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Another Property of Chebyshev Polynomials

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1. INTRODUCTION

Any study of best uniform approximation of real continuous functions on $[-1, +1]$ by polynomials contains much about Chebyshev polynomials and their numerously elegant properties and characterizations. And new properties and characterizations of Chebyshev polynomials continue to fascinate us (cf. DeVore [1], and Micchelli and Rivlin [4]). In this spirit, we present another property of Chebyshev polynomials which, though elementary, has not to our knowledge been mentioned previously in the literature.

To motivate our theoretical discussion, suppose we wish to find the best polynomial approximation $\tilde{p}_n(x; r)$, of fixed degree n, to $f(x) := e^x$ in the uniform norm on $[0, r]$, where r is a large positive number. Equivalently, by mapping to the interval $[-1, +1]$, we seek the best uniform polynomial approximation $p_n(t; r)$ of degree n to the normalized function

$$g(t; r) := 2f\left\{\frac{r}{2}(t + 1)\right\}\Big/f(r),\ t \in [-1, +1], \tag{1.1}$$

where $p_n(t; r)$ and $\tilde{p}_n(x; r)$ are obviously related through

$$f(r)\,p_n\left(\frac{2x - r}{r}; r\right) = 2\tilde{p}_n(x; r),\ (r(t + 1)/2 = x). \tag{1.2}$$

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Because of inherent monotonicity properties of e^x, it is not difficult to verify that the unique *linear* polynomial $p_1(t; r)$ of best uniform approximation to $g(t; r)$ on $[-1, +1]$ is explicitly given for any $r > 0$ by

$$p_1(t; r) = \left[e^{-r} + \left(\frac{1 - e^{-r}}{r}\right)\left\{1 + \ln\left(\frac{r}{1 - e^{-r}}\right)\right\}\right] + (1 - e^{-r}) \cdot t, \quad (1.3)$$

so that as $r \to +\infty$, $p_1(t; r)$ tends uniformly on $[-1, +1]$ to the Chebyshev polynomial $T_1(t) = t$, i.e., for the case $n = 1$,

$$\lim_{r\to\infty} \| p_n(\cdot, r) - T_n \|_{L_\infty[-1,+1]} = 0. \quad (1.4)$$

As we shall show (cf. (3.6) of Theorem 3), (1.4) similarly holds for *any* fixed nonnegative integer n, in part because the continuous functions $g(t; r)$ tend pointwise on $[-1, +1]$, as $r \to +\infty$, to the *discontinuous function*

$$f_0(t) := \begin{cases} 0, & -1 \leqslant t < 1, \\ 2, & t = 1. \end{cases} \quad (1.5)$$

With π_n denoting the set of all real polynomials of degree at most n, it is readily seen (cf. Lemma 1) that there is *no* unique best uniform approximation to f_0 in π_n over $[-1, +1]$, but the set of best approximations to f_0 in π_n *does* contain T_n, the nth Chebyshev polynomial (of the first kind), for *every* $n \geqslant 0$.

To state our main result for sequences of continuous functions on $[-1, +1]$, we give some needed notation. For brevity, we write $\| \cdot \|_{[-1,+1]}$ for $\| \cdot \|_{L_\infty[-1,+1]}$. Next, given a sequence $\{f_k\}_{k=1}^\infty$ of continuous functions on $[-1, +1]$, let $p_{n,k} \in \pi_n$ be the unique best uniform approximation to f_k on $[-1, +1]$ in π_n, i.e.,

$$E_n(f_k) := \inf_{p\in\pi_n} \| f_k - p \|_{[-1,+1]} = \| f_k - p_{n,k} \|_{[-1,+1]} \ \forall k \geqslant 1, \forall n \geqslant 0. \quad (1.6)$$

It is well known that there exist $n + 2$ distinct points of alternation in $[-1, +1]$ for $p_{n,k} - f_k$, and by discarding at most one of these points on the extreme right, $n + 1$ consecutive alternation points $x_j^{(n,k)}$ can be found such that

$$\text{(i)} \quad -1 \leqslant x_n^{(n,k)} < x_{n-1}^{(n,k)} < \cdots < x_1^{(n,k)} < x_0^{(n,k)} \leqslant 1, \quad (1.7)$$
$$\text{(ii)} \quad p_{n,k}(x_j^{(n,k)}) - f_k(x_j^{(n,k)}) = (-1)^j E_n(f_k), \ 0 \leqslant j \leqslant n.$$

With this notation, and with the function f_0 of (1.5), we then state

THEOREM 1. *Let* $\{f_k\}_{k=1}^\infty$ *be any sequence of continuous functions on* $[-1, +1]$ *such that* (cf. (1.5))

$$\lim_{k\to\infty} \| f_k - f_0 \|_{[-1,t]} = 0 \qquad \forall t \in [-1, +1), \quad (1.8)$$

$$\lim_{k\to\infty} f_k(1) = 2, \quad (1.9)$$

$$\lim_{k\to\infty} \| f_k - 1 \|_{[-1,+1]} = 1, \quad (1.10)$$

and (*cf.* (1.7))

$$\limsup_{k\to\infty} x_1^{(n,k)} < 1 \qquad \forall n \geqslant 0. \tag{1.11}$$

Then,

$$\lim_{k\to\infty} \| p_{n,k} - T_n \|_{[-1,+1]} = 0 \qquad \forall n \geqslant 0. \tag{1.12}$$

We remark that (1.8) and (1.9) imply of course the pointwise convergence of $\{f_k\}_{k=1}^{\infty}$ to f_0 in $[-1, +1]$.

Because the assumption of (1.11) is a priori difficult to verify, a stronger but more convenient hypothesis, (1.13), can be made, which, with (1.8) and (1.9), imply both (1.10) and (1.11). This results in

THEOREM 2. *Let* $\{f_k\}_{k=1}^{\infty}$ *be any sequence of continuous functions on* $[-1, +1]$ *such that* (1.8) *and* (1.9) *are both satisfied. In addition, assume*:

$$\left\{\begin{array}{l}\text{there is an } \alpha \text{ with } -1 \leqslant \alpha < 1 \text{ such that } f_k \text{ is nondecreasing} \\ \text{on } [\alpha, 1]\ \forall k \geqslant 1.\end{array}\right\} \tag{1.13}$$

Then, the conclusion (1.12) *is valid.*

2. PROOF OF MAIN RESULTS

To prove Theorems 1 and 2, we establish a number of lemmas. First, with the definition of f_0 in (1.5), it is immediate that, for any $p_n \in \pi_n$ and any $n \geqslant 0$,

$$\| f_0 - p_n \|_{[-1,+1]} \geqslant \max\{| 2 - p_n(1)|; | p_n(1)|\} \geqslant 1,$$

from which the next result easily follows.

LEMMA 1. $E_n(f_0) = 1$ *for all* $n \geqslant 0$, *and* $q_n \in \pi_n$ *is a best approximation to* f_0 *in* π_n *(over* $[-1, +1]$*) iff* $\| q_n \|_{[-1,+1]} = 1$ *and* $q_n(1) = 1$. *In particular, each Chebyshev polynomial* T_n, $n \geqslant 0$, *is a best approximation to* f_0 *in* π_n.

LEMMA 2. *If* q_n *is a nonconstant best approximation to* f_0 *in* π_n *and if* $| f_0 - q_n |$ *takes on the value* $E_n(f_0)$ *in* n *distinct points in* $[-1, +1)$, *then* $q_n = T_n$.

Proof. By hypothesis, there exist n distinct points $\{x_j\}_{j=1}^{n}$ in $[-1, +1)$ $|(f_0 - q_n)(x_j)| = E_n(f_0) = 1$, i.e., $| q_n(x_j)| = 1$ for $1 \leqslant j \leqslant n$. From Lemma 1, $| q_n(1)| = 1$ and also $\| q_n \|_{[-1,+1]} = 1$. Thus, $| q_n |$ assumes its maximum of unity on $[-1, +1]$ in $n + 1$ distinct points, from which, using a result of Rivlin [6, p. 73], it follows that either q_n is a constant, which contradicts the hypothesis, or $q_n = \pm T_n$. Since $q_n(1) = 1$ from Lemma 1, then only $q_n = T_n$ is possible. ∎

LEMMA 3. *Let* $\{f_k\}_{k=1}^{\infty}$ *be a sequence of continuous functions on* $[-1, +1]$ *satisfying* (1.8)–(1.10). *Then* (*cf.* (1.6)–(1.7)), *for each* $n \geqslant 0$, *there is a subsequence* $\{p_{n,s_k}\}_{k=1}^{\infty}$ *of* $\{p_{n,k}\}_{k=1}^{\infty}$ *in* π_n, *a* $q_n \in \pi_n$, *and n points* $-1 \leqslant \hat{x}_n \leqslant \hat{x}_{n-1} \leqslant \cdots \leqslant \hat{x}_1 \leqslant 1$ *for which*

$$\lim_{k\to\infty} \| p_{n,s_k} - q_n \|_{[-1,+1]} = 0, \tag{2.1}$$

and

$$\lim_{k\to\infty} x_j^{(n,k)} = \hat{x}_j, \ 1 \leqslant j \leqslant n. \tag{2.2}$$

Moreover, q_n *is a best approximation to* f_0 *in* π_n.

Proof. By definition, it follows that

$$E_n(f_k) \leqslant E_0(f_k) \leqslant \| f_k - 1 \|_{[-1,+1]} \qquad \forall n \geqslant 0, \qquad \forall k \geqslant 1. \tag{2.3}$$

When coupled with the hypothesis of (1.10), this implies that

$$\limsup_{k\to\infty} E_n(f_k) \leqslant 1. \tag{2.4}$$

Then, because $\| p_{n,k} - f_k \|_{[-1,+1]} = E_n(f_k)$ from (1.6), it follows from the hypothesis (1.10) and (2.3) that $\{p_{n,k}\}_{k=1}^{\infty}$ is a bounded subset of π_n. Thus, by the Bolzano-Weierstrass Theorem, there exist a subsequence $\{p_{n,s_k}\}_{k=1}^{\infty}$ and a $q_n \in \pi_n$ such that (2.1) and (2.2) are satisfied. To show that q_n is a best approximation to f_0 in π_n, we first see from the triangle inequality that, for any fixed t with $-1 \leqslant t < 1$,

$$| q_n(t)| \leqslant |(q_n - p_{n,s_k})(t)| + |(p_{n,s_k} - f_{s_k})(t)| + | f_{s_k}(t)| .$$

The first term on the right tends to zero as $k \to \infty$ because of (2.1), the last term tending to zero because of (1.8). The second term on the right is bounded above by $E_n(f_{s_k})$, so that with (2.3) and (1.10), then $| q_n(t)| \leqslant 1$. But as t was arbitrary in $[-1, +1)$, then $\| q_n \|_{[-1,+1]} \leqslant 1$. Next, we have that

$$| p_{n,s_k}(1) - f_{s_k}(1)| \leqslant \| p_{n,s_k} - f_{s_k} \|_{[-1,+1]} = E_n(f_{s_k}),$$

so that from (2.3),

$$| p_{n,s_k}(1) - f_{s_k}(1)| \leqslant \| f_{s_k} - 1 \|_{[-1,+1]} .$$

Thus, from (2.1), (1.9), and (1.10), it follows that $| q_n(1) - 2 | \leqslant 1$, whence $q_n(1) \geqslant 1$. But as $\| q_n \|_{[-1,+1]} \leqslant 1$, then $q_n(1) = 1$ and $\| q_n \|_{[-1,+1]} = 1$, whence, from Lemma 1, q_n is a best approximation to f_0 in π_n. ∎

LEMMA 4. *Let* $\{f_k\}_{k=1}^{\infty}$ *be a sequence of continuous functions on* $[-1, +1]$ *satisfying* (1.8)–(1.10). *Then,*

$$\lim_{k\to\infty} E_n(f_k) = 1. \tag{2.5}$$

Proof. Clearly, from (2.3) and (1.6),

$$| p_{n,s_k}(1) - f_{s_k}(1)| \leqslant E_n(f_{s_k}) \leqslant \| f_{s_k} - 1 \|_{[-1,+1]} .$$

Then, since $p_{n,s_k}(1)$ tends to $q_n(1) = 1$ from (2.1), $f_{s_k}(1)$ tends to 2 from (1.9), and $\| f_{s_k} - 1 \|_{[-1,+1]}$ tends to unity from (1.10), it follows that

$$\lim_{k\to\infty} E_n(f_{s_k}) = 1. \tag{2.6}$$

If $\lim_{k\to\infty} E_n(f_k) \neq 1$, there is a subsequence $\{f_{r_k}\}_{k=1}^{\infty}$ of $\{f_k\}_{k=1}^{\infty}$ for which, with (2.4), $\lim_{k\to\infty} E_n(f_{r_k}) = \alpha < 1$. But then, by what has been established, $\{f_{r_k}\}_{k=1}^{\infty}$ will have a subsequence satisfying (2.6), a contradiction. ∎

LEMMA 5. *Let* $\{f_k\}_{k=1}^{\infty}$ *be a sequence of continuous functions on* $[-1, +1]$ *satisfying* (1.8)–(1.11). *Then* (*cf.* (1.7) *and* (2.1)),

$$\lim_{k\to\infty} p_{n,s_k}(x_j^{(n,s_k)}) = (-1)^j , 1 \leqslant 1 \leqslant n. \tag{2.7}$$

Consequently (*cf.* (2.2)),

$$q_n(\hat{x}_j) = (-1)^j, \qquad 1 \leqslant j \leqslant n, \tag{2.8}$$

and the x_j*'s are distinct with* $-1 \leqslant \hat{x}_n < \hat{x}_{n-1} < \cdots < \hat{x}_1 < 1$.

Proof. From (1.7(ii)), we have that

$$p_{n,s_k}(x_j^{(n,s_k)}) = f_{s_k}(x_j^{(n,s_k)}) + (-1)^j E_n(f_{s_k}), 1 \leqslant j \leqslant n. \tag{2.9}$$

With the hypothesis of (1.11), there is a τ with $\tau < 1$ such that $x_j^{(n,s_k)} \leqslant \tau < 1$ for all $1 \leqslant j \leqslant n$, all $k \geqslant 1$. Thus, on applying the hypothesis of (1.8), the first term on the right of (2.9) necessarily tends to zero as $k \to \infty$, while the second term tends to $(-1)^j$ from Lemma 4, proving (2.7). Next, (2.8) follows directly from (2.7) and (2.1)–(2.2) of Lemma 3, which then provides the distinctness of the $\hat{x}_j$'s in $[-1, +1)$. ∎

Proof of Theorem 1. As a consequence of Lemma 5, we can apply Lemma 2 to deduce that $q_n = T_n$ for each $n \geqslant 0$. Hence, (2.1) becomes

$$\lim_{k\to\infty} \| p_{n,s_k} - T_n \|_{[-1,+1]} = 0.$$

It remains to show that (1.12) is actually valid. Suppose, on the contrary, that (1.12) fails to be true for some $n \geqslant 0$. Then, there is subsequence $\{r_k\}_{k=1}^{\infty}$ and an $\epsilon > 0$ for which

$$\| p_{n,r_k} - T_n \|_{[-1,+1]} \geqslant \epsilon \qquad \forall k \geqslant 1. \tag{2.10}$$

But, by the proof of Lemma 3, there is a subsequence $\{r_{k_j}\}_{j=1}^{\infty}$ of $\{r_k\}_{k=1}^{\infty}$ and a $\tilde{q}_n \in \pi_n$ such that $\lim_{j\to\infty} \| p_{n,r_{k_j}} - \tilde{q}_n \|_{[-1,+1]} = 0$ where, as in Lemma 3, $\tilde{q}_n$ is a best approximation of f_0. But the proofs of the subsequent lemmas similarly hold, so that by the same reasoning, $\tilde{q}_n = T_n$, which contradicts (2.10). ∎

To prove Theorem 2, it suffices from Theorem 1 to simply establish

LEMMA 6. *Let* $\{f_k\}_{k=1}^{\infty}$ *be a sequence of continuous functions on* $[-1, +1]$ *satisfying* (1.8), (1.9), *and* (1.13). *Then,* (1.10) *and* (1.11) *are satisfied.*

Proof. It is easy to verify that (1.8), (1.9), and (1.13) together imply (1.10). Next, setting $\sigma := \limsup_{k\to\infty} x_1^{(n,k)}$, we wish to show first (cf. (1.11)) that $\sigma < 1$. Letting $\{k_j\}_{j=1}^{\infty}$ be any subsequence for which $\sigma = \lim_{j\to\infty} x_1^{(n,k_j)}$, we may assume that $x_1^{(n,k_j)} \geqslant \alpha$ of (1.13) for all $j \geqslant 1$, for otherwise, (1.11) is trivially true. From (1.7)(i), we have that $\alpha \leqslant x_1^{(n,k_j)} < x_0^{(n,k_j)} \leqslant 1$. Hence, on subtracting the cases $j = 0$ and $j = 1$ in (1.7)(ii),

$$\begin{aligned} p_{n,k_j}(x_0^{(n,k_j)}) - p_{n,k_j}(x_1^{(n,k_j)}) &= (f(x_0^{(n,k_j)}) - f(x_1^{(n,k_j)}) + 2E_n(f_{k_j}) \\ &\geqslant 2E_n(f_{k_j}) \qquad \forall j \geqslant 1, \end{aligned} \tag{2.11}$$

since f_{k_j} is by hypothesis (1.13) nondecreasing on $[\alpha, 1]$. Consequently, (2.11) can be expressed as

$$p'_{n,k_j}(\xi_1^{(n,k_j)})(x_0^{(n,k_j)} - x_1^{(n,k_j)}) \geqslant 2E_n(f_{k_j}) \qquad \forall j \geqslant 1, \tag{2.12}$$

for some $\xi_1^{(n,k_j)} \in [-1, +1]$. Then, since $1 - x_1^{(n,k_j)} \geqslant x_0^{(n,k_j)} - x_1^{(n,k_j)} > 0$, (2.12) implies that

$$1 - x_1^{(n,k_j)} \geqslant \frac{2E_n(f_{k_j})}{\| p'_{n,k_j} \|_{[-1,+1]}} \qquad \forall j \geqslant 1. \tag{2.13}$$

Next, as in the proof of Lemma 3, the boundedness of $\{ p_{n,k_j}\}_{j=1}^{\infty}$ implies that there is an $M > 0$ for which

$$\| p_{n,k_j} \|_{[-1,+1]} \leqslant M \qquad \forall j \geqslant 1,$$

so that by Markoff's inequality (cf. Meinardus [3])

$$\| p'_{n,k_j} \|_{[-1,+1]} \leqslant Mn^2 \qquad \forall j \geqslant 1.$$

Inserting this in (2.13) then yields

$$(1 - x_1^{(n,k_j)}) \geqslant \frac{2E_n(f_{k_j})}{Mn^2} \qquad \forall j \geqslant 1. \tag{2.14}$$

But as Lemma 4 implies that $\lim_{k\to\infty} E_n(f_k) = 1$, this gives that

$$1 - \sigma \geqslant 2/Mn^2,$$

or

$$1 > 1 - \frac{2}{Mn^2} \geqslant \sigma := \limsup_{k\to\infty} x_1^{(n,k)},$$

i.e., (1.11) is satisfied. ∎

3. Applications and Additional Remarks

First, though the hypotheses of Theorem 1 may seem lengthy, it is interesting to point out that if any *one* of the four hypotheses of (1.8)–(1.11) is dropped, a counterexample to the conclusion (1.12) of Theorem 1 can be constructed.

Our original motivation for this problem came from investigations of best polynomial approximation to continuous functions defined on $[0, +\infty)$. To close the cycle and apply the result of Theorem 2, let $C_+[0, +\infty)$ denote the subset of all real continuous functions $f(x)$ on $[0, +\infty)$ for which there exists a real number $\tau(f) \geqslant 0$ such that $f(x)$ is positive and nondecreasing for all $x \geqslant \tau(f)$. Next, for any $f \in C_+[0, +\infty)$, define its associated nonnegative and nondecreasing function $m_f(r)$ by

$$m_f(r) := \max_{0\leqslant x\leqslant r} |f(x)| = \|f\|_{[0,r]} \qquad \forall r \geqslant 0. \tag{3.1}$$

Note that $f \in C_+[0, +\infty)$ implies that $m_f \in C_+[0, +\infty)$. With this notation, we say that $f \in C_+[0, +\infty)$ is *order positive* on $[0, +\infty)$ if, for each fixed γ satisfying $0 \leqslant \gamma < 1$,

$$\lim_{r\to+\infty} \frac{m_f(\gamma r)}{m_f(r)} = 0. \tag{3.2}$$

As is easily seen, $f(x) = e^x$ and $h(x) = e^x - 10e^{x/2} \sin x$ are order positive on $[0, +\infty)$. In addition, any entire function of positive order and of perfectly regular growth (cf. Valiron [7, p. 45]), having only nonnegative Maclaurin coefficients, is necessarily order positive on $[0, +\infty)$. Note, however, that *no* polynomial can have this property.

Next, if f is order positive on $[0, +\infty)$, it follows from (3.2) that m_f, and hence f, is unbounded on $[0, +\infty)$. Consequently, on setting

$$\mathfrak{M}_f := \{r \geqslant 0 : m_f(r) = f(r)\}, \tag{3.3}$$

then $\mathfrak{M}_f$ is an unbounded subset of $[0, +\infty)$ which contains all r sufficiently large. This brings us to the following construction. Assuming f is order positive on $[0, +\infty)$, then, with the functions $g(t; r)$ of (1.1), set

$$g_k(t) := g(t; r_k) = 2f\left\{\frac{r_k}{2}(t+1)\right\}\Big/f(r_k),\ t \in [-1, +1],\ k \geqslant 1, \quad (3.4)$$

where $\{r_k\}_{k=1}^{\infty}$ is any fixed subset of $\mathfrak{M}_f$ satisfying

$$0 \leqslant \tau(f) < r_1 < r_2 < \cdots,\ \text{with} \lim_{k\to\infty} r_k = +\infty. \quad (3.5)$$

Because f is order positive, it is easily seen that $\{g_k\}_{k=1}^{\infty}$ is a sequence of continuous functions on $[-1, +1]$ satisfying (1.8) and (1.9), and that g_k is positive and nondecreasing on $[-1 + 2\tau(f)/r_k, 1]$, so that (1.13) is satisfied with $\alpha = 2\tau(f)/r_1$. Hence, Theorem 2 can be applied, but as this application can be made for *every* subset $\{r_k\}_{k=1}^{\infty}$ of $\mathfrak{M}_k$ satisfying (3.5), we also then have

THEOREM 3. *If f is order positive $[0, +\infty)$, let the functions $g(t; r)$, $r \geqslant \tau(f)$, be defined as in* (1.1), *and let $p_n(\cdot, r)$ be the unique best approximation in π_n to $g(\cdot; r)$ on $[-1, +1]$. Then,*

$$\lim_{r\to\infty} \| p_n(\cdot; r) - T_n \|_{[-1,+1]} = 0 \qquad \forall n \geqslant 0. \quad (3.6)$$

If f is order positive on $[0, +\infty)$, then it follows from (3.6) of Theorem 3 and known properties of Chebyshev polynomials that $p_n(\cdot; r)$, the unique best uniform approximation in π_n to $g(\cdot; r)$ on $[-1, +1]$, is evidently positive and strictly increasing on $[1, +\infty)$ for every $n \geqslant 1$, provided that r is sufficiently large. But since $\tilde{p}_n(x; r)$, the unique best uniform approximation in π_n to $f(x)$ on $[0, r]$, is related to $p_n(t; r)$ through

$$f(r)\, p_n\left\{\frac{2x - r}{r}; r\right\} = 2\tilde{p}_n(x; r), \qquad r(t+1)/2 = x, \quad (3.7)$$

then, as a consequence of Theorem 3, we have

COROLLARY 1. *If f is order positive on $[0, +\infty)$, then for each positive integer n, there is an $s = s(n) \geqslant \tau(f)$, such that $\tilde{p}_n(\cdot; r)$, the unique best uniform approximation in π_n to f on $[0, r]$, is positive and strictly increasing on $[r, +\infty)$ for all $r \geqslant s$.*

We remark that sufficient conditions on f to insure increasing and nonnegative polynomial approximations on the right of the interval of approximation have similarly been considered in [2].

Finally, the function f_0 of (1.5) has the property (cf. Lemma 1) that

$$\text{(i)}\quad E_n(f) = 1 \qquad \forall n \geqslant 0,$$

and

$$\text{(ii)}\quad \|f - T_n\|_{[-1,+1]} = 1 \qquad \forall n \geqslant 0, \tag{3.8}$$

and this was key in our development. However, f_0 is not the only function on $[-1, +1]$ satisfying (3.8). For example, changing the definition in the f_0 in the point $x = -\frac{1}{2}$ so that $f(-\frac{1}{2}) = \frac{1}{2}$, gives a new function also satisfying (3.8). It is then of interest to exactly characterize those functions f defined on $[-1, +1]$ for which (3.8) is valid, for this allows an obvious parallel derivation of results analogous to Theorems 1 and 2. To sketch this characterization, note that (3.8)(ii) implies that

$$-1 + T_n(x) \leqslant f(x) \leqslant 1 + T_n(x), \qquad \forall x \in [-1, +1], \qquad \forall n \geqslant 0,$$

whence

$$-1 + \sup_{n\geqslant 0}\{T_n(x)\} := L(x) \leqslant f(x) \leqslant U(x) := 1 + \inf_{n\geqslant 0}\{T_n(x)\}\ \forall x \in [-1, +1]. \tag{3.9}$$

Since $-1 \leqslant T_n(x) \leqslant 1$ for all $x \in [-1, +1]$ and all $n \geqslant 0$ while $T_0(x) := 1$, it follows that $L(x) := 0$ and $0 \leqslant U(x) \leqslant 2$, so that

$$0 \leqslant f(x) \leqslant U(x) \leqslant 2 \qquad \forall x \in [-1, +1]. \tag{3.10}$$

Next, it is known (cf. Pólya–Szegö [5, vol. 1, p. 71]) that if s is irrational, then the sequence $\{ns - [[ns]]\}_{n=0}^{\infty}$ is uniformly distributed in $[0, 1]$ where $[[ns]]$ denotes the integer part of ns. This implies that $U(x) = 0$ for any $x = \cos\theta$ for which θ/π is irrational, and thus, from (3.10), $f(x) = 0$ in such points. In a similar fashion, one then establishes

PROPOSITION 1. *For f defined on $[-1, +1]$ to satisfy* (3.8), *it is necessary and sufficient that*

(i) $f(1) = 2$;

(ii) $f(x) = 0$ *for any* $x = \cos\theta$ *with* $\theta \in [0, \pi]$ *for which* θ/π *is irrational*;

(iii) $f(x) = 0$ *for any* $x = \cos\theta$ *with* $\theta \in [0, \pi]$ *for which* $\theta = \dfrac{r\pi}{m}$, *with r and m in lowest terms and r* odd; (3.11)

(iv) $0 \leqslant f(x) \leqslant 1 - \cos\left(\dfrac{\pi}{2m+1}\right)$ *for any* $x = \cos\theta$ *with* $\theta \in [0, \pi]$ *for which* $\theta = \left(\dfrac{2r\pi}{2m+1}\right)$.

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