C)^{-1}\| \|C - D\| \|(w - D)^{-1}\| \\
&\leqslant \frac{\sqrt{8}}{\operatorname{dist}(\Gamma, \sigma(C))} \|C - D\| \frac{1}{\operatorname{dist}(\Gamma, \sigma(D))} \\
&= \frac{\sqrt{8}\|C - D\|}{(1 + \alpha - \|C\|)(1 + \alpha - \|D\|)} \\
&= \frac{\sqrt{8}\|A\|\|C - D\|}{\|B\|} \\
&\leqslant \frac{\sqrt{8}\|C - D\|}{\sqrt{R}}.
\end{aligned}$$

Here we used the fact $\|E^{-1} - F^{-1}\| = \|E^{-1}(E - F)F^{-1}\| \leqslant \|E^{-1}\|\|E - F\|\|F^{-1}\|$ which holds for any invertible operators E and F, as well as the fact that for each $w \in \Gamma$, $\|(w - C)^{-1}\| = 1/\operatorname{dist}(w, \sigma(C))$.

This immediately implies that

$$\frac{1}{\sqrt{\|A\|}} \left\| \sqrt{A} - \sqrt{B} \right\| \leqslant \frac{\sqrt{8}\,\|A - B\|}{\sqrt{R}\,\|A\|}$$

which is inequality (1) of the lemma. Statement (2) of the lemma follows directly from (1) and the fact that $R < \|A\|$. □

It should be mentioned that a better estimate, $\left\| \sqrt{A} - \sqrt{B} \right\| \leqslant \left(1/2\sqrt{R}\right)\|A - B\|$, has been established in [3,10], although these results require a compactness assumption such as traceclass. The above lemma applies to all non-negative operators and hence it implies a Lipschitz continuity result for the absolute value function on $\mathscr{B}_R$, the set of all linear operators of norm greater than R. To see this, fix any operators A and B and observe

$$\||A|^2 - |B|^2\| \leqslant \|A^*\|\|A - B\| + \|A^* - B^*\|\|B\| = (\|A\| + \|B\|)\|A - B\|.$$

The above bound is sharp as can be seen from the operators A and B which are zero except for $A_{(1,1)} = 2$ and $B_{(1,1)} = 1$. More importantly, the bound implies if

$\|B\| \leqslant \|A\|$ then $\||A|^2 - |B|^2\| \leqslant 2\|A\|\|A - B\|$. We apply this in proving the following lemma.

Lemma 2. *Fix R with $0 < R < 1/4$. The absolute value map $A \to |A| = \sqrt{A^*A}$ is Lipschitz continuous on the set B_R. In fact, for all $A, B \in B_R$,*

$$\||A| - |B|\| \leqslant \frac{4\sqrt{2}\,\|A - B\|}{R}.$$

Proof. Let $A, B \in B_R$, with $R < \|B\| \leqslant \|A\|$. Note $|A|^2$ and $|B|^2$ lie in $B^+_{R^2}$. Hence by combining Eq. (1) of Lemma 1 with the above calculation we see

$$\begin{aligned}\||A| - |B|\| &= \left\|\sqrt{|A|^2} - \sqrt{|B|^2}\right\| \leqslant \frac{2\sqrt{2}}{\sqrt{R^2}\sqrt{\|A\|^2}}\||A|^2 - |B|^2\| \\ &\leqslant \frac{4\sqrt{2}\,\|A\|}{R\|A\|}\|A - B\|.\end{aligned}$$

This establishes the lemma. □

Note that the above lemmas hold for arbitrarily small positive choices of R. Moreover, standard methods easily extend the previous lemmas to the case where both operators have norms greater than or equal to R. These observations establish our main theorem.

Theorem. *Fix $R > 0$. There are Lipschitz constants M_R and N_R such that*

$$\left\|\sqrt{A} - \sqrt{B}\right\| \leqslant M_R\|A - B\| \quad \forall A, B \in \mathscr{L}^{\mathrm{P}}(\mathscr{H}) \text{ with norm at least } R$$

and

$$\||A| - |B|\| \leqslant N_R\|A - B\| \quad \forall A, B \in \mathscr{L}(\mathscr{H}) \text{ with norm at least } R.$$

Using the above theorem we can now construct subsets $\mathscr{I}_R$ of $\mathscr{L}^{\mathrm{S.A.}}(\mathscr{H})$ which properly contain $\mathscr{L}^{\mathrm{P}}(\mathscr{H})$ and on which the absolute value map is norm Lipschitz continuous.

Example. Let $R > 0$ and define

$$\mathscr{I}_R = \{A \in \mathscr{L}^{\mathrm{S.A.}}(\mathscr{H})\colon \sigma(A) \cap (-R, 0) = \emptyset\},$$

where $\sigma(A)$ denotes the spectrum of A. Let N_R be the Lipschitz constant for the absolute value map which is given in the above theorem. We now show that the absolute value map on $\mathscr{I}_R$ has Lipschitz constant $K_R = \max\{1, N_R\}$.

If A and B are both non-negative then the Lipschitz constant 1 works trivially. On the other hand, if neither operator is non-negative then the definition of $\mathscr{I}_R$ implies that $R \leqslant \|A\|, \|B\|$ and the Lipschitz constant N_R works. Now suppose that B is

non-negative and A is not. In this case, $R \leqslant \|A\|$ and so if $R \leqslant \|B\|$ the constant N_R works.

It remains only to consider the case where $0 < \|B\| \leqslant R \leqslant \|A\|$. Let $\alpha = R/\|B\|$. Then $1 \leqslant \alpha \leqslant \|A\|/\|B\|$. In particular, $\|\alpha A\| \geqslant \|A\| \geqslant R$ and $\|\alpha B\| = R$. Thus,

$$\alpha \||A| - |B|\| = \||\alpha A| - |\alpha B|\| \leqslant N_R \|\alpha A - \alpha B\| = \alpha N_R \|A - B\|.$$

So, the Lipschitz constant N_R works in this case as well.

Remark. Note that the above example does more than show that Lipschitz continuity of the absolute value map can hold on proper supersets of the positive operators. It provides an interesting insight into logarithmic Lipschitz bound on the absolute value map on $\mathscr{L}^{\mathrm{S.A.}}(\mathscr{H})$ which was established in [8]. More precisely, it was proved in [8] that for all $A, B \in \mathscr{L}^{\mathrm{S.A.}}(\mathscr{H})$,

$$\||A| - |B|\| \leqslant \left[\frac{4}{\pi} + \frac{2}{\pi} \log \frac{\|A\| + \|B\|}{\|A - B\|}\right] \|A - B\|.$$

We note that on the subsets $\mathscr{I}_R$ the absolute value is Lipschitz continuous even though the above logarithmic bound function tends to infinity. This suggests that developing new logarithmic Lipschitz bounds for the absolute map on $\mathscr{L}^{\mathrm{S.A.}}(\mathscr{H})$ could be an important line of inquiry.

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