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THEOREMS OF STEIN-ROSENBERG TYPE

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§1. Introduction.

To obtain a solution of the linear system of equations

$$A\underline{x} = \underline{b}, \tag{1.1}$$

where A is an $n \times n$ complex matrix, it is often convenient to consider the splitting of A,

$$A = D - L - U, \tag{1.2}$$

where D, L, and U are $n \times n$ matrices with D nonsingular. Here, we do not assume that D is diagonal, nor that L and U are triangular. Associated with the splitting (1.2) are the following well-known successive overrelaxation (SOR) iteration matrices $\mathcal{L}_\omega$, defined by

$$\mathcal{L}_\omega := (D - \omega L)^{-1}\{(1-\omega)D + \omega U\} \tag{1.3}$$

for all complex relaxation factors ω with ω sufficiently small, and the extrapolated Jacobi matrix

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$$J_\omega := I - \omega D^{-1}A. \tag{1.4}$$

Historically, the Stein-Rosenberg Theorem [3] plays an important role in the comparison of the iterative methods of (1.3) and (1.4), when J_1 is assumed to be a nonnegative matrix (written $J_1 \geq \mathcal{O}$) and when $D^{-1}L$ and $D^{-1}U$ are respectively strictly lower triangular and strictly upper triangular. If $\rho(F)$ denotes the spectral radius of any $n \times n$ matrix F, then the Stein-Rosenberg Theorem effectively gives us under these assumptions that (cf. Young [6, p. 120])

$$\rho(\mathcal{L}_\omega) \leq \rho(J_\omega) < 1 \text{ for all } 0 < \omega \leq 1 \text{ if } \rho(J_1) < 1, \tag{1.5}$$

and that

$$\rho(\mathcal{L}_\omega) \geq \rho(J_\omega) > 1 \text{ for all } 0 < \omega \leq 1 \text{ if } \rho(J_1) > 1. \tag{1.6}$$

Thus, on defining in general

$$\Omega_{\mathcal{L}} := \{\omega \in \mathbf{C}: \rho(\mathcal{L}_\omega) < 1\}, \text{ and } \mathfrak{D}_{\mathcal{L}} := \{\omega \in \mathbf{C}: \rho(\mathcal{L}_\omega) > 1\}, \tag{1.7}$$

and

$$\Omega_J := \{\omega \in \mathbf{C}: \rho(J_\omega) < 1\}, \text{ and } \mathfrak{D}_J := \{\omega \in \mathbf{C}: \rho(J_\omega) > 1\}, \tag{1.8}$$

we deduce from (1.5) and (1.6) that

Theorem A. Assuming $J_1 \geq \mathcal{O}$ and that D is nonsingular with $D^{-1}L$ and $D^{-1}U$ respectively strictly lower triangular and strictly upper triangular, then

$$\Omega_{\mathcal{L}} \cap \Omega_J \supset (0,1] \quad \text{if} \quad \rho(J_1) < 1, \text{ and} \tag{1.9}$$

$$\mathfrak{D}_{\mathcal{L}} \cap \mathfrak{D}_J \supset (0,1] \quad \text{if} \quad \rho(J_1) > 1. \tag{1.10}$$

The question we address in this paper is the finding of necessary and sufficient conditions such that $\Omega_{\mathcal{L}} \cap \Omega_J \neq \emptyset$ and $\mathfrak{D}_{\mathcal{L}} \cap \mathfrak{D}_J \neq \emptyset$, without assuming that $J_1 \geq \mathcal{O}$ or that $D^{-1}L$ and $D^{-1}U$ are respectively strictly lower and strictly upper triangular matrices.

Some preliminary results are given in §2, while our main results are given in §3. Then, in §4, some remarks and examples are given.

§2. Relationships between $\mathcal{L}_\omega$ and J_ω.

In this section, we establish some formal identities relating the matrices $\mathcal{L}_\omega$ and J_ω. In particular, we deduce in Theorem 2.2 an expression relating the eigenvalues of $\mathcal{L}_\omega$ and J_ω for ω small.

As an easily verified consequence of the definitions of (1.3) and (1.4), we have

Lemma 2.1. For the splitting $A = D - L - U$ of (1.2), assume that D is nonsingular. Then,

$$\mathcal{L}_\omega = J_\omega - \omega^2(D^{-1}L)(I - \omega D^{-1}L)^{-1}(D^{-1}A) \qquad (2.1)$$

for all complex ω with ω sufficiently small.

If $\sigma(F) := \{\lambda \in \mathbb{C}: \det(\lambda I - F)\}$ denotes the spectrum of any $n \times n$ matrix F, then, using (2.1) of Lemma 2.1, we establish

Theorem 2.2. For the splitting $A = D - L - U$ of (1.2), assume that D is nonsingular. Then, for each $\lambda \in \sigma(\mathcal{L}_\omega)$, there exists a $\mu \in \sigma(J_\omega)$ such that

$$|\lambda-\mu| = O(|\omega|^{1+\frac{1}{n}}), \text{ for all } \omega \text{ sufficiently small.} \qquad (2.2)$$

Proof. Consider the matrix

$$Q(\omega) := D^{-1}A + \omega D^{-1}L(I - \omega D^{-1}L)^{-1}D^{-1}A, \qquad (2.3)$$

which is defined for all ω sufficiently small. Because ω is assumed small, a classical result of Ostrowski [2, p. 334] gives us that for each $\xi \in \sigma(Q(\omega))$, there is a $\gamma \in \sigma(D^{-1}A)$ such that

$$|\xi-\gamma| = O(|\omega|^{1/n}) \text{ for all } \omega \text{ sufficiently small,} \tag{2.4}$$

where n is the order of the matrices in (1.2). Note, however, from (1.4) and (2.1) that we can express $Q(\omega)$ and $D^{-1}A$ as

$$Q(\omega) = \frac{1}{\omega}(I-\mathcal{L}_\omega) \text{ for all } \omega \neq 0 \text{ sufficiently small,} \tag{2.5}$$

and

$$D^{-1}A = \frac{1}{\omega}(I - J_\omega) \text{ for all } \omega \neq 0. \tag{2.6}$$

Hence, each $\xi \in \sigma(Q(\omega))$ and each $\gamma \in \sigma(D^{-1}A)$ can be expressed from (2.5) and (2.6) as

$$\xi = \frac{1}{\omega}(1 - \lambda) \text{ with } \lambda \in \sigma(\mathcal{L}_\omega), \tag{2.7}$$

and

$$\gamma = \frac{1}{\omega}(1 - \mu) \text{ with } \mu \in \sigma(J_\omega), \tag{2.8}$$

and substituting (2.7) and (2.8) in (2.4) yields (2.2) for $\omega \neq 0$ sufficiently small. Of course, (2.2) trivially holds for $\omega = 0$, since $\mathcal{L}_0 = J_0 = I$ from (1.3) and (1.4). ■

We remark that the exponent of $|\omega|$ in (2.2) of Theorem 2.2 is, in general, best possible, as simple examples show. However, with further assumptions on the matrices in the splitting of (1.2), the exponent of $|\omega|$ can be increased to 2, as we now show.

<u>Theorem 2.3</u>. For the splitting $A = D - L - U$ of (1.2), assume that D is nonsingular, and that

i) $D^{-1}A$ commutes with $D^{-1}L$, or

ii) $D^{-1}A$ is diagonalizable.

Then, for each $\lambda \in \sigma(\mathcal{L}_\omega)$, there exists a $\mu \in \sigma(J_\omega)$ such that

$$|\lambda - \mu| = O(|\omega|^2), \text{ for all } \omega \text{ sufficiently small.} \tag{2.9}$$

Proof. Assuming i), it follows from (1.2) that $D^{-1}A$ commutes with $D^{-1}L$ as well as $D^{-1}U$, whence $\mathcal{L}_\omega$ and J_ω also commute from (2.1). Then, (2.9) follows from (2.1) and the fact that if M and N are commuting matrices, then $\sigma(M+N) \subseteq \sigma(M) + \sigma(N)$. Assuming ii), a slight modification of a result in Stewart [4, p. 304] gives (2.9). ■

§3. Main Results.

With the aid of Theorem 2.2, we can develop our analogs of the Stein-Rosenberg Theorem.

Theorem 3.1. For the splitting $A = D - L - U$ of (1.2), assume that D is nonsingular. Further, assume that there exists a real $\hat{\theta}$ with $0 \le \hat{\theta} < 2\pi$ for which

$$\min \operatorname{Re}\{e^{i\hat{\theta}}\xi : \xi \in \sigma(D^{-1}A)\} > 0. \tag{3.1}$$

Then, for $\omega = re^{i\hat{\theta}}$ with $r > 0$ sufficiently small, $\mathcal{L}_\omega$ and J_ω simultaneously converge. Thus,

$$\Omega_{\mathcal{L}} \cap \Omega_J \neq \phi. \tag{3.2}$$

Proof. Since $J_\omega = I - \omega D^{-1}A$ from (1.4), it follows that any $\mu \in \sigma(J_\omega)$ can be expressed, for $\omega = re^{i\hat{\theta}}$, as

$$\mu = 1 - \omega\xi = 1 - re^{i\hat{\theta}}\xi, \text{ where } \xi \in \sigma(D^{-1}A),$$

so that

$$|\mu|^2 = 1 - 2r \operatorname{Re}(e^{i\hat{\theta}}\xi) + r^2|\xi|^2. \tag{3.3}$$

Using (3.1), then $\rho(J_\omega) < 1$ for all $r > 0$ sufficiently small.

Continuing, using (2.2) of Theorem 2.2, it follows that for each $\lambda \in \sigma(\mathcal{L}_\omega)$, there is a $\mu \in \sigma(J_\omega)$ such that $|\lambda - \mu| = O(r^{1+\frac{1}{n}})$, or

$$|\lambda| = |\mu| + o(r^{1+\frac{1}{n}}).$$

Using (3.3), it follows that

$$|\lambda|^2 = 1 - 2r \operatorname{Re}(e^{i\hat{\theta}}\xi) + o(r^{1+\frac{1}{n}}), \tag{3.4}$$

which, from hypothesis (3.1), gives that $\rho(\mathcal{L}_\omega) < 1$ for all $r > 0$ sufficiently small. Consequently, $\Omega_{\mathcal{L}} \cap \Omega_J$ contains all $\omega = re^{i\hat{\theta}}$ for $r > 0$ sufficiently small, which establishes (3.2). ∎

In the converse direction, we have

Theorem 3.2. For the splitting $A = D - L - U$ of (1.2), assume that D is nonsingular, and assume for each real θ with $0 \le \theta < 2\pi$ that

$$\min \operatorname{Re}\{e^{i\theta}\xi : \xi \in \sigma(D^{-1}A)\} \le 0. \tag{3.5}$$

Then, J_ω diverges for all complex numbers ω.

Proof. Any $\mu \in \sigma(J_\omega)$ can, as in the preceding result, be expressed as $\mu = 1 - \omega\xi$ where $\xi \in \sigma(D^{-1}A)$. For any $\omega = re^{i\theta}$, we have, as in (3.3), that

$$|\mu|^2 = 1 - 2r \operatorname{Re}(e^{i\theta}\xi) + r^2|\xi|^2 \ge 1,$$

the last inequality following from hypothesis (3.5). Thus, J_ω diverges for all ω. ∎

Now, set

$$K(D^{-1}A) := \text{closed convex hull of } \sigma(D^{-1}A), \tag{3.6}$$

and let $\mathring{K}(D^{-1}A)$ denote its interior. As a characterization of (3.5) of Theorem 3.2 in terms of $K(D^{-1}A)$, we have

Theorem 3.3. For the splitting $A = D - L - U$ of (1.2), assume that D is nonsingular. Then, $0 \in K(D^{-1}A)$ iff (3.5) is valid for each θ with $0 \le \theta < 2\pi$.

Proof. If $0 \in K(D^{-1}A)$, then $0 \in e^{i\theta}K(D^{-1}A)$ for each θ, $0 \le \theta < 2\pi$, from which it follows that (3.5) is valid for each real θ. Conversely, if $0 \notin \mathring{K}(D^{-1}A)$, then it is geometrically evident that there exists a $\hat{\theta}$ with $0 \le \hat{\theta} < 2\pi$ for which (3.1) holds for $\theta = \hat{\theta}$. But then, from Theorems 3.1 and 3.2, (3.5) cannot hold. ■

As an immediate consequence of Theorems 3.1-3.3, we have the following analog of (1.9) of the Stein-Rosenberg Theorem A.

Theorem 3.4. For the splitting A = D - L - U of (1.2), assume D is nonsingular. Then,

$$\Omega_{\mathcal{L}} \cap \Omega_J \neq \phi \text{ iff } 0 \notin K(D^{-1}A). \qquad (3.7)$$

As a consequence of Theorem 3.4, we have

Corollary 3.5. For the splitting A = D - L - U of (1.2), assume that D is nonsingular, and that $D^{-1}A$ is strongly stable, i.e., Re $\xi > 0$ for any $\xi \in \sigma(D^{-1}A)$. Then, $\mathcal{L}_\omega$ and J_ω are simultaneously convergent for all $\omega > 0$ sufficiently small.

Proof. The hypothesis that $D^{-1}A$ is strongly stable insures that (3.1) is valid with $\theta = 0$. Then, apply Theorem 3.1. ■

In the converse direction, we similarly establish the analog of (1.10) of Theorem A.

Theorem 3.6. For the splitting A = D - L - U of (1.2), assume D is nonsingular. If $0 \in \mathring{K}(D^{-1}A)$, then

$$\mathfrak{D}_J \cap \mathfrak{D}_{\mathcal{L}} \supseteq \{\omega \in \mathbb{C} : |\omega| < r_0 \text{ and } \omega \neq 0\} \text{ for some } r_0 > 0. \qquad (3.8)$$

Conversely, if (3.8) is valid, then $0 \in K(D^{-1}A)$.

Proof. If $0 \in \mathring{K}(D^{-1}A)$, then for every θ with $0 \le \theta < 2\pi$, there is a $\xi \in \sigma(D^{-1}A)$ such that $\mathrm{Re}(e^{i\theta}\xi) < 0$. Thus, by Theorem 3.2, J_ω diverges for all ω. Moreover, from (3.4), we deduce that $\rho(\mathcal{L}_\omega) > 1$ for all $\omega = re^{i\theta}$ with $0 \le \theta < 2\pi$, and with $r > 0$ sufficiently small. Hence, (3.8) is valid.

Conversely, assume (3.8) is valid. Then, for each θ with $0 \le \theta < 2\pi$ and each $r_0 \ge r > 0$, there exists a $\xi \in \sigma(D^{-1}A)$ with $\xi \ne 0$ such that $|1 - re^{i\theta}\xi| > 1$, whence

$$1 - 2r\,\mathrm{Re}(e^{i\theta}\xi) + r^2|\xi|^2 > 1.$$

For $r > 0$ sufficiently small, this evidently implies that $\mathrm{Re}(e^{i\theta}\xi) \le 0$. But as θ was arbitrary, then $0 \in K(D^{-1}A)$. ∎

The following example indicates the sharpness of the above results.

Example 3.1. Consider the matrix A and its splitting (1.2) defined by

$$A := \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}; \quad D := I; \quad L := \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}; \quad U := \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}.$$

Then, $\sigma(D^{-1}A) = \{\pm i\}$, so that $K(D^{-1}A) = \{i\tau: -1 \le \tau \le +1\}$, whence $0 \in K(D^{-1}A)$, but $0 \notin \mathring{K}(D^{-1}A)$. Thus, by Theorems 3.2 and 3.3, J_ω diverges for all ω. However, the characteristic polynomial for the associated matrix $\mathcal{L}_\omega$ is just $\lambda^2+(2\omega^2-2)\lambda+1-\omega^2$, and, as its discriminant is $4\omega^2(\omega^2-1)$, then $\rho(\mathcal{L}_\omega) = \sqrt{1-\omega^2} < 1$ for all $0 < \omega \le 1$. Thus, $\Omega_{\mathcal{L}} \supset (0,1]$, while $\Omega_J = \phi$.

§4. Remarks and Examples.

In the case when $\Omega_{\mathcal{L}} \cap \Omega_J \ne \phi$, our Stein-Rosenberg-type Theorem 3.4 does not give us that the successive over-relaxation matrix $\mathcal{L}_\omega$ is iteratively faster (cf. [5]) than the corresponding Jacobi matrix J_ω. Indeed, within the framework in which our results were derived, such a result could not be true, as the following examples show.

Example 4.1. Consider

$$A := \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad D := I.$$

In this case, A is strongly stable since $\sigma(D^{-1}A) = \{1 \pm i\}$. Furthermore, one easily sees that

$$\rho(J_\omega) = \{2\omega^2 - 2\omega + 1\}^{1/2}, \quad \omega \text{ real}, \tag{4.1}$$

$$\Omega_J \cap \mathbb{R} = (0,1), \tag{4.2}$$

$$\min\{\rho(J_\omega) : \omega \text{ real}\} = \rho(J_{1/2}) = 2^{-1/2} \doteq 0.7071. \tag{4.3}$$

Now, set

$$L^{(1)} := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad U^{(1)} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix},$$

from which it follows that the associated matrix $\mathcal{L}_\omega^{(1)}$ has $(\lambda + \omega - 1)^2 + \omega^2\lambda$ as its characteristic polynomial. From this, one easily obtains that

$$\Omega_{\mathcal{L}^{(1)}} \cap \mathbb{R} = (0,1), \tag{4.4}$$

$$\rho(\mathcal{L}_\omega^{(1)}) < \rho(J_\omega) < 1 \quad \text{for all } \omega \in (0,1), \tag{4.5}$$

and that

$$\min\{\rho(\mathcal{L}_\omega^{(1)} : \omega \text{ real}\} = \rho(\mathcal{L}^{(1)}_{2(\sqrt{2}-1)}) \doteq 0.1716. \tag{4.6}$$

On the other hand, setting

$$L^{(2)} := \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}; \quad U^{(2)} = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix},$$

the associated matrix $\mathcal{L}_\omega^{(2)}$ has $\lambda^2 - (\omega^2 - 2\omega + 2)\lambda + (3\omega^2 - 2\omega + 1)$ as its characteristic polynomial. From this, one again easily obtains that

$$\Omega_{\mathcal{L}^{(2)}} \cap \mathbb{R} = (0, 2/3), \tag{4.7}$$

$$\rho(J_\omega) < \rho(\mathcal{L}_\omega^{(2)}) < 1 \text{ for all } \omega \in (0, 2/3), \tag{4.8}$$

and that

$$\min\{\rho(\mathcal{L}_\omega^{(2)}) : \omega \text{ real}\} = \rho(\mathcal{L}_{1/3}^{(2)}) \doteq 0.8165. \tag{4.9}$$

Note that (4.8) is the reversed inequality of (4.5).

Finally, it is interesting and appropriate to consider two well-known examples [1; 5, p. 74], due to Professor L. Collatz who is being honored with this volume, which are associated with the convergence and divergence of $\mathcal{L}_1$ and J_1.

Example 4.2. With D:= I, set

$$A := \frac{1}{2}\begin{bmatrix} 2 & -1 & 1 \\ 2 & 2 & 2 \\ -1 & -1 & 2 \end{bmatrix}; \quad L := \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}; \quad U := \frac{1}{2}\begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then, it was shown by Professor Collatz that $\rho(J_1) > 1$, while $\rho(\mathcal{L}_1) < 1$. However, $\sigma(D^{-1}A) = \{1;\ 1 \pm i\sqrt{5}/2\}$, so that $D^{-1}A$ is strictly stable. Thus, by Corollary 3.5, there is an $\omega > 0$ such that

$$\Omega_J \cap \Omega_{\mathcal{L}} \supseteq (0,\ \omega).$$

A short calculation shows that the largest such ω is 8/9, whence

$$\Omega_J \cap \Omega_{\mathcal{L}} \supseteq (0,\ 8/9).$$

On the other hand, consider

Example 4.3. With D: I, set

$$A := \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix}; \quad L := \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -2 & -2 & 0 \end{bmatrix}; \quad U := \begin{bmatrix} 0 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

In this example, it was shown by Professor Collatz that $\rho(J_1) < 1$, while $\rho(\mathcal{L}_1) > 1$. However, $\sigma(D^{-1}A) = \{1, 1, 1\}$, so that $D^{-1}A$ is again strictly stable, and hence, by Corollary 3.5, there is an $\hat{\omega} > 0$ such that

$$\Omega_J \cap \Omega_{\mathcal{L}} \supseteq (0, \hat{\omega}).$$

A short calculation shows that the largest such $\hat{\omega}$ is approximately 0.4873, whence

$$\Omega_J \cap \Omega_{\mathcal{L}} \supseteq (0, 0.4873).$$

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