/ 20



# LACUNARY TRIGONOMETRIC INTERPOLATION ON EQUIDISTANT NODES

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Dedicated to Professor A. Zygmund on his $79^{th}$ birthday, December 26, 1979

This paper studies the interpolation at equidistant nodes of a given function and certain of its derivatives by trigonometric polynomials. Essentially, unique interpolation is possible only if even derivative values and odd derivative values are used in equal quantities (cf. Theorems 1 and 2). In addition, explicit forms for the fundamental polynomials are derived.

## 1. INTRODUCTION

We shall say that an interpolation problem is <u>regular</u> on some given nodes if it is uniquely solvable on those nodes. Recently, we have shown [1] that the problem of $(0, m_1, \ldots, m_q)$-interpolation by algebraic polynomials is regular on the n-th roots of unity with some natural growth conditions on the $m_i$'s. Our object here is to solve the problem of regularity of $(0, m_1, \ldots, m_q)$-interpolation by trigonometric polynomials on the equidistant nodes $X := \{x_k\}_0^{n-1}$, where $x_k := \frac{2k\pi}{n}$, $k = 0, 1, \ldots, n-1$.

For our problem, it is convenient to distinguish between the following two classes of trigonometric polynomials: For m a positive integer,

$$(1.1)\quad \mathcal{T}_m := \{T(x) = a_0 + \sum_{\nu=1}^{m} (a_\nu \cos \nu x + b_\nu \sin \nu x) : \ a_\nu, b_\nu \text{ are complex numbers}\},$$

$$(1.2)\quad \mathcal{T}_{m,\varepsilon} := \{T(x) = a_0 + \sum_{\nu=1}^{m-1} (a_\nu \cos \nu x + b_\nu \sin \nu x) + a_m \cos\left(mx - \frac{\pi\varepsilon}{2}\right) : \ a_\nu, b_\nu \text{ are complex numbers}\},$$

where $\varepsilon = 0$ or $1$. For m a positive integer, the inclusion $\mathcal{T}_{m,\varepsilon} \subseteq \mathcal{T}_m \subseteq \mathcal{T}_{m+1,\varepsilon}$ evidently holds.

We propose to solve the following problems:

PROBLEM A. Given q positive integers $m_1, m_2, \ldots, m_q$ and the nodes $X = \{\frac{2k\pi}{n}\}_0^{n-1}$, $n \geq 1$, determine conditions on the $m_i$'s which will guarantee the existence of a unique trigonometric polynomial $T(x)$ of the appropriate type and order (depending on n and q) such that

$$(1.3) \quad T^{(m_\nu)}(x_k) = a_{k\nu}; \; k = 0, 1, \ldots, n-1; \; \nu = 0, 1, \ldots, q, \quad \text{with } m_0 := 0,$$

for any given data $\{a_{k\nu}\}$.

For example, it is easy to see that if $q \geq 2$, then all the $m_i$'s cannot be even, nor can they be all odd if the interpolation problem A is regular. For, if all the $m_i$'s were even, the functions sin nx and the identically zero function would both satisfy all the conditions of (1.3) with zero data, thereby contradicting the uniqueness of interpolation. Similarly, if all the $m_i$'s are odd, the function 1-cos nx satisfies (1.3) with zero data.

PROBLEM B. If Problem A has a unique solution, find an explicit form of the interpolatory polynomial. In other words, find the explicit form of the associated fundamental polynomials, defined by

$$(1.4) \quad \rho_{i,m_\nu}^{(m_j)}(x_k) := \delta_{j,\nu}\delta_{k,i}; \; i, k = 0, 1, \ldots, n-1; \; j, \nu = 0, 1, \ldots, q.$$

In order to solve these problems, we shall denote by $o_q$ and $e_q$ the number of odd $m_i$'s and even $m_i$'s, respectively, in $\{m_1, m_2, \ldots, m_q\}$, and, with $[[\tau]]$ denoting the integer part of the real number $\tau$, we shall use throughout the notation

$$(1.5) \quad M := [[(nq + 1)/2]] .$$

With this notation, we shall establish the following two theorems:

THEOREM 1. If $n = 2r + 1$ is odd, the problem of $(0, m_1, \ldots, m_q)$-interpolation by trigonometric polynomials is regular on $X = \{\frac{2k\pi}{n}\}_0^{n-1}$, precisely when

$$(1.6)\quad \begin{cases} o_q - e_q = 0 & \text{if } q = 2p, \\ \qquad\quad = \pm 1 & \text{if } q = 2p+1. \end{cases}$$

Moreover, when $q = 2p$, interpolation is within the class $\mathcal{T}_{M+r}$, while if $q = 2p + 1$, interpolation is within the class $\mathcal{T}_{M+r,\varepsilon}$ where $\varepsilon = 1$ or $0$, according as $o_q - e_q = +1$ or $-1$, respectively.

THEOREM 2. If $n = 2r$ is even, the problem of $(0,m_1,\cdots,m_q)$-trigonometric interpolation is regular on X precisely when

$$(1.7)\quad \begin{cases} o_q - e_q = 0 & \text{if } q = 2p, \\ \qquad\quad = 1 & \text{if } q = 2p+1. \end{cases}$$

Moreover, the interpolation is within the class $\mathcal{T}_{M+r,\varepsilon}$ where $\varepsilon = 0$ or $1$, according as $q$ is even or odd, respectively.

Only special cases of Theorems 1 and 2 are known in the literature. The simplest case of Lagrange interpolation, i.e., $q = 0$, on X, as well as the Hermite interpolation case of $q = 1$ and $m_1 = 1$ on X, can be found in Zygmund [11, p. 1 and p. 23]. In [5], Kiš settled the (0,2) case on X, the first lacunary case so settled. The results of Kiš were subsequently extended by Sharma and Varma [7, 8], Varma [9], Čuprigin [2], and Zeel' [10]. The results of these authors are all particular cases of Theorems 1 and 2.

Certain determinants, analogous to Vandermonde determinants but bearing on our lacunary problem, occur in our proofs, and a discussion of these determinants is relegated to §2. In §3, we prove Theorem 1 by establishing two lemmas which formalize an inductive proof of Theorem 1. The proof of Theorem 2 is along completely similar lines, and will not, for reasons of brevity, be given. Problem B is finally addressed in §4.

It is worth remarking that our work here has been recently extended by A. Sharma, P. W. Smith, and J. Tzimbalario [6].

## 2. SOME DETERMINANTS

In order to facilitate matters later, we show in this section that certain determinants are nonzero. We begin with the following known result:

LEMMA 1 (cf. Gantmacher [3, p. 99]). *Let* $m_1 < m_2 < \cdots < m_q$ *be distinct real numbers, and let* $t_1 < t_2 < \cdots < t_q$ *be positive numbers. Then, the determinant*

$$(2.1)\quad \begin{vmatrix} t_1^{m_1} & t_1^{m_2} & \cdots & t_1^{m_q} \\ t_2^{m_1} & t_2^{m_2} & \cdots & t_2^{m_q} \\ \vdots & & & \vdots \\ t_q^{m_1} & t_q^{m_2} & \cdots & t_q^{m_q} \end{vmatrix}$$

*is positive*.

Next, we need an analogous result, which is formulated as

LEMMA 2. *Let* $m_1 < m_2 < \cdots < m_k$ *be distinct positive even integers, and let* $m_{k+1} < m_{k+2} < \cdots < m_q$ *be distinct positive odd integers, where* $0 \le k < q$. *For any* $q+1$ *positive numbers* $t_1 < t_2 < \cdots < t_{q+1}$, *the following determinant*,

$$(2.2)\quad B := \begin{pmatrix} 1 & (-t_1)^{m_1} & \cdots & (-t_1)^{m_k} & (-t_1)^{m_{k+1}} & \cdots & (-t_1)^{m_q} \\ 1 & t_2^{m_1} & \cdots & t_2^{m_k} & t_2^{m_{k+1}} & \cdots & t_2^{m_q} \\ 1 & (-t_3)^{m_1} & \cdots & (-t_3)^{m_k} & (-t_3)^{m_{k+1}} & \cdots & (-t_3)^{m_q} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 1 & ((-1)^{q+1}t_{q+1})^{m_1} & \cdots & ((-1)^{q+1}t_{q+1})^{m_k} & ((-1)^{q+1}t_{q+1})^{m_{k+1}} & \cdots & ((-1)^{q+1}t_{q+1})^{m_q} \end{pmatrix}$$

*is nonzero. More precisely*,

$$(2.3)\quad \operatorname{sgn} B = (-1)^{(q-k)(q+k-1)/2}.$$

*In addition, the determinant in* (2.2), *in which the* $(j+1)$-*st column is replaced by* $\{t_1^{m_j}, (-t_2)^{m_j}, t_3^{m_j}, \cdots, ((-1)^q t_{q+1})^{m_j}\}^T$ *for each* $j=1, 2, \cdots, q$, *is similarly nonzero, with sign* $(-1)^{(q-k)(q+k+1)/2}$.

Proof. We shall use the Laplace expansion of B of (2.2) in terms of the first k+1 columns of B. Then (cf. Karlin [4, p. 6],

$$(2.4)\quad B = \Sigma(-1)^{i_1+i_2+\dots+i_{k+1}+\frac{(k+1)(k+2)}{2}} B\begin{pmatrix} i_1,\dots,i_{k+1} \\ 1,\dots,k+1 \end{pmatrix} \times B\begin{pmatrix} i_1',\dots,i_{q-k}' \\ k+2,\dots,q+1 \end{pmatrix},$$

where $B\begin{pmatrix} i_1,\dots,i_{k+1} \\ 1,\dots,k+1 \end{pmatrix}$ is the determinant of the rows $i_1,\dots,i_{k+1}$ and columns 1, 2, ..., k+1 of B, and $B\begin{pmatrix} i_1',\dots,i_{q-k}' \\ k+2,\dots,q+1 \end{pmatrix}$ is the determinant formed from the complementary rows $i_1',\dots,i_{q-k}'$ and complementary columns k+2, ..., q+1 of B. Here, the sum (2.4) is taken over all integers $i_1, \dots, i_{k+1}$ for which $1 \le i_1 < i_2 < \dots < i_{k+1} \le q+1$. By definition,

$$(2.5)\quad i_1+i_2+\dots+i_{k+1}+i_1'+i_2'+\dots+i_{q-k}' = 1+2+\dots+q+1 = (q+1)(q+2)/2.$$

Now, applying Lemma 1, $\operatorname{sgn} B\begin{pmatrix} i_1,\dots,i_{k+1} \\ 1,\dots,k+1 \end{pmatrix} = 1$ for all $1 \le i_1 < \dots < i_{k+1} \le q+1$. Next, note that the j-th row of $B\begin{pmatrix} i_1',\dots,i_{q-k}' \\ k+2,\dots,q+1 \end{pmatrix}$ contains all negative (positive) entries if $i_j'$ is odd (even). Then, multiplying each element of row $i_\nu'$ by $(-1)^{i_\nu'}$ in the determinant $B\begin{pmatrix} i_1',\dots,i_{q-k}' \\ k+2,\dots,q+1 \end{pmatrix}$ for each $\nu = 1, 2, \dots, q-k$, gives a determinant which is positive from Lemma 1. Consequently,

$$\operatorname{sgn} B\begin{pmatrix} i_1',\dots,i_{q-k}' \\ k+2,\dots,q+1 \end{pmatrix} = (-1)^{\sum_{\nu=1}^{q-k} i_\nu'}.$$

Thus, each term of the sum (2.4) is nonzero and has the same sign, namely,

$$(-1)^{\{\sum_{\nu=1}^{k+1} i_\nu + (k+1)(k+2)/2 + \sum_{\nu=1}^{q-k} i_\nu'\}}$$

which, using (2.5), is equivalent to the desired result of (2.3). The proof for the alternate form of the determinant B is completely similar. ∎

Now, with $m_1, m_2, \cdots, m_q$ any distinct, not necessarily ordered, positive integers, and with (1.5), let

$$(2.6)\quad \Delta_{M-j,q} := \begin{vmatrix} 1 & (M-j)^{m_1} & \cdots & (M-j)^{m_q} \\ 1 & (M-j-n)^{m_1} & & (M-j-n)^{m_q} \\ \vdots & \vdots & & \vdots \\ 1 & (M-j-qn)^{m_1} & \cdots & (M-j-qn)^{m_q} \end{vmatrix}.$$

Similarly, let $\Delta^*_{M-j,q}$ denote the determinant obtained by deleting the last row and last column in $\Delta_{M-j,q}$. Then, as applications of Lemma 2, we have

LEMMA 3. <u>For</u> $n \geq 3$ <u>and</u> $q = 2p$,

$$(2.7)\quad \Delta_{M-j,q} \neq 0 \text{ and } \Delta^*_{M-j,q} \neq 0 \text{ for all } j=1,2,\cdots,[\![(n-1)/2]\!].$$

<u>Similarly</u>, <u>for</u> $n = 2r \geq 4$ <u>and</u> $q = 2p+1$,

$$(2.8)\quad \Delta_{M-j,q} \neq 0 \text{ and } \Delta^*_{M-j,q} \neq 0 \text{ for all } j=1,2,\cdots,[\![(n-1)/2]\!],$$

<u>while for</u> $n=2r+1 \geq 3$ <u>and</u> $q = 2p+1$,

$$(2.9)\quad \Delta_{M-1-j,q}\neq 0 \text{ and } \Delta^*_{M-j-1,q}\neq 0 \text{ for all } j=0,1,\cdots,[\![(n-1)/2]\!].$$

<u>Proof</u>. To establish (2.7), assume $n \geq 3$ and $q = 2p$, so that (cf. (1.5)) $M = np$. Then, the numbers $\{M-j-\lambda n\}_{\lambda=0}^{q}$ form a strictly decreasing sequence, of which the first p terms are positive and the remaining terms negative, for any j with $1 \leq j \leq n-1$. These numbers can be arranged, in order of increasing absolute values, as follows:

$$-j,\ n-j,\ -n-j,\ 2n-j,\ \cdots,\ -pn-j.$$

Setting $t_1, t_2, \cdots, t_{q+1}$ to be their successive absolute values, i.e.,

$$t_{2\ell+1} = \ell n+j \text{ for } \ell=0,1,\cdots,p;\ t_{2\ell} = \ell n-j \text{ for } \ell=1,2,\cdots,p,$$

then, with the hypothesis that $1 \leq j \leq [\![(n-1)/2]\!]$, it follows that $0 < t_1 < t_2 < \cdots < t_{q+1}$. Now, the determinant $\Delta_{M-j,q}$ of (2.6) in this case, is, after a suitable interchange of rows and columns, just the determinant B of (2.2) which is nonzero from Lemma 2. Thus, $\Delta_{M-j,q} \neq 0$ and similarly

$\Delta^*_{M-j,q} \neq 0$, establishing (2.7). Establishing (2.8) and (2.9) is similar, except that the latter part of Lemma 2 is applied in each case. ∎

Another determinant, which closely resembles that of (2.6) when j=0, must also be considered. It is the occurrence of this type of determinant which is responsible for the conditions on $o_q - e_q$ in (1.6) and (1.7) in Theorems 1 and 2. With $m_1, m_2, \cdots, m_q$ distinct positive integers and with (1.5), define

$$(2.10) \qquad \Phi_{M,q} := \begin{pmatrix} 1 & M^{m_1} \cos \tfrac{1}{2}\pi m_1 & M^{m_2} \cos \tfrac{1}{2}\pi m_2 & \cdots & M^{m_q} \cos \tfrac{1}{2}\pi m_q \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & (M-pn)^{m_1} \cos \tfrac{1}{2}\pi m_1 & (M-pn)^{m_2} \cos \tfrac{1}{2}\pi m_2 & \cdots & (M-pn)^{m_q} \cos \tfrac{1}{2}\pi m_q \\ 0 & [M-(p+1)n]^{m_1} \sin \tfrac{1}{2}\pi m_1 & [M-(p+1)n]^{m_2} \sin \tfrac{1}{2}\pi m_2 & \cdots & [M-(p+1)n]^{m_q} \sin \tfrac{1}{2}\pi m_q \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & (M-qn)^{m_1} \sin \tfrac{1}{2}\pi m_1 & (M-qn)^{m_2} \sin \tfrac{1}{2}\pi m_2 & \cdots & (M-qn)^{m_q} \sin \tfrac{1}{2}\pi m_q \end{pmatrix}$$

Similarly, let $\Phi^*_{M,q}$ denote the determinant obtained by deleting the last row and last column of $\Phi_{M,q}$. We next establish

LEMMA 4. <u>Let</u> $m_1, m_2, \cdots, m_p$ <u>be distinct positive even integers, and let</u> $m_{p+1}, m_{p+2}, \cdots, m_q$ <u>be distinct positive odd integers. For</u> $q = 2p+1$ <u>and</u> $n > 1$, <u>then</u>

$$(2.11) \qquad \Phi_{M,q} \neq 0 \text{ and } \Phi^*_{M,q} \neq 0.$$

<u>Further, if</u> $n = 2r$ <u>and if</u> $\Phi_{M+r-n,q}$ <u>and</u> $\Phi^*_{M+r-n,q}$ <u>are defined by (2.10) with M replaced by</u> M+r-n, <u>we similarly have that</u>

$$(2.12) \qquad \Phi_{M+r-n,q} \neq 0 \text{ and } \Phi^*_{M+r-n,q} \neq 0.$$

<u>Proof</u>. First, note that $\Phi_{M,q}$ of (2.10) is a determinant of order $q+1 = 2p+2$. From the hypotheses on the positive integers $\{m_i\}_{i=1}^q$, it follows that the matrix in (2.10) reduces to block-diagonal form, so that

$$\Phi_{M,q} = \pm V_1 \cdot V_2,$$

where $V_1$ and $V_2$ are generalized Vandermonde determinants of order p+1, given by

$$V_1 := \begin{pmatrix} 1 & M^{m_1} & \cdots & M^{m_p} \\ 1 & (M-n)^{m_1} & \cdots & (M-n)^{m_p} \\ \vdots & & & \vdots \\ 1 & (M-pn)^{m_1} & \cdots & (M-pn)^{m_p} \end{pmatrix} \quad V_2 := \begin{pmatrix} [M-(p+1)n]^{m_{p+1}} & \cdots & [M-(p+1)n]^{m_q} \\ [M-(p+2)n]^{m_{p+1}} & \cdots & [M-(p+2)n]^{m_q} \\ \vdots & & \vdots \\ [M-qn]^{m_{p+1}} & \cdots & [M-qn]^{m_q} \end{pmatrix}$$

Applying Lemma 1 with $n > 1$, then $V_1 \neq 0$ and $V_2 \neq 0$, whence $\Phi_{M,q} \neq 0$. Since $\Phi^*_{M,q}$ has the same structure as that of $\Phi_{M,q}$, then $\Phi^*_{M,q} = V_1 V_2^*$, where $V_2^*$ is obtained from $V_2$ by omitting its last row and last column. From Lemma 1 again, $\Phi^*_{M,q} \neq 0$. In the same way, (2.12) follows. ∎

## 3. THE CASE n ODD: n = 2r+1

Theorem 1 is an easy consequence of the following two lemmas. Here, $m_1, m_2, \cdots, m_q$ are again distinct, not necessarily ordered, positive integers.

LEMMA 5. *If* $n = 2r+1$, $q = 2p+1$, *and if* $(0, m_1, \cdots, m_{q-1})$-*interpolation is regular on* X *with respect to* $\mathcal{T}_{M-1}$, *then* $(0, m_1, \cdots, m_q)$-*interpolation is regular on* X *with respect to* $\mathcal{T}_{M+r,\varepsilon}$, *provided that* $\varepsilon=+1$ *if* $m_q$ *is odd, and* $\varepsilon=0$ *if* $m_q$ *is even*.

LEMMA 6. *If* $n = 2r+1$, $q = 2p$, *and if* $(0, m_1, \cdots, m_{q-1})$-*interpolation is regular on* X *with respect to* $\mathcal{T}_{M,\varepsilon}$ *where* $\varepsilon$ *is as in Theorem* 1, *then* $(0, m_1, \cdots, m_q)$-*interpolation is regular on* X *with respect to* $\mathcal{T}_{M+r}$, *provided that* $o_q - e_q = 0$.

We first outline the proof of Theorem 1, assuming the validity of Lemmas 5 and 6, and then we shall turn to the proofs of these lemmas. Since Lagrange interpolation, corresponding to the case $q = 0$, is evidently regular on X with respect to $\mathcal{T}_r$, we may apply Lemma 5 to deduce that $(0, m_1)$-interpolation is regular on X with respect to $\mathcal{T}_{2r+1,\varepsilon}$, where $\varepsilon = 1$ if $m_1$ is odd and $\varepsilon = 0$ if $m_1$ is even. Now, applying Lemma 6 with $q = 2$, we deduce that $(0, m_1, m_2)$-interpolation is regular on X with respect to $\mathcal{T}_{3r+1}$, provided that $o_2-e_2=0$, i.e., $m_1$ and $m_2$ are of different parity. Thus, by repeated alternate use of Lemmas 5 and 6, we see that $(0,m_1,\cdots,m_q)$-interpolation is regular on X as described in Theorem 1, provided that $m_1$, $m_2$, $\cdots$, $m_q$ satisfy (1.6).

Before we turn to the proofs of Lemmas 5 and 6, we observe that in each of these lemmas, it is assumed that $(0, m_1, \cdots, m_{q-1})$-interpolation is regular on X with respect to an appropriate $\mathcal{T}_s$ or $\mathcal{T}_{s,\varepsilon}$. This implies that there exists a linear operator $\mathcal{L}_n$ of the form

$$(3.1)\quad \mathcal{L}_n(x;\ f) := \sum_{\nu=0}^{q-1}\sum_{k=0}^{n-1} f^{(m_\nu)}(x_k)\,\rho_{k,m_\nu}(x), \qquad (m_0 := 0),$$

mapping any sufficiently differentiable function of period $2\pi$ into a trigonometric polynomial, where the $\rho_{k,m_\nu}(x)$ are the associated fundamental trigonometric polynomials of $(0, m_1, \cdots, m_{q-1})$-interpolation on X, i.e., (cf. (1.4)),

$$(3.2)\quad \rho_{k,m_\nu}^{(m_j)}(x_i) = \delta_{j,\nu}\delta_{k,i};\ i,\ k=0,1,\cdots,n-1;\ j,\nu=0,1,\cdots,q-1.$$

Proof of Lemma 5. As the problem in (1.3) is linear, we consider instead of (1.3) the associated homogeneous system defined by

$$(3.3)\quad T^{(m_\nu)}(x_k) = 0,\ k = 0,1,\cdots,n-1;\ \nu = 0,1,\cdots,n-1,$$

where $T(x) \in \mathcal{T}_{M+r,\varepsilon}$. The lemma is proved if we can show that $T(x) \equiv 0$.

First, for any $T(x) \in \mathcal{T}_{M+r,\varepsilon}$, we can write $T(x)$ as the sum

$$(3.4)\quad T(x) = A(x) + B_{M,r}(x),$$

where $A(x) \in \mathcal{T}_{M-1}$, and where

$$(3.5)\quad B_{M,r}(x) = \sum_{j=0}^{r-1}\{a_j\cos(M+j)x + b_j\sin(M+j)x\} + c_r\cos[(M+r)x - \tfrac{\pi\varepsilon}{2}],$$

with $\varepsilon = 1$ or 0, according as $m_q$ is odd or even. Now, by the regularity hypothesis, the linear operator $\mathcal{L}_n$ of (3.1) is a projection on $\mathcal{T}_{M-1}$, and (3.3) gives $\mathcal{L}_n(x;\, T(x)) \equiv 0$. Thus, applying $\mathcal{L}_n$ to (3.4) gives

$$A(x) = -\mathcal{L}_n(x;\, B_{M,r}(x)),$$

so that from (3.4),

$$(3.6) \quad T(x) = B_{M,r}(x) - \mathcal{L}_n(x; B_{M,r}) := \sum_{j=0}^{r-1} \{a_j \omega_j(x) + b_j \mu_j(x)\} + c_r \omega_{r,\varepsilon}(x),$$

where we set

$$(3.7) \quad \begin{cases} \omega_j(x) := \cos(M+j)x - \mathcal{L}_n(x;\, \cos(M+j)x), \quad j = 0,1,\cdots,r-1, \\ \mu_j(x) := \sin(M+j)x - \mathcal{L}_n(x;\, \sin(M+j)x), \quad j = 0,1,\cdots,r-1, \\ \omega_{r,\varepsilon}(x) := \cos\left[(M+r)x - \frac{\pi\varepsilon}{2}\right] - \mathcal{L}_n\left(x;\, \cos\left[(M+r)x - \frac{\pi\varepsilon}{2}\right]\right). \end{cases}$$

We claim that

$$(3.8) \quad \begin{cases} \omega_j^{(m_q)}(x_k) = \Delta_{M-j-1,q} \cos\left[(r-j)x_k + \frac{\pi m_q}{2}\right] / \Delta^*_{M-j-1,q}, \\ \qquad\qquad j = 0,1,\cdots,r-1, \\ \mu_j^{(m_q)}(x_k) = \Delta_{M-j-1,q} \sin\left[(r-j)x_k + \frac{\pi m_q}{2}\right] / \Delta^*_{M-j-1,q}, \\ \qquad\qquad j = 0,1,\cdots,r-1, \\ \omega_{r,\varepsilon}^{(m_q)}(x_k) = \Delta_{M-r-1,q} \cos\left[(m_q - \varepsilon)\frac{\pi}{2}\right] / \Delta^*_{M-r-1,q}, \end{cases}$$

for $k = 0, 1, \cdots, n-1$, where $\Delta_{M-j-1,q}$ and $\Delta^*_{M-j-1,q}$ are defined in §2 (cf. (2.6)). We shall establish (3.8) later in this section, and we continue with the preceding argument.

Applying the remaining conditions of (3.5), namely

$$T^{(m_q)}(x_k) = 0, \quad k = 0, 1, \cdots, n-1,$$

to the representation of (3.6), we see from (3.8) that the sum

$$(3.9) \quad \sum_{j=0}^{r-1} \left[\Delta_{M-j-1,q}\left\{a_j \cos\left[(r-j)x + \frac{\pi m_q}{2}\right] + b_j \sin\left[(r-j)x + \frac{\pi m_q}{2}\right]\right\} / \Delta^*_{M-j-1,q}\right] + c_r \Delta_{M-r-1,q} \cos\left[(m_q - \varepsilon)\frac{\pi}{2}\right] / \Delta^*_{M-r-1,q},$$

which is a trigonometric polynomial of degree r, vanishes at all n nodes of X. Since $n = 2r+1$, this means that this sum must vanish identically. Since, by (2.9) of Corollary 3, $\Delta_{M-j-1,q} \neq 0$ for all $j = 0, 1, \ldots, r-1$, and since $m_q - \varepsilon$ is, (cf. Theorem 1) by definition, even, it follows that all the coefficients $a_j$, $b_j$, and $c_r$ of (3.9) are all zero. But then, from (3.6), $T(x) \equiv 0$. ■

<u>Proof of (3.8)</u>. To calculate the $a_j$'s of (3.5), we observe from the definition of M that, since n and q are both odd, $\cos(M+j)x = \cos[M-j-1-qn]x$, so that from (3.1), we have

$$(3.10)\quad \mathcal{L}_n(x;\ \cos(M+j)x) = \sum_{\nu=0}^{q-1} (M-j-1-qn)^{m_\nu} I_{j,\nu}(x),$$

where

$$(3.11)\quad I_{j,\nu}(x) := \sum_{k=0}^{n-1} \cos[(M-j-1)x_k + \frac{\pi m_\nu}{2}]\cdot \rho_{k,m_\nu}(x), \qquad j = 0, 1, \ldots, r-1.$$

Next, we further observe that, as $2M = nq+1$, then

$$-M < M-j-1-\lambda n < M, \text{ for } j = 0, 1, \ldots, r;\ \lambda = 0, 1, \ldots, q-1,$$

so that by the reproducing character of the operator $\mathcal{L}_n$ on $\mathcal{T}_{M-1}$, we further have the identities

$$(3.12)\quad \cos(M-j-1-\lambda n)x = \sum_{\nu=0}^{q-1} (M-j-1-\lambda n)^{m_\nu} I_{j,\nu}(x); \qquad j = 0, 1, \ldots, r;\ \lambda = 0, 1, \ldots, q-1.$$

Combining (3.12) for the cases $j = 0, 1, \ldots, r-1$ with (3.10), this implies that the following determinants are zero:

$$(3.13)\quad \begin{vmatrix} 1 & (M-j-1)^{m_1} & \cdots & (M-j-1)^{m_{q-1}} & \cos(M-j-1)x \\ 1 & (M-j-1-n)^{m_1} & \cdots & (M-j-1-n)^{m_{q-1}} & \cos(M-j-1-n)x \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & [M-j-1-(q-1)n]^{m_1} & & [M-j-1-(q-1)n]^{m_{q-1}} & \cos[M-j-1-(q-1)n]x \\ 1 & [M-j-1-qn]^{m_1} & \cdots & [M-j-1-qn]^{m_{q-1}} & \mathcal{L}_n(x;\ \cos(M+j)x) \end{vmatrix}$$

$$= 0,\quad j=0,1,\ldots,r-1.$$

Expanding this determinant in terms of its last column and noting that the cofactor of $\mathcal{L}_n(x;\ \cos(M+j)x)$ is just $\Delta^*_{M-j-1,q}$

from the definition of $\omega_j(x)$ in (3.7) that

$$\omega_j(x) = \Delta/\Delta^*_{M-j-1,q}, \quad j=0,1,\ldots,r-1,$$

where is the determinant obtained by putting $\cos(M-j-1-qn)x$ in the determinant of (3.13) in place of $\mathcal{L}_n(x;\cos(M+j)x)$. Then, since all columns of $\Delta$, except the last, are independent of $x$, we see that differentiating $\omega_j(x)$ $m_q$-times and putting $x_k$ for $x$ gives, with the definition of $\Delta_{M-j-1,q}$ in (2.6), the desired first equation of (3.8). The second and third equations of (3.8) are similarly deduced, the derivation of the third relation in (3.8) making use of the fact that (3.12) is valid for $j = r$. ■

In a similar fashion, we give the

Proof of Lemma 6. Here, $n = 2r+1$ and $q = 2p$, so that $M=np$. Again, we consider the homogeneous system defined by

$$(3.14)\qquad T^{(m_\nu)}(x_k) = 0,\ \nu = 0,1,\cdots,q;\ k = 0,1,\cdots,n-1,$$

where $T(x)\in \mathcal{T}_{M+r}$. By hypothesis, the linear operator $\mathcal{L}_n$ of the form (3.1) exists and maps $f$ into $\mathcal{T}_{M,\varepsilon}$, where each $\rho_{k,m_\nu}(x)$ of (3.1) is now in $\mathcal{T}_{M,\varepsilon}$. Also, since we require that $o_q - e_q = 0$, we may suppose that $m_1,\cdots,m_p$ have the same parity as $\varepsilon$, while $m_{p+1},\cdots,m_{2p}$ have the opposite parity to $\varepsilon$.

For $T(x)\in\mathcal{T}_{M+r}$, we write $T(x) = A(x) + B_{M,r}(x)$, where $A(x)\in\mathcal{T}_{M,\varepsilon}$, and where

$$B_{M,r}(x)=b_0\sin(Mx-\tfrac{\pi\varepsilon}{2}) + \sum_{j=1}^{r}\{a_j\cos(M+j)x+b_j\sin(M+j)x\}.$$

Following the method of proof of Lemma 5, we see again that the conditions $\nu=0,1,\cdots,q-1$; $k=0,1,\cdots,n-1$ of (3.14) imply that

$$(3.15)\qquad \begin{aligned} T(x) &= B_{M,r}(x)-\mathcal{L}_n(x;B_{M,r}(x)) \\ &= \sum_{j=1}^{r}\{a_j\omega_j(x) + b_j\mu_j(x)\} + b_0\mu_{0,\varepsilon}(x), \end{aligned}$$

where

$$(3.16)\qquad \begin{cases} \omega_j(x) = \cos(M+j)x - \mathcal{L}_n(x;\cos(M+j)x),\ j=1,2,\cdots,r, \\ \mu_j(x) = \sin(M+j)x - \mathcal{L}_n(x;\sin(M+j)x),\ j=1,2,\cdots,r. \\ \mu_{0,\varepsilon}(x) = \sin(Mx-\tfrac{\pi\varepsilon}{2}) - \mathcal{L}_n(x;\sin(Mx-\tfrac{\pi\varepsilon}{2})). \end{cases}$$

We shall show that

$$(3.17)\quad \begin{cases} \omega_j^{(m_q)}(x_k)=\Delta_{M-j,q}\cos(jx_k-\frac{\pi m_q}{2})/\Delta^*_{M-j,q}, & j=1,2,\cdots,r \\ \mu_j^{(m_q)}(x_k)=\Delta_{M-j,q}\sin(jx_k-\frac{\pi m_q}{2})/\Delta^*_{M-j,q}, & j=1,2,\cdots,r, \\ \mu_{0,\varepsilon}^{(m_q)}(x_k)=-\Phi_{M,q}\sin[(m_q+\varepsilon)\frac{\pi}{2}]/\Phi^*_{M,q}, & \end{cases}$$

where $\Delta_{M-j,q}$ is given in (2.6), $\Phi_{M,q}$ is given in (2.10), and $\Delta^*_{M-j,q}$ and $\Phi^*_{M,q}$ are obtained from $\Delta_{M-1,q}$ and $\Phi_{M,q}$, respectively, by omitting their last row and last column.

We shall establish (3.17) later in this section, and we continue with the preceding argument. Applying the remaining conditions of (3.14), namely

$$T^{(m_q)}(x_k) = 0, \qquad k = 0, 1, \cdots, n-1,$$

to the representation of (3.15), we deduce from (3.17) that the sum

$$(3.18)\quad \begin{aligned} &-b_0\Phi_{M,q}\sin[(m_q+\varepsilon)\tfrac{\pi}{2}]/\Phi^*_{M,q}+\sum_{j=1}^{r}[\Delta_{M-j,q}\{a_j\cos(jx-\tfrac{\pi m_q}{2})+ \\ &b_j\sin(jx-\tfrac{\pi m_q}{2})\}/\Delta^*_{M-j,q}], \end{aligned}$$

which is a trigonometric polynomial of degree r, vanishes at all n = 2r+1 nodes of X, and thus vanishes identically. By Lemmas 3 and 4 (cf. (2.8) and (2.11)), the various determinants which appear in (3.18), i.e., $\Delta_{M-j,q}$, $\Delta^*_{M-j,q}$, $\Phi_{M,q}$, and $\Phi^*_{M,q}$, are all nonzero. In addition, the hypothesis of Lemma 6 and the condition $o_q-e_q = 0$, together imply that $m_q+\varepsilon$ is odd, so that $\sin(m_q+\varepsilon)\frac{\pi}{2}$ is nonzero. Thus, as the trigonometric polynomial in (3.18) vanishes identically, then the $a_j$'s $(j=1,2,\cdots,r)$ and the $b_j$'s $(j=0,1,\cdots,r)$ are all zero, whence $T(x) \equiv 0$ from (3.15). ∎

Proof of (3.17). Since q = 2p, then 2M = nq, so that $\cos(M+j)x = \cos(M-j-qn)x$ for all $j=1,2,\cdots,r$. Thus, with (3.1), we can write

$$\mathscr{L}_n(x;\ \cos(M+j)x) = \sum_{\nu=0}^{q-1}(M-j-qn)^{m_\nu}\,\hat{I}_{j,\nu}(x),$$

where

$$\hat{I}_{j,\nu}(x) := \sum_{k=0}^{n-1} \cos[(M-j)x_k + \frac{\pi m_\nu}{2}] \cdot \rho_{k,m_\nu}(x), \quad j=1,2,\cdots,r.$$

Since $\mathcal{L}_n$ reproduces the functions $\cos(M-j-\lambda n)x$ and $\sin(M-j-\lambda n)x$ for $\lambda=0,\cdots,q-1$, we can use the resulting identities similar to (3.12) to obtain $\omega_j(x)$ as $(\Delta^*_{M-1,q})^{-1}$ times a determinant, defined by adjoining the following extra row to $\Delta^*_{M-j,q}$, namely,

$$\{1, (M-j-qn)^{m_1}, (M-j-qn)^{m_2}, \cdots (M-j-qn)^{m_{q-1}}, \cos(M-j-qn)x\},$$

and the following extra column:

$$\{\cos(M-j)x, \cos(M-j-n)x, \cdots, \cos(M-j-qn)x\}^T.$$

Then, on differentiating $m_q$-times this resulting expression for $\omega_j(x)$ and putting x equal to $x_k$, the first relation of (3.17) is obtained. The proof of the second relation in (3.17) is similar and is omitted.

In order to derive the last formula in (3.17), we observe that

$$\mathcal{L}_n(x; \sin(Mx-\frac{\pi\varepsilon}{2})) = \sum_{\nu=0}^{q-1} M^{m_\nu} \sin(m_\nu-\varepsilon)\frac{\pi}{2}\cdot J_\nu(x), \tag{3.19}$$

where $J_\nu(x) := \sum_{k=0}^{n-1} \rho_{k,m_\nu}(x)$. We also note the identities

$$(3.20)\quad \begin{cases} \cos(Mx-\frac{\pi\varepsilon}{2}) = \sum_{\nu=0}^{q-1} M^{m_\nu} \cos(m_\nu-\varepsilon)\frac{\pi}{2}\cdot J_\nu(x), \\ \cos(M-\lambda n)x = \sum_{\nu=0}^{q-1} (M-\lambda n)^{m_\nu} \cos\frac{\pi m_\nu}{2}\cdot J_\nu(x), \quad \lambda=1,2,\cdots,p, \\ \sin(M-\lambda n)x = \sum_{\nu=0}^{q-1} (M-\lambda n)^{m_\nu} \sin\frac{\pi m_\nu}{2}\cdot J_\nu(x), \quad \lambda=p+1,\cdots,2p-1. \end{cases}$$

Since $m_1, \cdots, m_p$ have the same parity as $\varepsilon$ and since $m_{p+1},\cdots,m_q$ have the opposite parity, we shall treat the case when $\varepsilon = 0$ and $m_1,\cdots,m_p$ are even; the other case is analogously treated.

Thus, we assume that $\varepsilon = 0$, and that $m_1,\cdots,m_p$ are even. Now, the q identities given in (3.20) are linearly independent since the associated coefficient matrix is exactly $\Phi^*_{M,q}$ of Lemma 4 (cf. (2.10)), and is thus nonzero. On combining the q+1 equations (3.19) and (3.20), and setting $\mu_{0,0}(x) = \sin Mx - \mathcal{L}_n(x; \sin Mx)$, we obtain

$$\mu_{0,0}(x) =$$

$$\frac{-1}{\Phi^*_{M,q}} \begin{vmatrix} 1 & M^{m_1}\cos\frac{1}{2}\pi m_1 & \cdots & M^{m_{q-1}}\cos\frac{1}{2}\pi m_{q-1} & \cos Mx \\ 1 & (M-n)^{m_1}\cos\frac{1}{2}\pi m_1 & \cdots & (M-n)^{m_{q-1}}\cos\frac{1}{2}\pi m_{q-1} & \cos(M-n)x \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & (M-pn)^{m_1}\cos\frac{1}{2}\pi m_1 & \cdots & (M-pn)^{m_{q-1}}\cos\frac{1}{2}\pi m_{q-1} & \cos(M-pn)x \\ 0 & (M-(p+1)n)^{m_1}\sin\frac{1}{2}\pi m_1 & \cdots & (M-(p+1)n)^{m_{q-1}}\sin\frac{1}{2}\pi m_{q-1} & \sin(M-(p+1)n)x \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & (M-qn)^{m_1}\sin\frac{1}{2}\pi m_1 & \cdots & (M-qn)^{m_{q-1}}\sin\frac{1}{2}\pi m_{q-1} & \sin(M-qn)x \end{vmatrix}$$

From this, it is easy to see that

$$\mu_{0,0}^{(m_q)}(x_k) = \Phi_{M,q}\sin\frac{\pi m_q}{2}/\Phi^*_{M,q}, \quad k = 0, 1, \cdots, n-1.$$

Similarly, if $\varepsilon = 1$ and $m_1, \cdots, m_p$ are odd, we have

$$\mu_{0,0}^{(m_q)}(x_k) = \Phi_{M,q}\cos\frac{\pi m_q}{2}/\Phi^*_{M,q}, \quad k = 0, 1, \cdots, n-1,$$

which completes the proof of (3.17). ∎

## 4. EXPLICIT FORMS FOR THE FUNDAMENTAL POLYNOMIALS

The determinant $\Delta_{M-1,q}$ of (2.6), which occurred in the solution of Problem A, can also be effectively used to answer Problem B. We assume that the conditions of Theorem 1 or 2 are satisfied, so that the $(0,m_1,\cdots,m_q)$-interpolation problem is regular on the nodes $x_k = \frac{2k\pi}{n}$. Then, the linear system given by (1.4) has a unique solution $\rho_{i,m_\nu}(x)$ within the proper trigonometric class as defined by our Theorems.

As our nodes $x_i$ are equidistant, it is clear that

$$\rho_{i,m_\nu}(x) = \rho_{0,m_\nu}(x-x_i) \text{ for } i = 1, \cdots, n-1.$$

It is therefore sufficient to give explicit forms for the function $\rho_{0,m_\nu}(x)$ which is defined by the linear system

$$(4.1)\quad \begin{cases} \rho_{0,m_\nu}^{(m_j)}(x_k) = 0, & k = 0, 1, \cdots, n-1 \text{ and } j \neq \nu, \\ \rho_{0,m_\nu}^{(m_\nu)}(x_k) = \delta_{0,k}, & k = 0, 1, \cdots, n-1. \end{cases}$$

Using the determinants $\Delta_{M-j,q}$ as given in (2.6), we define certain trigonometric polynomials $N_{M-j}(x; m_\nu, q)$ which, for each $\nu = 0,1,\cdots,q$, are obtained from $\Delta_{M-j,q}$ by replacing its $(\nu+1)$-st column by

$$\{\cos[(M-j)x-\frac{\pi m_\nu}{2}],\ \cos[(M-j-n)x-\frac{\pi m_\nu}{2}], \cdots,\ \cos[(M-j-qn)x-\frac{\pi m_\nu}{2}]\}^T.$$

With these polynomials, we have

THEOREM 3. Let $n = 2r+1$, and let $m_1, m_2, \cdots, m_q$ be positive integers which satisfy condition (1.6) of Theorem 1. Then, for $\nu = 0,1,\cdots,q$, we have

$$(4.2)\quad \rho_{0,m_\nu}(x) = \begin{cases} \frac{1}{n}\frac{N_M(x;\ m_\nu,\ q)}{\Delta_{M,q}} + \frac{2}{n}\sum_{j=1}^{r}\frac{N_{M-j}(x;\ m_\nu,\ q)}{\Delta_{M-j,q}} & \text{for } q = 2p, \\ \frac{1}{n}\frac{N_{M-r-1}(x;\ m_\nu, q)}{\Delta_{M-r-1,q}} + \frac{2}{n}\sum_{j=1}^{r}\frac{N_{M-j}(x;\ m_\nu,\ q)}{\Delta_{M-j,q}} & \text{for } q = 2p+1. \end{cases}$$

Proof. The proof follows easily by observing that $\rho_{0,m_\nu}(x)$, as given by (4.2), satisfies (4.1). Indeed, we observe that

$$N_{M-j}^{(m_\nu)}(x_k;\ m_\nu,\ q) = \cos[(M-j)x_k]\cdot\Delta_{M-j,q}$$

and so, if $q = 2p$, then $M = pn$ and it follows that

$$\rho_{0,m_\nu}^{(m_\nu)}(x_k) = \frac{1}{n}[1+2\sum_{j=1}^{r}\cos j\ x_k] = \begin{cases} 0 & \text{if } k \neq 0; \\ 1 & \text{if } k = 0, \end{cases}$$

as desired. If $q = 2p+1$, then $M = pn + m + 1$, so that

$$\rho_{0,m_\nu}^{(m_\nu)}(x_k) = \frac{1}{n}[1 + 2 \sum_{j=1}^{r} \cos(r + 1 - j)x_k] = \begin{cases} 0 & \text{if } k \neq 0; \\ 1 & \text{if } k = 0. \end{cases}$$

Thus, Theorem 3 is verified directly. The case when $n = 2r$ is exactly similar, and we can give

THEOREM 4. *If* $n = 2r$ *and if* $m_1, \cdots, m_q$ *are positive integers which satisfy condition* (1.7) *of Theorem* 2, *then for* $\nu = 0, 1, \cdots, q-1$,

$$\rho_{0,m_\nu}(x) = \frac{1}{n}\frac{N_M(x; m_\nu, q)}{\Delta_{M,q}} + \frac{2}{n}\sum_{j=1}^{r-1}\frac{N_{M-j}(x; m_\nu, q)}{\Delta_{M-r,q}} + \frac{1}{n}\frac{N_{M-r}(x; m_\nu, q)}{\Delta_{M-r,q}}. \tag{4.3}$$

## 5. CONCLUSION

A natural problem which now arises is the problem of *convergence*; more precisely, as the number of nodes is allowed to increase, how well do our interpolating trigonometric polynomials approximate the function we interpolate?

We fix q positive integers $m_1, \cdots, m_q$ which satisfy condition (1.6) or (1.7), according as n is odd or even. Then, according to our theorems, there are defined linear operators $\mathcal{L}_n(x; f)$, given by

$$\mathcal{L}_n(x; f) = \sum_{k=0}^{n-1} f(x_k)\rho_{k,0}(x) + \sum_{\nu=1}^{q}\sum_{k=0}^{n-1} \beta_{k,\nu}\rho_{k,m_\nu}(x) \tag{5.1}$$

where $\rho_{k,0}$ and $\rho_{k,m_\nu}$ are the fundamental polynomials given in §5, and the $\{\beta_{k,\nu}\}_{k=0,\ \nu=1}^{n-1,\ q}$ are certain given numbers. If f is continuous and periodic with period $2\pi$, we are interested in finding conditions on the $\beta_{k,\nu}$ which will ensure that

$$\lim_{n\to\infty} \mathcal{L}_n(x; f) = f(x).$$

We hope to return to this problem.

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