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ON TWO CONJECTURES ON THE ZEROS OF GENERALIZED BESSEL POLYNOMIALS

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I. INTRODUCTION

In a path-finding paper, Krall and Frink [3] studied properties of the so-called Bessel polynomials (BP), and they defined there a generalization which has become known under the name generalized Bessel polynomials (GBP).

In his recent monograph, Grosswald [2] has given a systematic treatment of the GBP, including a chapter on the location of their zeros. Because of the still growing interest in GBP's for their many applications, it is important to study the location of the zeros of GBP's even more closely than has been done up to now. Using techniques developed in Saff

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and Varga [7]-[11], it is possible to improve many previous results on the location of the zeros of GBP. The full statements of these results, along with their proofs, will appear elsewhere (cf. de Bruin, Saff, and Varga [1]).

The purpose of this note is to focus on two outstanding conjectures associated with the zeros of GBP. The first, to be answered affirmatively in Section II, concerns a conjecture of Grosswald [2, p. 163, no. 6] on the stability of the zeros of GBP. The second conjecture, due to Luke [4, p. 194], concerns the asymptotic behavior of the unique (negative) real zero of odd-degree GBP. With our asymptotic results, given in Section III, this conjecture is shown to be incorrect, and the true asymptotic formula is obtained.

We now give the definition of GBP by means of an explicit formula.

Definition 1.1. For any real number a and for any nonnegative integer n, the GBP $y_n(z; a)$ is given by

$$y_n(z;\ a) := \sum_{k=0}^{n} \binom{n}{k} (n+a-1)_k \left(\frac{z}{2}\right)^k, \tag{1.1}$$

where, for any real number x and for any positive integer k,

$$(x)_0 := 1, \text{ and } (x)_k := x(x+1)\cdots(x+k-1). \tag{1.2}$$

It is immediately clear that the GBP $y_n(z; a)$ is of exact degree n iff $a \notin \{-2n+2, -2n+3, \cdots, -n+1\}$.

II. STABILITY AND THE GROSSWALD CONJECTURE

First, we give the new result of

Theorem 2.1. For any $n \geq 2$ with $n+a-1>0$, all zeros of the GBP $y_n(z; a)$ lie in the sector

$$S(n,a) := \{z = re^{i\theta} \in \mathbb{C} : |\theta| > \cos^{-1}(-a/(2n+a-2)),\ -\pi < \theta \leq \pi\}. \tag{2.1}$$

This result immediately implies the following known stability result (Martinez [5]) for the zeros of GBP.

Corollary 2.2. For any $n \geq 2$ and any $a \geq 0$, all zeros of the GBP $y_n(z; a)$ lie in the open left-half plane.

Using the methods of Saff and Varga [7] and [8], further improvements on Corollary 2.2 can be made:

Theorem 2.3. For each real number a, there exists a positive integer $n_0(a)$ such that all zeros of the GBP $y_n(z; a)$ lie in the open left-half plane for all $n \geq n_0(a)$. For $a \geq -1$, and for $a \geq -2$, one can take $n_0(a) = 2$ and $n_0(a) = 4$, respectively, while for $a < -2$, one can take $n_0(a) = 1 + [\![2^{3-a}]\!]$, where $[\![\cdot]\!]$ denotes the greatest integer function.

This last result then establishes a conjecture of Grosswald [2, p. 162, no. 6] on the stability of zeros of GBP's. Concerning the sharpness of Theorem 2.3 for the cases $a \geq -1$ and $a \geq -2$, the reader is referred to the more detailed results of [1].

III. ASYMPTOTICS AND THE LUKE CONJECTURE

Olver [6] proved that the zeros of the normalized ordinary BP $y_n(z/n; 2)$ tend, as $n \to \infty$, to a curve Γ in the closed left-hand plane, defined by

$$\Gamma := \{z \in \mathbb{C} : |\omega(z)| = 1 \text{ and } \operatorname{Re} z \leq 0\}, \tag{3.1}$$

where

$$\omega(z) := \frac{e^{\sqrt{1+z^{-2}}}}{z\{1+\sqrt{1+z^{-2}}\}}, \tag{3.2}$$

so that $\{\pm i\}$ are endpoints of Γ. Using the asymptotic methods of Saff and Varga [11], this can be substantially sharpened for the case $a = 2$, as well as generalized to any real a.

Theorem 3.1. *For any fixed real* a, $\hat{z}$ *is a limit point of zeros of the normalized GBP* $y_n(2z/(2n+a-2);\ a)$, *as* $n \to \infty$, *iff* $\hat{z} \in \Gamma$. *If* γ *is a closed arc of* $\Gamma \setminus \{\pm i\}$ *with endpoints* μ_1 *and* μ_2 *(with* $\frac{\pi}{2} < \arg \mu_1 \le \arg \mu_2 < \frac{3\pi}{2}$), *where* $\omega(\mu_j) = e^{i\phi_j}$, $j = 1, 2$, $(\frac{\pi}{2} < \phi_2 \le \phi_1 < \frac{3\pi}{2})$, *let* $\tau_n(\gamma)$ *denote the number of zeros of* $y_n(z;\ a)$ *which satisfy* $\arg \mu_1 \le \arg z \le \arg \mu_2$. *Then,*

$$\lim_{n\to\infty} \tau_n(\gamma)/n = (\phi_1 - \phi_2)/\pi. \tag{3.3}$$

Moreover, for any fixed $a \ge 2$, *there exists an integer* $n_1(a)$ *such that for each* $n \ge n_1(a)$, *every zero* z *of the normalized GBP* $y_n(2z/(2n+a-2);\ a)$ *satisfies* $|\omega(z)| > 1$ *and* $\mathrm{Re}\ z < 0$.

The last statement of the above theorem has the following geometrical interpretation. Letting $\gamma := \{z = iy \in \mathbb{C} : -1 \le y \le 1\}$, then, under the conditions given in Theorem 3.1, these zeros all lie *inside* the closed curve $\Gamma \cup \gamma$.

Finally, we turn to the behavior of the unique (negative) real zero of an odd-degree GBP $y_n(z;\ a)$, denoted by $\alpha_n(a)$. Our new result is

Theorem 3.2. *For any fixed real number* a,

$$\frac{2}{\alpha_n(a)} = (2n+a-2)\hat{r} + K(\hat{r};\ a) + \mathcal{O}\left(\frac{1}{2n+a-2}\right), \text{ as } n \to \infty, \tag{3.4}$$

where $\hat{r}$ *is the unique negative root of*

$$-\hat{r}\, e^{\sqrt{1+\hat{r}^2}} = 1 + \sqrt{1+\hat{r}^2} \quad (\hat{r} \doteq -0.662\ 743\ 419), \tag{3.5}$$

and where

$$K(\hat{r};\ a) := \hat{r}[\sqrt{1+\hat{r}^2} + (2-a)\ \ln(\hat{r} + \sqrt{1+\hat{r}^2})]/\sqrt{1+\hat{r}^2}. \tag{3.6}$$

With the approximate value of $\hat{r}$ from (3.5), we have

Corollary 3.3. *For any fixed real number* a,

$$\frac{2}{\alpha_n(a)} \doteq 2n\,\hat{r} - 1.006\ 289\ 950\ a + 1.349\ 836\ 480 + \mathcal{O}\left(\frac{1}{2n+a-2}\right), \quad \underline{\text{as}}\ n \to \infty. \tag{3.7}$$

Recalling now the conjecture of Luke [4, p. 194] that

$$\frac{2}{\alpha_n(a)} \sim 2n\,\hat{r} - a + (\pi+1)/\pi, \ a > 0, \text{ as } n \to \infty, \tag{3.8}$$

we see that, since $(\pi+1)/\pi \doteq 1.318\ 309\ 886$, neither of the constant terms in this conjecture is correct, although these conjectured constants had a maximum relative error of only 2%.

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