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ESTRATTO

SEMI-INFINITE AND INFINITE STRIPS FREE OF ZEROS

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§ 1. - INTRODUCTION.

For a class of entire functions, which includes e^z, we shall show the existence of a semi-infinite strip of non-zero width, symmetric about the non-negative real axis, such that for each entire function in this class, the function and all its partial sums have no zeros in this strip. We shall then show the existence of an infinite strip about the imaginary axis for a class of entire functions, which includes $\sin z$ and $\cos z$, for which the same results are true. It will then be shown that the width of the strips obtained is the best possible result.

§ 2. - PRELIMINARY LEMMAS.

Let $f(z)$ be en entire function with $f(z) = \sum_{k=0}^{\infty} a_k z^k$, where the coefficients are real numbers. Then, the n-th partial sum of $f(z)$, $\mathfrak{s}_n(z)$, is:

$$\mathfrak{s}_n(z) = \sum_{k=0}^{n} a_k z^k ; \qquad \mathfrak{s}_n(x+iy) = \sum_{k=0}^{n} a_k \sum_{P=0}^{k} C_{k,P}\, x^P (iy)^{k-P} \tag{1.}$$

It is easy to verify that $\mathfrak{I}(\mathfrak{s}_n(x+iy)) = y\, P_{n-1}(x, y)$ where

$$P_{n-1}(x,y) = \sum_{j=0}^{n-1} x^j r_{n,j}(y)$$

$$r_{n,j}(y) = \sum_{k=0}^{\left[\frac{n-j-1}{2}\right]} C_{j+2k+1,\,j}\, a_{j+2k+1}\,(-1)^k\, y^{2k}\,. \tag{2.}$$

We seek to find the most general conditions on the coefficients, a_k, such that $r_{n,j} \geq 0$ for all $j = 0, 1, \ldots, n-1$; $n = 1, 2, \ldots$ We establish:

Lemma 1. For $r_{n,j} \geq 0$, the coefficients, a_k, $k \geq 1$, are non-negative numbers.

Proof. Suppose $a_m < 0$ for $m \geq 1$. From (2.) above, we have $r_{m,m-1} = a_m\, C_{m,m-1} < 0$. Contradiction.

Lemma 2. For $y^2 > 0$, $r_{n,j} \geq 0$, then for $m \geq 1$, $a_m = 0$ implies $a_{m+2k} = 0$; $k = 0, 1, \ldots$

Proof. Suppose $a_m = 0$ where $m \geq 1$. Then from (2.),

$$r_{m+2,\,m-1} = -\,a_{m+2}\, C_{m+2,\,m-1}\, y^2\,.$$

From the previous lemma, $a_{m+2} \geq 0$. Hence, if $r_{m+2,m-1} \geq 0$, then we conclude that $a_{m+2} = 0$ since $y^2 > 0$. Extending the argument now to a_{m+2}, we have the desired result.

Lemma 3. For $r_{n,j} \geq 0$ and $a_m > 0$, $a_{m+2} > 0$ for $m \geq 1$, then

$$0 \leq y^2 \leq 6 \left(\frac{a_m}{a_{m+2}}\right) \frac{1}{(m+1)\,(m+2)}\,.$$

Proof. Consider $r_{m+2,m-1}$. From (2.), we have

$$r_{m+2,\,m-1} = C_{m,\,m-1} \cdot a_m - C_{m+2,\,m-1} \cdot a_{m+2} \cdot y^2 \geq 0\,.$$

Solving for y^2, we have the desired result.

Let $f(z)$ be an entire function with $f(z) = \sum_{k=0}^{\infty} (-1)^k a_{2k}\, z^{2k}$ where the coefficients, a_{2k}, are real numbers. Then, the $2n$—th partial sum of $f(z)$, $S_{2n}(z)$, can be written as:

$$S_{2n}(x+iy) = \sum_{k=0}^{n} (-1)^k\, a_{2k} \sum_{P=0}^{2k} C_{2k,\,P}\,(iy)^P\, x^{2k-P} \tag{3.}$$

It is easy to verify that

$$\Re\,[\mathfrak{S}_{2n}(x+iy)] = \sum_{j=0}^{n} y^{2j}\, s_{n,j}(x)$$

$$s_{n,j}(x) = \sum_{k=0}^{[n-j]} (-1)^k\, a_{2j+2k}\, C_{2j+2k,\,2j}\, x^{2k}. \qquad (4.)$$

The proofs of the following lemmas are similar to those preceding.

Lemma 4. For $s_{n,j} \geq 0$; $j = 0, 1, \ldots, n$; $n \geq 0$, then $a_{2k} \geq 0$ for all $k \geq 0$.

Lemma 5. For $x^2 > 0$, $s_{n,j} \geq 0$, then for $k \geq 0$, $a_{2k} = 0$ implies $a_{2k+2j} = 0$ for all $j \geq 0$.

Lemma 6. For $s_{n,j} \geq 0$, then if $a_{2m} > 0$, $a_{2m+2} > 0$ for $m \geq 0$, then

$$0 \leq x^2 \leq 2\left(\frac{a_{2m}}{a_{2m+2}}\right)\frac{1}{(2m+1)(2m+2)}\,.$$

Let $f(z)$ be an entire function with $f(z) = \sum_{k=0}^{\infty} (-1)^k a_{2k+1} z^{2k+1}$ where the coefficients are real numbers. Then the $(2n+1)$ — st partial sum of $f(z)$, $\mathfrak{S}_{2n+1}(z)$, can be written as:

$$\mathfrak{S}_{2n+1}(x+iy) = \sum_{k=0}^{n} (-1)^k a_{2k+1} \sum_{P=0}^{2k+1} C_{2k+1,\,P}\,(iy)^P x^{2k+1-P}. \qquad (5.)$$

It is easy to verify that $\Re\,[\mathfrak{S}_{2n+1}(x+iy)] = x \sum_{j=0}^{n} y^{2j}\, t_{n,j}(x)$

$$t_{n,j}(x) = \sum_{k=0}^{[n-j]} (-1)^k\, a_{2j+2k+1}\, C_{2j+2k+1,\,2j}\, x^{2k}. \qquad (6.)$$

We have:

Lemma 7. For $t_{n,j}(x) \geq 0$; $j = 0, 1, \ldots, n$; $n \geq 0$, then $a_{2k+1} \geq 0$ for $k \geq 0$.

Lemma 8. For $x^2 > 0$, $t_{n,j}(x) \geq 0$, then $a_{2k+1} = 0$, $k \geq 0$ implies $a_{2k+2j+1} = 0$; $j \geq 0$.

Lemma 9. For $t_{n,j} \geq 0$, then if $a_{2k+1} > 0$, $a_{2k+3} > 0$; $k \geq 0$, then

$$0 \leq x^2 \leq 6\left(\frac{a_{2k+1}}{a_{2k+3}}\right)\frac{1}{(2k+2)(2k+3)}\,.$$

§ 3. - Main Theorems.

Theorem 1. Let $g(z) = \sum_{k=0}^{\infty} \beta_k z^k$ be an entire function, not identically zero, such that if $\beta_k = \alpha_k e^{i\varPsi_k}$ where $|\beta_k| = \alpha_k$ then:

a.) if $\alpha_k = 0$, $k \geq 1$, then $\alpha_{k+2j} = 0$; $j \geq 0$.

b.) $A_1 = \text{glb.} \left[\left(\frac{\alpha_m}{\alpha_{m+2}}\right) \frac{1}{(m+1)(m+2)}\right] > 0$ for $\alpha_m > 0$, $\alpha_{m+2} > 0$.

c.) $\varPsi_k = k\theta_0$ for all $k \geq 0$ where θ_0 is some fixed

angle. Then, there exists a semi-infinite strip, V_1, symmetric about $\theta = \theta_0$, of width W_1 where $W_1 = 2\sqrt{6 A_1} > 0$, such that $g(z)$ and all its partial sums have no zeros in V_1.

Proof. Let $z' = e^{+i\theta_0} z$. Then $g(e^{-i\theta_0} z') = f(z') = f(x+iy)$ is an entire function with non-negative real coefficients, α_k. If there exists a real number $M > 0$ such that for all $y^2 < M$, $r_{n,j}(y)$ is non-negative, than by Descartes' rule of signs, $P_{n,j}(x, y)$, considered as a polynomial in x, has no positive real roots; hence, for $x > 0$, $y^2 < M$ we have $P_{n-1}(x, y) \neq 0$. Then, $\mathfrak{I}(\mathfrak{s}_n[x+iy)] \neq 0$ for $x > 0$, $y^2 < M$ except when $y = 0$. But for $y = 0$, $\mathfrak{s}_n(z)$ reduces to: $\mathfrak{s}_n(x) = \sum_{k=0}^{n} \alpha_k z^k$. Since the $\alpha_k's$ are non-negative real numbers, $\mathfrak{s}_n(x)$ can have no zeros for $x > 0$. Hence, for $x > 0$, $0 \leq y^2 < M$, $\mathfrak{s}_n(z)$, for all $n \geq 1$, can have no zeros in the semi-infinite strip V_1, symmetric about the positive real axis; $z \in V_1$ implies $\mathfrak{R}(z) > 0$, $[\mathfrak{I}(z)]^2 < M$.

The width of V_1 is clearly $2M^{1/2}$. By Hurwitz's theorem, the zeros of $f(z)$ are limit points of zeros of $\mathfrak{s}_n(z)$. Then, $f(z)$ has no zeros in V_1.

Consider all quotients of the form:

$$\left(\frac{\alpha_m}{\alpha_{m+2}}\right) \frac{1}{(m+1)(m+2)}$$

where $\alpha_m > 0$, $\alpha_{m+2} > 0$.

Let A_1 be the glb. of these quotients. These quotients are clearly positive numbers, and by b.) of the hypothesis, $A_1 > 0$. Let $M = 6A_1$.

To show that $r_{n,j} \geq 0$ for all $0 \leq y^2 < M$, it will suffice to show that

$$N_{k,d} = \sum_{k=2d}^{2d+1} a_{j+2k+1} C_{j+2k+1,j} (-1)^k y^{2k} \geq 0 \tag{7.}$$

for all $d \geq 0$, $y^2 < M$, for if $\left[\frac{n-j-1}{2}\right]$ is odd in (2.), then $r_{n,j}$ is a sum of terms of the type $N_{k,d}$; if $\left[\frac{n-j-i}{2}\right]$ is even, then $r_{n,j}$ is a sum of terms of the type $N_{k,d}$, plus a final term which is clearly non-negative for all choices of y.

Expanding (7.), we have:

$$\frac{y^{2d}(j+4d+1)!}{j!\,(4d+3)!}\Big\{-(j+4d+2)(j+4d+3)\,a_{j+4d+3}\,y^2 + a_{j+4d+1}(4d+2)(4d+3)\Big\}. \tag{8.}$$

By hypothesis, if $a_{j+4d+1} = 0$, then $a_{j+4d+3} = 0$ and $N_{k,d} = 0$ for all values of y. If $a_{j+4d+1} > 0$ and $a_{i+4d+3} = 0$ then $N_{k,d} > 0$ for all values of y. If $a_{j+4d+1} > 0$, $a_{j+4d+3} > 0$ then $N_{k,d} > 0$ if

$$y^2 < \left(\frac{a_{j+4d+1}}{a_{j+4d+3}}\right) \frac{(4d+2)\,(4d+3)}{(j+4d+2)\,(j+4d+3)}.$$

But

$$\left(\frac{a_{j+4d+1}}{a_{j+4d+3}}\right) \frac{(4d+2)(4d+3)}{(j+4d+2)(j+4d+3)} \geq 6\left(\frac{a_{j+4d+1}}{a_{j+4d+3}}\right) \cdot \frac{1}{(j+4d+2)\,(j+4d+3)} \geq 6\,A_1.$$

Then, in all cases, for $0 \leq y^2 < M$, $N_{k,d} \geq 0$. Hence, with the results of lemmas 1, 2, and 3, we conclude that $r_{n,j} \geq 0$, which completes the proof.

The proofs of the following theorems all similar to the proof given in the previous theorem.

Theorem 2. Let $g(z) = \sum_{k=0}^{\infty} (-1)^k \beta_{2k} z^{2k}$ be an entire function, not idedentically zero, such that if $\beta_{2k} = a_{2k}\, e^{i\Psi_{2k}}$ where $|\beta_{2k}| = a_{2k}$, then:

a.) if $a_{2k} = 0$ for $k \geq 0$, then $a_{2k+2j} = 0$ for $j \geq 0$.

b.) $A_2 = \text{glb.} \left[\left(\frac{a_{2k}}{a_{2k+2}} \right) \frac{1}{(2k+1)(2k+2)} \right] > 0$ for $a_{2k} > 0$, $a_{2k+2} > 0$.

c.) $\Psi_{2k} = 2k\theta_0$ for $k \geq 0$ where θ_0 is some fixed angle. Then, there exists an infinite strip, V_2, symmetric about $\theta = \theta_0 \pm \frac{\pi}{2}$ of width W_2 where $W_2 = 2\sqrt{2A_2} > 0$, such that $g(z)$ and all its partial sums have no zeros in V_2.

Theorem 3. Let $g(z) = \sum_{k=0}^{\infty} (-1)^k \beta_{2k+1} z^{2k+1}$ be an entire function, not identically zero, such that if $\beta_{2k+1} = a_{2k+1} e^{i\Psi_{2k+1}}$ where $|\beta_{2k+1}| = a_{2k+1}$ then:

a.) if $a_{2k+1} = 0$; $k \geq 0$, then $a_{2k+2j+1} = 0$; $j \geq 0$.

b.) $A_3 = \text{glb.} \left[\left(\frac{a_{2k+1}}{a_{2k+3}} \right) \frac{1}{(2k+2)(2k+3)} \right] > 0$ for $a_{2k+1} > 0$, $a_{2k+3} > 0$.

c.) $\Psi_{2k+1} = (2k+1)\,\theta_0$ for $k \geq 0$ where θ_0 is some fixed angle. Then, there exists an infinite strip, V_3, symmetric about $\theta = \theta_0 \pm \pi/2$ of width W_3 where $W_3 = 2\sqrt{6A_3} > 0$, such that $g(z)$ and all its partial sums have no zeros in V_3 except at $z=0$, at which point, $g(z)$ and all its partial sum are zero.

Remark: The bounds, A_i, related to the width of the strips, cannot be increased. We see that $f(z) = \cos z$; $g(z) = \sin z$ satisfy the hypothesis of theorems 2 and 3, respectively. We have that $A_2 = A_3 = 1$ for $\cos z$ and $\sin z$. For $\cos z$, $S_2(z) = 1 - \frac{z^2}{2}$; $S_2(z)$ has zeros at $(\sqrt{2}, 0)$, $(-\sqrt{2}, 0)$. For $\sin z$, $S_3(z) = z - \frac{z^3}{6}$; $S_3(z)$ has zeros at $(0, 0)$, $(+\sqrt{6}, 0)$, $(-\sqrt{6}, 0)$. But the semi-width of the strip, as given by theorem 2, for $\cos z$ is $\sqrt{2}$; the semi-width for the strip for $\sin z$ is $\sqrt{6}$. Clearly, these strips cannot be increased in width without including zeros of the $\mathfrak{s}_n(z)$.

§ 4.

We will now show the relationship between the theorems given in § 3.

Def. If $g(z)$ satisfies the hypothesis of theorem k; $k=1, 2, 3$ then $g(z) \in S_k$; $S=S_1 U S_2 U S_3$.

Lemma 10. If $g(z) \in S_k$, then $\gamma g(\delta z) \in S_k$, where γ, δ are non-zero complex constants.

Proof. Obvious.

Lemma 11. If $g(z) \in S_1$ and γ is a non-zero complex constant, then:

a.) $\gamma\,[g(z) + g(-z)] \in S_2$

b.) $\gamma\,[g(z) - g(-z)] \in S_3$

Proof. Obvious.

Lemma 12. Under the operations defined in lemma 11, S_1 generates S.

Proof. Suppose $g(z) \in S_2$ where $g(z) = \sum_{k=0}^{\infty} (-1)^k a_{2k} z^{2k}$ and $a_{2k} \geq 0$, for convenience. Let $h(z) = \sum_{k=0}^{\infty} \beta_k z^k$ where

1.) $\beta_{2k} = a_{2k}$ for all $k \geq 0$.

2.) $\beta_{2k+1} = \dfrac{1}{(2k+1)!}$

Clearly, $h(z)$ is an entire function, and $h(z) \in S_1$.

But $\frac{1}{2}\,[h(z) + h(-z)] = g(z)$. The method of construction is similar for $g(z) \in S_3$.

We combine the results of the previous lemmas to obtain the following:

Theorem 4. Let $f(z)$ satisfy the hypothesis of theorem 1. Then:

1.) there exists a semi-infinite strip, V_1, of non-zero width, symmetric about $\theta = \theta_0$ such that $f(z)$ and its partial sums have no zeros in V_1.

2.) there exists an infinite strip, V_2, of non-zero width, symmetric about $\theta = \theta_0 \pm \pi/2$ such that $g(z) = \frac{1}{2}[f(z) + f(-z)]$ and all its partial sums, have no zeros in V_2.

3.) there exists an infinite strip, V_3, of non-zero width, symmetric about $\theta = \theta_0 \pm \pi/2$ such that $h(z) = \frac{1}{2i}[f(z) - f(-z)]$ and all its partial sums, have no zeros in V_3, except at $z = 0$, at which point, $h(z)$ and all its partial sums vanish.

Remark. e^z satisfies Theorem 7; hence, there exist infinite strips for $\sin z$, $\cos z$.