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HERMITE-BIRKHOFF INTERPOLATION IN THE nTH ROOTS OF UNITY

BY

A. S. CAVARETTA, JR., A. SHARMA[1] AND R. S. VARGA[2]

Dedicated to Professor G. G. Lorentz on his seventieth birthday
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ABSTRACT. Consider, as nodes for polynomial interpolation, the nth roots of unity. For a sufficiently smooth function $f(z)$, we require a polynomial $p(z)$ to interpolate f and certain of its derivatives at each node. It is shown that the so-called Pólya conditions, which are necessary for unique interpolation, are in this setting also sufficient.

1. Introduction. While there is considerable literature on the Hermite-Birkhoff problem of interpolation on the real line (cf. Lorentz and Riemenschneider [3], Sharma [8], and van Rooij et al. [10]), the corresponding problem where the nodes are on the unit circle has received far less attention (cf. Kiš [1] and Sharma [6], [7]).

There is a distinction between these problems, since examples are known where the Hermite-Birkhoff (written H-B) interpolation problem is not poised on the real line, but the corresponding H-B problem on the circle is poised, and, conversely. To illustrate this, the H-B problem in three distinct points z_1, z_2, z_3, corresponding to the incidence matrix

$$\begin{array}{c} z_1 \\ z_2 \\ z_3 \end{array} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

is to determine a polynomial $p_2(z) = a_0 + a_1 z + a_2 z^2$ which satisfies

$$p_2(z_1) = \mu_1; \qquad p_2'(z_2) = \mu_2; \qquad p_2(z_3) = \mu_3,$$

for any given arbitrary complex numbers $\{\mu_i\}_{i=1}^3$. The determinant $\Delta_1(z_1, z_2, z_3)$ of the associated 3×3 matrix for the unknown coefficients $\{a_i\}_{i=0}^2$ for this problem is

$$\Delta_1(z_1, z_2, z_3) = (z_3 - z_1)\{z_1 + z_3 - 2z_2\}. \tag{1.1}$$

From this, it directly follows that this H-B problem is *poised on the unit circle*, i.e., $\Delta_1(z_1, z_2, z_3) \neq 0$ for any three distinct points z_1, z_2, z_3 on the unit circle. The associated problem on any line however is *not poised*, as choosing $2z_2 = z_1 + z_3$

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shows. Conversely, for the H-B problem, corresponding to the incidence matrix

$$\begin{array}{c} z_1 \\ z_2 \\ z_3 \end{array}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

the determinant $\Delta_2(z_1, z_2, z_3)$ for this incidence matrix is

$$\Delta_2(z_1, z_2, z_3) = 2(z_3 - z_1)\{(z_3 - z_2)^2 + (z_2 - z_1)^2 - (z_3 - z_2)(z_2 - z_1)\}. \tag{1.2}$$

In this case, this H-B problem is *real poised* since, for any three real points with $z_1 < z_2 < z_3$, $\Delta_2(z_1, z_2, z_3) > 0$, but is *not poised* on the unit circle since $\Delta_2(\hat{z}_1, \hat{z}_2, \hat{z}_3) = 0$ for $\hat{z}_1 = 1$, $\hat{z}_2 = e^{i\pi/3}$, $\hat{z}_3 = e^{2i\pi/3}$.

This note concerns the H-B interpolation problem whose incidence matrix is given by

$$\begin{array}{c} \\ z_1 \\ z_2 \\ \vdots \\ z_n \end{array}\begin{array}{c} \begin{array}{ccccccccccccccccc} 0 & & & & m_1 & & & & m_2 & & & & m_q & & & & \end{array} \\ \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \vdots & & & & \vdots & & & & \vdots & & & \vdots \\ 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{pmatrix}. \end{array}$$

For short we refer to this problem as the $(0, m_1, m_2, \ldots, m_q)$ case. In §3, we prove the

THEOREM. *For any nonnegative integer q, let $\{m_i\}_{i=0}^q$ be any nonnegative integers satisfying*

$$0 = m_0 < m_1 < m_2 < \cdots < m_q, \tag{1.3}$$

and let n be any positive integer for which

$$m_k \leq kn \quad \textit{for all } k = 0, 1, \ldots, q. \tag{1.4}$$

Then, the H-B interpolation problem $(0, m_1, m_2, \ldots, m_q)$ in the nth roots of unity $\{z_i\}_{i=1}^n$ is uniquely solvable for any given data.

We remark that special cases of this Theorem are known in the literature. The H-B interpolation problem $(0, 1, 2, \ldots, q)$ is just the classical case of Hermite interpolation, which is of course real and also circle poised. Next, Kiš [1] showed that the H-B interpolation problems $(0, 2)$ and $(0, 1, 2, \ldots, r, r + 2)$, for r any nonnegative integer, are uniquely solvable (for all sufficiently large n) in the roots of unity. The first result of Kiš was generalized by Sharma [7] to the $(0, m)$ case for any positive integer m. Sharma [6] also observed that the H-B problem $(0, m_1, m_2)$, the special case $q = 2$ of our Theorem, is uniquely solvable in the roots of unity for any positive integers $m_1 < m_2$, and gave an explicit proof of this in the case *sufficient*. We also obtain at the end of §3 explicit formulae for the fundamental polynomials.

2. A necessary lemma. To handle the determinants which we encounter in the proof of the Theorem, we need the following

LEMMA. *For any nonnegative integer q and any nonnegative integers $\{a_i\}_{i=0}^{q}$ and $\{\alpha_i\}_{i=0}^{q}$ satisfying*

$$\begin{cases} 0 \leqslant a_0 < a_1 < \cdots < a_q, \\ 0 \leqslant \alpha_0 < \alpha_1 < \cdots < \alpha_q, \\ \alpha_i \leqslant a_i \quad \textit{for } i = 0, 1, \ldots, q, \end{cases} \tag{2.1}$$

we define

$$M = M\begin{pmatrix} a_0, a_1, \ldots, a_q \\ \alpha_0, \alpha_1, \ldots, \alpha_q \end{pmatrix} := \begin{bmatrix} \binom{a_0}{\alpha_0} & \binom{a_0}{\alpha_1} & \cdots & \binom{a_0}{\alpha_q} \\ \binom{a_1}{\alpha_0} & \binom{a_1}{\alpha_1} & \cdots & \binom{a_1}{\alpha_q} \\ \vdots & \vdots & & \vdots \\ \binom{a_q}{\alpha_0} & \binom{a_q}{\alpha_1} & \cdots & \binom{a_q}{\alpha_q} \end{bmatrix}. \tag{2.2}$$

Then, we have

$$\det M > 0. \tag{2.3}$$

We remark that the result of this Lemma can be found in a paper by Zia-Uddin [11]. Zia-Uddin's proof, apparently due to A. C. Aitken, is however much more complicated. We also are indebted to Dr. C. A. Micchelli for suggesting the approach used below.

PROOF OF THE LEMMA. Consider the following two-point Pólya problem in the points $t = 0$ and $t = 1$, corresponding to the incidence matrix schematically shown below:

$$\begin{array}{c} \\ t = 0 \\ t = 1 \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccccccccccc} & & & & a_0 & & & & a_1 & & & & a_q \end{array} \\ \begin{pmatrix} 1 & & \cdots & & 1\,0\,1 & \cdots & & 1\,0\,1 & \cdots & & 1\,0 & & \\ 0 & \cdots & 0\,1\,0 & \cdots & & 0\,1\,0 & \cdots & & 0\,1\,0 & & & \cdots & 0 \end{pmatrix} \\ \begin{array}{ccccccccccccc} & & \alpha_0 & & & & \alpha_1 & & & & \alpha_q & & \end{array} \end{array} \tag{2.4}$$

Because $\alpha_i \leqslant a_i$ for all $0 \leqslant i \leqslant q$ from (2.1), it follows that the above incidence matrix satisfies the (weak) Pólya condition (cf. [4]). But, as this is a two-point interpolation problem, the Pólya condition is both necessary and sufficient for unique solvability (cf. [4]). Now, consider the particular polynomial

$$p(t) := \sum_{i=0}^{q} d_i t^{a_i}. \tag{2.5}$$

By definition, it follows that

$$\frac{1}{j!}p^{(j)}(t)\big|_{t=0} = 0 \quad \text{for any } j \neq \alpha_i,\ 0 \leqslant i \leqslant q,$$

whence $p(t)$ satisfies the interpolation of homogeneous data in the point $t = 0$ for the problem of (2.4). On the other hand, imposing the homogeneous interpolation conditions of (2.4) at the point $t = 1$ implies that

$$\frac{1}{(\alpha_j)!}p^{(\alpha_j)}(t)\big|_{t=1} = 0 \quad \text{for each } 0 \leqslant j \leqslant q. \tag{2.6}$$

In terms of (2.5) and (2.2), (2.6) can be expressed in matrix form simply as

$$M \cdot \left[d_0, d_1, \ldots, d_q\right]^T = \mathbf{0}.$$

But, as there is unique solvability for this problem, then $\det M \neq 0$. Thus, it remains to show that $\det M > 0$. It is well known (cf. Schoenberg [5]) that the infinite triangular Pascal matrix

$$\mathcal{P} := \begin{bmatrix} 1 & & & & \\ & & & 0 & \\ 1 & 1 & & & \\ 1 & \binom{2}{1} & \binom{2}{2} & & \\ 1 & \binom{3}{1} & \binom{3}{2} & \binom{3}{3} & \\ & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

is totally positive, so that the determinant of any square submatrix of $\mathcal{P}$ is necessarily nonnegative. Since the matrix M of (2.2) can be seen to be a square submatrix of $\mathcal{P}$ and since $\det M \neq 0$ from the discussion above, then $\det M > 0$. $\square$

3. Proof of the Theorem. We shall prove this Theorem by induction on q. The Theorem is obviously true for $q = 0$ and any $n \geqslant 1$, since this is the case of Lagrange interpolation. Suppose then that the Theorem is true for any $q - 1$ integers $m_1, m_2, \ldots, m_{q-1}$ satisfying (1.3), and suppose that (1.4) is valid. Then, if ω is any primitive nth root of unity, there is a unique linear interpolation formula

$$L_n(z; f) = \sum_{\nu=0}^{q-1} \sum_{k=0}^{n-1} f^{(m_\nu)}(\omega^k)\alpha_{k,m_\nu}(z) \tag{3.1}$$

which reproduces polynomials in π_{qn-1} (where π_r denotes the set of all complex polynomials of degree at most r). Here, the $\alpha_{k,m_\nu}(z)$ form the unique fundamental polynomials associated with the H-B interpolation problem $(0, m_1, \ldots, m_{q-1})$, i.e., $\alpha_{k,m_\nu} \in \pi_{qn-1}$ satisfies

We will now show that $P(z) \equiv 0$. We can express $P(z)$ as

$$P(z) = z^{qn}Q(z) + R(z), \tag{3.4}$$

where $Q(z) \in \pi_{n-1}$ and $R(z) \in \pi_{qn-1}$. Set

$$Q(z) = \sum_{\nu=0}^{n-1} a_\nu z^\nu. \tag{3.5}$$

Applying the conditions of (3.3) to (3.4) for $0 \leqslant \nu \leqslant q-1, 0 \leqslant k \leqslant n-1$, gives

$$R^{(m_\nu)}(\omega^k) = -(z^{qn}Q(z))^{(m_\nu)}_{z=\omega^k}, \qquad 0 \leqslant \nu \leqslant q-1; 0 \leqslant k \leqslant n-1. \tag{3.6}$$

Using the induction hypothesis, we apply the operator L_n of (3.1) to $R(z)$. Then the linearity and reproducing properties of L_n, together with (3.5) and (3.6), give that

$$R(z) = L_n(z; R(z)) = -L_n(z; z^{qn}Q(z)) = -\sum_{\nu=0}^{n-1} a_\nu L_n(z; z^{\nu+qn}). \tag{3.7}$$

Setting $(a)_m := a(a-1)\cdots(a-m+1)$ and $(a)_0 := 1$, we see from (3.1) that

$$L_n(z; z^{\nu+qn}) = \sum_{j=0}^{q-1} (\nu+qn)_{m_j} I_{\nu,j}(z), \tag{3.8}$$

where

$$I_{\nu,j}(z) := \sum_{k=0}^{n-1} \omega^{k(\nu-m_j)} \alpha_{k,m_j}(z). \tag{3.9}$$

Next, the reproducing property of L_n also gives (cf. (3.8)) that

$$z^{\nu+\lambda n} = L_n(z; z^{\nu+\lambda n}) = \sum_{j=0}^{q-1} (\nu+\lambda n)_{m_j} I_{\nu,j}(z); \qquad 0 \leqslant \lambda \leqslant q-1; 0 \leqslant \nu \leqslant n-1. \tag{3.10}$$

Thus, from (3.8) and (3.10), we see that

$$\begin{bmatrix} L_n(z; z^{\nu+qn}) & 1 & (\nu+qn)_{m_1} & \cdots & (\nu+qn)_{m_{q-1}} \\ z^\nu & 1 & (\nu)_{m_1} & \cdots & (\nu)_{m_{q-1}} \\ z^{\nu+n} & 1 & (\nu+n)_{m_1} & \cdots & (\nu+n)_{m_{q-1}} \\ \vdots & \vdots & \vdots & & \vdots \\ z^{\nu+(q-1)n} & 1 & (\nu+(q-1)n)_{m_1} & \cdots & (\nu+(q-1)n)_{m_{q-1}} \end{bmatrix}$$

$$\cdot \begin{bmatrix} 1 \\ -I_{\nu,0}(z) \\ -I_{\nu,1}(z) \\ \vdots \\ -I_{\nu,q-1}(z) \end{bmatrix} = 0,$$

which implies that

$$\det\begin{bmatrix} L_n(z; z^{\nu+qn}) & 1 & (\nu+qn)_{m_1} & \cdots & (\nu+qn)_{m_{q-1}} \\ z^{\nu} & 1 & (\nu)_{m_1} & \cdots & (\nu)_{m_{q-1}} \\ z^{\nu+n} & 1 & (\nu+n)_{m_1} & \cdots & (\nu+n)_{m_{q-1}} \\ \vdots & \vdots & \vdots & & \vdots \\ z^{\nu+(q-1)n} & 1 & (\nu+(q-1)n)_{m_1} & \cdots & (\nu+(q-1)n)_{m_{q-1}} \end{bmatrix} = 0. \tag{3.11}$$

Now, as $(a)_m = \binom{a}{m} \cdot m!$, the cofactor $A_{1,1}$ of $L_n(z; z^{\nu+qn})$ in the above determinant is just (cf. (2.2))

$$\left(\prod_{j=1}^{q-1} (m_j!)\right) \cdot M\begin{pmatrix} \nu, & \nu+n, & \ldots, & \nu+(q-1)n \\ 0, & m_1, & \ldots, & m_{q-1} \end{pmatrix},$$

and hence is nonzero from the Lemma. Thus, on expanding the determinant in (3.11), it follows that

$$L_n(z; z^{\nu+qn}) = \sum_{\lambda=0}^{q-1} b_\lambda(\nu) z^{\nu+\lambda n}, \qquad 0 \leqslant \nu \leqslant n-1, \tag{3.12}$$

where

$$b_\lambda(\nu) := -A_{\lambda+2,1}/A_{1,1}, \qquad 0 \leqslant \lambda \leqslant q-1. \tag{3.13}$$

Here $A_{l,1}$ denotes the cofactor of the lth element of the first column of the matrix in (3.11), $1 \leqslant l \leqslant q+1$.

Next, from (3.4) and (3.7), we can write

$$P(z) = \sum_{\nu=0}^{n-1} a_\nu \{z^{\nu+qn} - L_n(z; z^{\nu+qn})\},$$

so that with (3.12),

$$P(z) = \sum_{\nu=0}^{n-1} a_\nu \left\{ z^{\nu+qn} - \sum_{\lambda=0}^{q-1} b_\lambda(\nu) z^{\nu+\lambda n} \right\}. \tag{3.14}$$

Applying the final condition (cf. (3.3) and (3.6)) that

$$P^{(m_q)}(\omega^k) = 0, \qquad 0 \leqslant k \leqslant n-1,$$

yields

$$\sum_{\nu=0}^{n-1} a_\nu c_\nu \omega^{\nu k} = 0, \qquad 0 \leqslant k \leqslant n-1, \tag{3.15}$$

where

$$\sum_{\lambda=0}^{q-1} b_\lambda(\nu) (\ldots)$$

$$c_\nu = M_{\nu,q}/M_{\nu,q-1}, \tag{3.17}$$

where

$$M_{\nu,q} = M_{\nu,q}(n) := \det\begin{bmatrix} 1 & (\nu)_{m_1} & (\nu)_{m_2} & \cdots & (\nu)_{m_q} \\ 1 & (\nu+n)_{m_1} & (\nu+n)_{m_2} & \cdots & (\nu+n)_{m_q} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & (\nu+qn)_{m_1} & (\nu+qn)_{m_2} & \cdots & (\nu+qn)_{m_q} \end{bmatrix}. \tag{3.18}$$

To complete the proof of our Theorem, we need only note that

$$M_{\nu,q} = \left(\prod_{j=1}^{q}(m_j!)\right)\cdot M\begin{pmatrix} \nu, & \nu+n, & \ldots, & \nu+nq \\ 0, & m_1, & \ldots, & m_q \end{pmatrix}$$

for any $0 \leqslant \nu \leqslant n-1$. Since $m_k \leqslant kn$ by hypothesis (1.4), the condition (2.1) of the Lemma is satisfied and so $M_{\nu,q} > 0$ in (3.18). Thus, $c_\nu > 0$, whence $a_\nu c_\nu = 0$ implies $a_\nu = 0$, $0 \leqslant \nu \leqslant n-1$. It follows that $P(z)$ vanishes identically, as desired. $\square$

Incidentally, we observe that explicit formulae for the fundamental polynomials $\alpha_{k,m_j}(z)$, $0 \leqslant k \leqslant n-1$, $0 \leqslant j \leqslant q$, can be easily obtained. First, from (3.2) (with $q-1$ replaced by q), it easily follows that

$$\alpha_{0,m_j}(z\cdot\omega^{-k}) = \omega^{-km_j}\alpha_{k,m_j}(z), \qquad \forall 0 \leqslant k \leqslant n-1,\ \forall 0 \leqslant j \leqslant q. \tag{3.19}$$

Thus, it suffices to determine explicitly $\alpha_{0,m_j}(z)$ for all $0 \leqslant j \leqslant q$. Set

$$N_j(z^n;\nu,q) := \det \begin{array}{c} \begin{array}{ccccccc} & & & (j+1)\text{st column} & & & \end{array} \\ \begin{bmatrix} 1 & (\nu)_{m_1} & \cdots & 1 & \cdots & (\nu)_{m_q} \\ 1 & (\nu+n)_{m_1} & \cdots & z^n & \cdots & (\nu+n)_{m_q} \\ \vdots & \vdots & & \vdots & & \vdots \\ 1 & (\nu+qn)_{m_1} & \cdots & z^{qn} & \cdots & (\nu+qn)_{m_q} \end{bmatrix} \end{array}, \tag{3.20}$$

which results from replacing the $(j+1)$st column of $M_{\nu,q}$ of (3.18) with $[1, z^n, z^{2n}, \ldots, z^{qn}]^T$. Then, it can be verified that

$$\alpha_{0,m_j}(z) = \frac{1}{n}\sum_{\nu=0}^{n-1}\frac{z^\nu N_j(z^n;\nu,q)}{M_{\nu,q}}, \qquad \forall 0 \leqslant j \leqslant q. \tag{3.21}$$

For example, for $z = \omega^k$ for any $0 \leqslant k \leqslant n-1$ and for any $j > 0$, it is evident that the matrix in (3.20) has identical first and $(j+1)$st columns, whence $N_j(\omega^{kn};\nu,q) = 0$ for all $0 \leqslant k \leqslant n-1$. Thus, $\alpha_{0,m_j}(\omega^k) = 0$, for all $0 \leqslant k \leqslant n-1$.

4. Some nonpoised problems. As a further consequence of the Lemma, we can improve upon a theorem of Sharma and Tzimbalario [9], concerning the nonpoisedness of certain three-point problems. Let E be a three-row incidence matrix with exactly $n+1$ ones. Let $i_1 < i_2 < \cdots < i_p$, $j_1 < j_2 < \cdots < j_q$ and $k_1 < k_2 < \cdots < k_r$ denote the positions of the 1's in the first, second, and third rows respectively; $p+q+r = n+1$. Suppose further that $l_1 < l_2 < \cdots < l_{p+r}$ denote the positions of the 0's in the second row. Following Sharma and Tzimbalario, we take the interpolation at the nodes α, 0, 1, with $\alpha < 0$, and denote by $D_E(\alpha)$ the determinant of the homogeneous problem. If $D_E(\alpha)$ changes in sign $(-\infty, 0)$, we say that E is *strongly nonpoised*. The Lemma of §2 allows for the following improved version of Sharma and Tzimbalario.

THEOREM. *Suppose*

$$\begin{cases} i_1 \leqslant l_1, \ldots, i_p \leqslant l_p, \\ k_1 \leqslant l_1, \ldots, k_r \leqslant l_r. \end{cases} \tag{4.1}$$

If $\sum_{m=1}^{p}(l_{r+m} - l_m) + pr \equiv 1 \pmod 2$, *then E is strongly nonpoised.*

Our condition (4.1) replaces a more restrictive condition of Sharma and Tzimbalario [9] which requires $l_1 > \max(i_p - p;\, k_r - r)$. We further remark that the result of [9] has been shown to be a special case of a criterion of G. G. Lorentz (cf. Lorentz and Riemenschneider [2]), but the exact interrelation of the above Theorem with the criterion of Lorentz is beyond the specific aims of this work, and is left as an open question.

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DEPARTMENT OF MATHEMATICS, KENT STATE UNIVERSITY, KENT, OHIO 44242