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Interpolation in the roots of unity: An extension of a theorem of J. L. Walsh

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Dedicated to Professor Alexander M. Ostrowski

§1. Introduction

Let A_ρ denote the collection of functions analytic in $|z|<\rho$ and having a singularity on the circle $|z|=\rho$. (We assume throughout that $1<\rho<\infty$.) Next, for each positive integer n, let $p_{n-1}(z;f)$ denote the Lagrange polynomial interpolant of $f(z)\in A_\rho$ in the n-th roots of unity, i.e.,

$$p_{n-1}(\omega;f)=f(\omega) \tag{1.1}$$

where ω is any n-th root of unity, and let

$$P_{n-1}(z;f):=\sum_{k=0}^{n-1} a_k z^k \tag{1.2}$$

be the $(n-1)$-st partial sum of the power series expansion for $f(z)=\sum_{k=0}^{\infty} a_k z^k$. Then, a well-known and beautiful result of J. L. Walsh [20, p. 153] can be stated as

THEOREM A. *Let $f(z)\in A_\rho$. Then*

$$\lim_{n\to\infty}(p_{n-1}(z;f)-P_{n-1}(z;f))=0, \qquad \forall |z|<\rho^2, \tag{1.3}$$

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the convergence being uniform and geometric for all $|z| \le Z < \rho^2$. *Moreover, the result of* (1.1) *is best possible* (*in the sense that* (1.3) *is not valid at each point of* $|z| = \rho^2$ *for all* $f \in A_\rho$).

What is surprising in Walsh's Theorem A is that the difference of $p_{n-1}(z;f)$ and $P_{n-1}(z;f)$ converges to zero on a *larger* set than the domain of definition of f. Because of this, one is motivated to ask if it is possible to *modify* the polynomial $P_{n-1}(z;f)$ in a *systematic* way from the power-series for f to produce a *new* polynomial in π_{n-1} (where π_{n-1} as usual denotes all complex polynomials of degree at most $n-1$) such that the difference between the Lagrange interpolation polynomial and this new polynomial converges to zero on a *larger* set than $|z| < \rho^2$ for all functions in A_ρ.

Our object in this paper is to examine other interpolation processes and certain linear projection operators, all in the spirit of Theorem A of Walsh. Section 2 deals with Lagrange interpolation, and gives a simple extension of Theorem A. Section 3 is devoted to Hermite interpolation. Somewhat surprisingly, it turns out that in Hermite interpolation, the region of convergence to zero of the difference of the Hermite interpolant and the corresponding Taylor polynomial, becomes *smaller* than that of (1.3) of Theorem A for the case of Lagrange interpolation. This naturally leads one to ask if a similar shrinking takes place when one uses lacunary (Hermite-Birkhoff) interpolation in the roots of unity. We then examine in detail three cases of such lacunary interpolation in §5–7. Building on results of Motzkin and Sharma [11], "next-to-interpolatory" polynomials are considered in §8 in relation to Theorem A, while in §9, we deal with interpolation by polynomials in z and z^{-1} (cf. Sharma [16]) in relation to Theorem A.

In §10, we return to Walsh's Theorem A and give another extension of this theorem. It appears that the results of earlier sections have similar extensions.

Walsh's Theorem A can also be extended from polynomial to rational interpolations, and will be treated elsewhere.

§2. Lagrange Interpolation

The following new result gives Walsh's Theorem A as the special case $l = 1$.

THEOREM 1. *Let* $f(z) \in A_\rho$. *If*

$$P_{n-1,j}(z;f) := \sum_{k=0}^{n-1} a_{k+jn} z^k, \qquad j = 0, 1, \ldots, \tag{2.1}$$

then, for each positive integer l,

$$\lim_{n\to\infty}\left\{p_{n-1}(z;f)-\sum_{j=0}^{l-1}P_{n-1,j}(z;f)\right\}=0,\qquad \forall |z|<\rho^{l+1}, \tag{2.2}$$

the convergence being uniform and geometric for all $|z|\le Z<\rho^{l+1}$. *Moreover, the result of* (2.2) *is best possible.*

We remark from (2.1) that each $P_{n-1,j}(z;f)$ is in π_{n-1}, and is formed by summing the n consecutive terms $\sum_{k=1}^{n-1} a_{k+jn}z^{k+jn}$ of the power series expansion for f, and then multiplying this sum by the factor z^{-jn}. In this way, $P_{n-1,j}(z;f)$ and the sum $\sum_{j=0}^{l-1} P_{n-1,j}(z;f)$ are *systematically* determined from the power series expansion for f.

Proof. From the definition of $p_{n-1}(z;f)$ in (1.1), it can be directly verified that $p_{n-1}(z;f)$ has the integral representation:

$$p_{n-1}(z;f)=\frac{1}{2\pi i}\int_{\Gamma}\frac{f(t)(t^n-z^n)}{(t-z)(t^n-1)}\,dt, \tag{2.3}$$

where Γ is any circle $|t|=R$ with $1<R<\rho$. Similarly, from (2.1) and the Cauchy integral formula, $P_{n-1,j}(z;f)$ has the integral representation:

$$P_{n-1,j}(z;f)=\frac{1}{2\pi i}\int_{\Gamma}\frac{f(t)(t^n-z^n)}{(t-z)t^{n(j+1)}}\,dt,\qquad j=0,1,\ldots. \tag{2.4}$$

Thus, it follows for each positive integer l that

$$p_{n-1}(z;f)-\sum_{j=0}^{l-1}P_{n-1,j}(z;f)=\frac{1}{2\pi i}\int_{\Gamma}\frac{f(t)(t^n-z^n)}{(t-z)(t^n-1)t^{ln}}\,dt. \tag{2.5}$$

To bound the integral in (2.5), let $|f(t)|\le M$ on $|t|=R$. Then, we have for $|t|=R$ and for all z with $|z|\le\mu$, $(\mu\ge\rho)$, that

$$\left|\frac{t^n-z^n}{t-z}\right|\le\frac{\mu^n+R^n}{\mu-R},$$

and the integral in (2.5) is bounded above in modulus by

$$\frac{MR(\mu^n+R^n)}{(\mu-R)(R^n-1)R^{ln}}.$$

On taking nth roots.

$$\limsup_{n\to\infty}\left\{\max\left|p_{n-1}(z;f)-\sum_{j=0}^{l-1}P_{n-1,j}(z;f)\right|:|z|\leq\mu\right\}^{1/n}\leq\frac{\mu}{R^{l+1}}$$

and, as quantity on the left is independent of R, we can let R tend to ρ, whence

$$\limsup_{n\to\infty}\left\{\max\left|p_{n-1}(z;f)-\sum_{j=0}^{l-1}P_{n-1,j}(z;f)\right|:|z|\leq\mu\right\}^{1/n}\leq\frac{\mu}{\rho^{l+1}}.$$

Thus, for any positive number τ with $\rho^{-l}\leq\tau<1$, we have

$$\limsup_{n\to\infty}\left\{\max\left|p_{n-1}(z;f)-\sum_{j=0}^{l-1}P_{n-1,j}(z;f)\right|:|z|\leq\tau\rho^{l+1}\right\}^{1/n}\leq\tau<1,$$

which establishes the desired uniform and geometric convergence of (2.2) of Theorem 1.

To show that the result of Theorem 1 is best possible, consider the particular function $\hat{f}(z):=(\rho-z)^{-1}\in A_\rho$, which was also used by Walsh [20, p. 154] to establish the sharpness of his Theorem A. In this case, it can be verified that

$$p_{n-1}(z;\hat{f})=\frac{\rho^n-z^n}{(\rho-z)(\rho^n-1)},\qquad P_{n-1,j}(z;\hat{f})=\frac{\rho^n-z^n}{(\rho-z)\rho^{(j+1)n}},\tag{2.6}$$

whence

$$p_{n-1}(z;\hat{f})-\sum_{j=0}^{l-1}P_{n-1,j}(z;\hat{f})=\frac{\rho^n-z^n}{(\rho-z)\rho^{ln}(\rho^n-1)}.$$

For $|z|=\rho^{l+1}$, the above expression yields

$$\lim_{n\to\infty}\left[\min\left\{\left|p_{n-1}(z;\hat{f})-\sum_{j=0}^{l-1}P_{n-1,j}(z;\hat{f})\right|:|z|=\rho^{l+1}\right\}\right]\geqslant\frac{1}{\rho^{l+1}+\rho}>0,$$

showing that (2.2) of Theorem 1 is not valid at *any* point of the circle $|z|=\rho^{l+1}$ in this case. ■

It is natural to ask if this phenomenon of "overconvergence" of Theorems A and 1 extends to polynomials of *best approximation* on the closed unit disk. Specifically, for any $f(z)\in A_\rho$, let $\hat{p}_{n-1}(z;f)$ be the unique best uniform approximation to f from π_{n-1} on $|z|\le 1$:

$$\|\hat{p}_{n-1}f\|_{|z|\le 1}=\inf\{\|q-f\|_{|z|\le 1}:q\in\pi_{n-1}\},$$

where $\|g\|_A:=\sup\{|g(z)|:z\in A\}$. Then, with $P_{n-1,0}(z;f)$ of (2.1) denoting the $(n-1)$-st partial sum of $f(z)=\sum_{k=0}^{\infty}a_kz^k$, one can ask if

$$\lim_{n\to\infty}(\hat{p}_{n-1}(z;f)-P_{n-1,0}(z;f))=0,$$

in some set which is again larger than the domain of definition of f, for all $f\in A_\rho$. The following *negative* answer to this question was kindly pointed out to us by Professor D. J. Newman. For $\hat{f}(z):=(\rho-z)^{-1}$, it is known (cf. Rivlin [13]) that

$$\hat{p}_{n-1}(z;\hat{f})=\frac{\rho^{n-1}-z^{n-1}}{(\rho-z)\rho^{n-1}}+\frac{z^{n-1}}{(\rho^2-1)\rho^{n-2}}$$

for all $n\ge 2$, so that with (2.6),

$$\hat{p}_{n-1}(z;\hat{f})-P_{n-1,0}(z;\hat{f})=\frac{1}{(\rho^2-1)\rho}\left(\frac{z}{\rho}\right)^{n-1}.$$

In this case, it is evident that

$$\lim_{n\to\infty}\{\hat{p}_{n-1}(z;\hat{f}\}-P_{n-1}(z;\hat{f})\}=0 \text{ } \textit{only} \text{ for } |z|<\rho.$$

Returning to Theorem 1, we see from (2.2) that, on letting $l\to\infty$,

$$p_{n-1}(z;f)=\sum_{j=0}^{\infty}P_{n-1,j}(z;f), \tag{2.7}$$

for any $f(z) \in A_\rho$. If we set

$$p_{n-1}(z;f) = \sum_{k=0}^{n-1} c_k z^k, \tag{2.8}$$

where $c_k = c_k(n,f)$, then, with the definition of $P_{n-1,j}(z;f)$ in (2.1), we deduce from (2.7) the known formula (cf. Gautschi [4], Meinardus [10])

$$c_k = \sum_{j=0}^{\infty} a_{k+jn}, \qquad k=0,1,\ldots,n-1. \tag{2.9}$$

It is also natural to ask if a sharpening of (2.2) of Theorem 1 is possible if the stronger hypothesis, that f in A_ρ is *continuous* in $|z| \leq \rho$, is made. The *affirmative* answer to this question, posed by Professor J. Szabados, is given below in Theorem 2. For notation, $C(D_\rho)$ denotes all functions continuous in the closed disk $D_\rho := \{z : |z| \leq \rho\}$.

THEOREM 2. *Let* $f(z) \in A_\rho \cap C(D_\rho)$. *With the definitions of* (1.1) *and* (2.1), *then, for each positive integer* l,

$$\lim_{n\to\infty} \left\{ p_{n-1}(z;f) - \sum_{j=0}^{l-1} P_{n-1,j}(z;f) \right\} = 0, \quad \forall\, |z| \leq \rho^{l+1}, \tag{2.10}$$

the convergence being uniform and geometric for all $|z| \leq Z < \rho^{l+1}$. *Moreover, the result of* (2.10) *is best possible* (*in the sense that* (2.10) *is not valid at each point of* $|z| > \rho^{l+1}$ *for all* $f \in A_\rho \cap C(D_\rho)$).

Proof. For any $f(z) \in A_\rho \cap C(D_\rho)$, let $s_{n-1}(z;f)$ be the best approximation to f from π_{n-1} on D_ρ, i.e.,

$$E_{n-1}(f) := \inf \{ \|f-q\|_{D_\rho} : q \in \pi_{n-1} \} = \|f - s_{n-1}\|_{D_\rho}. \tag{2.11}$$

It is known (cf. Walsh [20, p. 89]) that

$$\lim_{n\to\infty} E_{n-1}(f) = 0. \tag{2.12}$$

Next, it is evident from (1.1) and (2.1) that

$$p_{n-1}(z;s_{n-1}) = \sum_{j=0}^{l-1} P_{n-1}(z;s_{n-1}) \text{ for each } l = 1, 2, \ldots.$$

Because of the linearity of the operators involved, this implies that

$$p_{n-1}(z;f)-\sum_{j=0}^{l-1} P_{n-1,j}(z;f)=p_{n-1}(z;f-s_{n-1})-\sum_{j=0}^{l-1} P_{n-1,j}(z;f-s_{n-1}),$$

so that from (2.5),

$$p_{n-1}(z;f)-\sum_{j=0}^{l-1} P_{n-1,j}(z;f)=\frac{1}{2\pi i}\int_\Gamma \frac{(f(t)-s_{n-1}(t;f))(t^n-z^n)\,dt}{(t-z)(t^n-1)t^{ln}}. \tag{2.13}$$

Now, using (2.11), this integral can be bounded above by

$$\max\left\{\left|p_{n-1}(z;f)-\sum_{j=0}^{l-1} P_{n-1,j}(z;f)\right|:|z|\le\rho^{l+1}\right\}\le\frac{E_{n-1}(f)(\rho^{n(l+1)}+R^n)R}{(\rho^{l+1}-R)(R^n-1)R^{ln}},$$

and, as the quantity on the left is independent of R, we can again let R tend to ρ, whence

$$\max\left\{\left|p_{n-1}(z;f)-\sum_{j=0}^{l-1} P_{n-1,j}(z;f)\right|:|z|\le\rho^{l+1}\right\}\le\frac{E_{n-1}(f)(1+\rho^{-nl})}{\rho^{l}(1-\rho^{-l})(1-\rho^{-n})}.$$

But, as the right side, from (2.12), tends to zero as $n\to\infty$, we have the desired result of (2.10). That the convergence of (2.10) is uniform and geometric on $|z|\le Z<\rho^{l+1}$ follows of course from Theorem 1.

Finally, concerning the sharpness of (2.10), consider the specific function

$$g(z):=\sum_{n=2}^{\infty}\frac{(z/\rho)^n}{n(n-1)}=\left(1-\frac{z}{\rho}\right)\ln\left(1-\frac{z}{\rho}\right)+\frac{z}{\rho},$$

which is in $A_\rho\cap C(D_\rho)$. It can be verified that (2.10) fails for this specific function $g(z)$, for any real z with $z>\rho^{1+l}$. ■

We remark that the result of Theorem 2 is valid under even somewhat weaker hypotheses on f. Specifically, consider the subset of functions $f(z)$ in A_ρ for which $\int_0^{2\pi}|f(\rho e^{i\theta})|\,d\theta$ is finite. As Professor R. A. DeVore has kindly mentioned, on defining

$$E_{n-1,1}(f;\tau):=\inf\left\{\frac{1}{2\pi}\int_0^{2\pi}|f(\tau e^{i\theta})-q(\tau e^{i\theta})|\,d\theta : q\in\pi_{n-1}\right\}$$

for $0 \le \tau \le \rho$, then, by means of Cesàro $(C, 1)$-means of the partial sums of f, it can be shown that

$$\sup \{E_{n-1,1}(f; \tau) : 0 \le \tau \le \rho\} \to 0, \quad \text{as} \quad n \to \infty.$$

From the proof of Theorem 2, one then easily sees that the result of Theorem 2 applies equally well to this subset of A_ρ.

In subsequent developments and extensions of Theorem 1, it will be clear that analogous improvements can be made along the lines of Theorem 2 with the set $A_\rho \cap C(D_\rho)$ or with the subset of A_ρ for which $\int_0^{2\pi} |f(\rho e^{i\theta})| \, d\theta$ is finite. For brevity, these improvements will not be stated.

§3. Hermite Interpolation

The result of Theorem 1 can itself be generalized using Hermite interpolation. Let r be a fixed positive integer, and let n be any positive integer. We shall denote by $h_{rn-1}(z; f)$ the Hermite polynomial interpolant to $f, f', \ldots, f^{(r-1)}$ in the nth roots of unity, i.e.,

$$h_{rn-1}^{(j)}(\omega; f) = f^{(j)}(\omega), \qquad j = 0, 1, \ldots, r-1, \tag{3.1}$$

where ω is any n-th root of unity. If $f(z) = \sum_{k=0}^{\infty} a_k z^k$, we set

$$H_{rn-1,0}(z; f) := \sum_{k=0}^{rn-1} a_k z^k \tag{3.2}$$

to be corresponding $(rn-1)$-st partial sum of f, and we set

$$H_{rn-1,j}(z; f) := \beta_j(z) \sum_{k=0}^{n-1} a_{k+n(r+j-1)} \cdot z^k, \qquad j = 1, 2, \ldots, \tag{3.3}$$

where $\beta_j(z) = \beta_j(z; n, r)$ is defined by

$$\beta_j(z) := \sum_{k=0}^{r-1} \binom{r+j-1}{k} (z^n - 1)^k, \qquad j = 1, 2, \ldots. \tag{3.4}$$

We can now formulate

THEOREM 3. *Let $f(z) \in A_\rho$, and let $h_{rn-1}(z; f)$, $H_{rn-1,0}(z; f)$ and $H_{rn-1,j}(z, f)$ $(j \ge 1)$ be polynomials defined by* (3.1), (3.2), *and* (3.3), *respectively. Then, for each* *positive integer l,*

$$\lim_{n\to\infty}\left\{h_{rn-1}(z;f)-\sum_{j=0}^{l-1}H_{rn-1,j}(z;f)\right\}=0, \qquad \forall|z|<\rho^{1+l/r}, \tag{3.5}$$

the convergence being uniform and geometric for all $|z|\le Z<\rho^{1+l/r}$. Moreover, the result of (3.5) *is best possible.*

We remark from (3.2)–(3.4) that each $H_{rn-1,j}(z;f)$ is in π_{rn-1}, and, as in the Lagrange case of §2, is formed in a systematic way by summing the n consecutive terms $\sum_{k=0}^{n-1}a_{k+n(r+j-1)}z^{k+n(r+j-1)}$ of the power series expansion for f, and then multiplying this sum by the rational function $\beta_j(z)/z^{n(r+j-1)}$, this rational function being independent of f.

We further remark that the case $r=1$ of Theorem 2 gives Theorem 1. Note, moreover, that, contrary to what one might expect, the case $r>1$ of Hermite interpolation gives a *smaller* region (i.e., $\rho^{1+l/r}$ vs. ρ^{1+l}) of convergence to zero in (3.5) than the case $r=1$ of Lagrange interpolation in (2.2). Even on comparing the Lagrange and Hermite cases on the basis of the *number* of coefficients a_k of $f(z)=\sum_{k=0}^{\infty}a_kz^k$ used by both interpolation schemes, the domain of convergence to zero in (2.2) of the difference of the polynomials of Theorem 1 in the Lagrange case *still* exceeds that of (3.5) of the Hermite case for $r>1$. To see this, $\sum_{j=0}^{l-1}P_{n-1,j}(z;f)$ depends from (2.1) on the coefficients a_k of f for $0\le k\le ln-1$, while $\sum_{j=0}^{l'-1}H_{rn-1,j}(z;f)$ from (3.3) depends on the coefficients a_k of f for $0\le k\le n(r+l'-1)-1$. On equating the number of coefficients of f used, we have $l=r+l'-1$, or $l'=l+1-r$. Thus, the bounds for the radii of convergence to zero for the Lagrange and Hermite cases are, respectively,

$$\rho^{l+1} \quad\text{and}\quad \rho^{(l+1)/r}.$$

For the proof of Theorem 3, we require the following two lemmas.

LEMMA 1. *For all t with $|t|>1$, the following identity holds:*

$$\frac{z^{rn}}{t^{rn}}-\frac{(z^n-1)^r}{(t^n-1)^r}=\frac{(t^n-z^n)}{t^{rn}}\sum_{s=1}^{\infty}\frac{\beta_s(z)}{t^{sn}}. \tag{3.6}$$

Proof. First, we observe from the definition in (3.4) that

$$\beta_1(z)=z^{rn}-(z^n-1)^r.$$

With this observation, it can be verified, by easy manipulations, that (3.6) is equivalent to

$$\frac{(z^n-1)^r}{(t^n-1)^r}=\frac{(z^n-1)^r}{t^{rn}}+\frac{1}{t^{rn}}\sum_{s=1}^{\infty}\frac{(z^n\beta_s(z)-\beta_{s+1}(z))}{t^{sn}}.$$

Multiplying both sides by t^{rn} and expanding $(1-t^{-n})^{-r}$ on the left in powers of t^{-n}, it is sufficient to show that the coefficients of t^{-sn} on both sides are the same. For $s=0$, this is obvious, while for any $s\geq 1$, this requires that

$$\binom{r+s-1}{s}(z^n-1)^r=z^n\beta_s(z)-\beta_{s+1}(z).$$

But, this now follows directly from (3.4). ■

Next, for any $\alpha(x)\in\pi_p$, $\sum_{s=0}^{\infty}\alpha(s)z^s$ is analytic in $|z|<1$, so that for any given δ with $0\leq\delta<1$, there exists a constant $N=N(\delta;\alpha)$ such that $|\sum_{s=0}^{\infty}\alpha(s)z^s|\leq N$ for all $|z|\leq\delta$. Applying this to the particular polynomial $\binom{x+r+l-1}{k}$ and to $z=t^{-n}$ with $|t|>1$, gives

LEMMA 2. *For any integers l and r and any integer k with $0\leq k\leq r-1$, and any $|t|>1$, there exists a constant $N=N(|t|;l,r,\nu)$ such that*

$$\left|\sum_{s=0}^{\infty}\binom{r+s+l-1}{k}t^{-sn}\right|\leq N,\qquad\forall n\geq 1. \tag{3.7}$$

Proof of Theorem 3. The Hermite interpolant $h_{rn-1}(z;f)$ satisfying (3.1) can be verified to have the following integral representation

$$h_{rn-1}(z;f)=\frac{1}{2\pi i}\int_{\Gamma}\frac{f(t)[(t^n-1)^r-(z^n-1)^r]}{(t-z)(t^n-1)^r}\,dt, \tag{3.8}$$

where Γ is any circle $|t|=R$ with $1<R<\rho$. Similarly, from (3.2) and (3.3), we have

$$H_{rn-1,0}(z;f)=\frac{1}{2\pi i}\int_{\Gamma}\frac{f(t)(t^{rn}-z^{rn})}{(t-z)t^{rn}}\,dt, \tag{3.9}$$

With this observation, it can be verified, by easy manipulations, that (3.6) is equivalent to

$$\frac{(z^n-1)^r}{(t^n-1)^r}=\frac{(z^n-1)^r}{t^{rn}}+\frac{1}{t^{rn}}\sum_{s=1}^{\infty}\frac{(z^n\beta_s(z)-\beta_{s+1}(z))}{t^{sn}}.$$

Multiplying both sides by t^{rn} and expanding $(1-t^{-n})^{-r}$ on the left in powers of t^{-n}, it is sufficient to show that the coefficients of t^{-sn} on both sides are the same. For $s=0$, this is obvious, while for any $s\geq 1$, this requires that

$$\binom{r+s-1}{s}(z^n-1)^r=z^n\beta_s(z)-\beta_{s+1}(z).$$

But, this now follows directly from (3.4). ■

Next, for any $\alpha(x)\in\pi_p$, $\sum_{s=0}^{\infty}\alpha(s)z^s$ is analytic in $|z|<1$, so that for any given δ with $0\leq\delta<1$, there exists a constant $N=N(\delta;\alpha)$ such that $|\sum_{s=0}^{\infty}\alpha(s)z^s|\leq N$ for all $|z|\leq\delta$. Applying this to the particular polynomial $\binom{x+r+l-1}{k}$ and to $z=t^{-n}$ with $|t|>1$, gives

LEMMA 2. *For any integers l and r and any integer k with $0\leq k\leq r-1$, and any $|t|>1$, there exists a constant $N=N(|t|; l, r, \nu)$ such that*

$$\left|\sum_{s=0}^{\infty}\binom{r+s+l-1}{k}t^{-sn}\right|\leq N,\qquad\forall n\geq 1. \tag{3.7}$$

Proof of Theorem 3. The Hermite interpolant $h_{rn-1}(z;f)$ satisfying (3.1) can be verified to have the following integral representation

$$h_{rn-1}(z;f)=\frac{1}{2\pi i}\int_{\Gamma}\frac{f(t)[(t^n-1)^r-(z^n-1)^r]}{(t-z)(t^n-1)^r}\,dt, \tag{3.8}$$

where Γ is any circle $|t|=R$ with $1<R<\rho$. Similarly, from (3.2) and (3.3), we have

$$H_{rn-1,0}(z;f)=\frac{1}{2\pi i}\int_{\Gamma}\frac{f(t)(t^{rn}-z^{rn})}{(t-z)t^{rn}}\,dt, \tag{3.9}$$

and

$$H_{rn-1,j}(z;f)=\frac{\beta_j(z)}{2\pi i}\int_\Gamma \frac{f(t)(t^n-z^n)}{(t-z)t^{(r+j)n}}\,dt. \tag{3.10}$$

Then, from (3.8)–(3.10), we have

$$h_{rn-1}(z;f)-\sum_{j=0}^{l-1} H_{rn-1,j}(z;f)=\frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{(t-z)}K(t,z)\,dt, \tag{3.11}$$

where

$$K(t,z):=\frac{z^{rn}}{t^{rn}}-\frac{(z^n-1)^r}{(t^n-1)^r}-\frac{(t^n-z^n)}{t^{rn}}\sum_{j=1}^{l-1}\frac{\beta_j(z)}{t^{jn}}, \qquad l=1,2,\dots, \tag{3.12}$$

(where we use the convention $\sum_{j=1}^{l-1}\equiv 0$ throughout if $l=1$). Using Lemma 1, $K(t,z)$ can also be expressed as

$$\begin{aligned} K(t,z)&=\frac{(t^n-z^n)}{t^{(r+l)n}}\sum_{s=0}^{\infty}\beta_{s+l}(z)\cdot t^{-sn}\\ &=\frac{(t^n-z^n)}{t^{(r+l)n}}\sum_{k=0}^{r-1}(z^n-1)^k\sum_{s=0}^{\infty}\binom{r+s+l-1}{k}\cdot t^{-sn}, \end{aligned}$$

using the definition of $\beta_j(z)$ in (3.4). Now, since $|t|=R>1$ on Γ, it follows from Lemma 2 that

$$|K(t,z)|\le M\,\frac{(R^n+|z|^n)(|z|+1)^{r-1}}{R^{(r+l)n}}, \text{ where } M \text{ is independent of } n. \tag{3.13}$$

Using (3.11) and (3.13), we see, as in the proof of Theorem 1, that

$$\limsup_{n\to\infty}\left\{\max\left|h_{rn-1}(z;f)-\sum_{j=0}^{l-1}H_{rn-1,j}(z;f)\right|:|z|\le\tau\rho^{1+l/r}\right\}^{1/rn}\le\tau<1,$$

where τ is any positive number with $\rho^{-l/r}\le\tau<1$, which again gives the desired uniform and geometric convergence of (3.5).

Finally, direct computation with the function $\hat{f}(z):=(\rho-z)^{-1}$ again shows that the result of (3.5) is best possible. ■

As in §2, letting $l \to \infty$ in (3.5) gives that

$$h_{rn-1}(z; f) = \sum_{j=0}^{\infty} H_{rn-1,j}(z; f) \tag{3.14}$$

for any $f \in A_\rho$. If we set

$$h_{rn-1}(z; f) = \sum_{\nu=0}^{rn-1} c_\nu z^\nu, \quad \text{and} \quad \beta_j(z) = \sum_{\nu=0}^{r-1} \gamma_{j,\nu} z^{n\nu}, \qquad j = 1, 2, \ldots, \tag{3.15}$$

where $c_\nu = c_\nu(r, n, f)$ and where $\gamma_{j,\nu} = \gamma_{j,\nu}(r, n)$, then from (3.14) we obtain, in analogy with (2.9), that

$$c_{j+kn} = a_{j+kn} + \sum_{i=1}^{\infty} \gamma_{i,k} a_{j+r(n+i-1)},$$

for all $0 \le k \le r-1$ and all $0 \le j \le n-1$.

§4. A Different Approach to Hermite Interpolations

It is not at all transparent how the polynomials $\beta_j(z)$ in (3.4) are arrived at. In order to bring this out more clearly, we shall now derive the auxiliary polynomials $H_{rn-1,j}(z; f)$ of (3.3) from a different point of view which will prove useful in Hermite-Birkhoff interpolation, to be encountered in later sections.

For any $f(z) = \sum_{k=0}^{\infty} a_k z^k$ in A_ρ, we can write from (3.2) that

$$f(z) = H_{rn-1,0}(z; f) + S(z), \tag{4.1}$$

where

$$S(z) = \sum_{k=rn}^{\infty} a_k z^k = \sum_{s=0}^{\infty} \sum_{j=0}^{n-1} a_{j+(r+s)n} z^{j+(r+s)n}. \tag{4.2}$$

Now, h_{rn-1} is a linear projection mapping from A_ρ into π_{rn-1}, so that applying h_{rn-1} to (4.1) gives

$$h_{rn-1}(z; f) = H_{rn-1,0}(z; f) + h_{rn-1}(z; S). \tag{4.3}$$

In order to determine $h_{rn-1}(z; S)$, we first determine $h_{rn-1}(z; z^{j+(r+s)n})$, where

$0 \le j \le n-1$ and where $s \ge 0$. Writing

$$z^{j+(r+s)n} = z^j \sum_{k=0}^{r+s} \binom{r+s}{k} (z^n - 1)^k$$
$$= z^j \sum_{k=0}^{r-1} \binom{r+s}{k} (z^n - 1)^k + z^j \sum_{k=r}^{r+s} \binom{r+s}{k} (z^{n-1})^k =: g_1^{(j,r,s)}(z) + g_2^{(j,r,s)}(z),$$

where $g_1^{(j,r,s)}(z) \in \pi_{rn-1}$, then applying h_{rn-1} to $z^{j+(r+s)n}$ yields

$$h_{rn-1}(z; z^{j+(r+s)n}) = g_1^{(j,r,s)}(z) + h_{rn-1}(z; g_2^{(j,r,s)}) = g_1^{(j,r,s)}(z),$$

since $g_2^{(j,r,s)}(z)$ vanishes in the n-th roots of unity, along with its first $r-1$ derivatives, whence $h_{rn-1}(z; g_2^{(j,r,s)}) \equiv 0$. Recalling the definition of (3.4), we further see that

$$h_{rn-1}(z; z^{j+(r+s)n}) = z^j \beta_{s+1}(z). \tag{4.4}$$

Now, applying h_{rn-1} to (4.2) and using (4.4),

$$h_{rn-1}(z; S) = \sum_{s=0}^{\infty} \sum_{j=0}^{n-1} a_{j+(r+s)n} h_{rn-1}(z; z^{j+(r+s)n})$$
$$= \sum_{s=0}^{\infty} \beta_{s+1}(z) \sum_{j=0}^{n-1} a_{j+(r+s)n} z^j = \sum_{s=1}^{\infty} H_{rn-1,s}(z; f),$$

the last equality following from the definition (3.3). Hence, with (4.3), (4.5) yields

$$h_{rn-1}(z; f) = H_{rn-1,0}(z; f) + \sum_{s=1}^{\infty} H_{rn-1,s}(z; f).$$

which establishes (3.14) in a different manner.

§5. Some Hermite-Birkhoff Interpolation Schemes: the (0, m) Case

From the foregoing, it is not clear a priori whether theorems analogous to Theorem 3 will hold when we replace Hermite interpolation in the n-th roots of unity by general Hermite-Birkhoff interpolation (called H-B interpolation for brevity) in the n-th roots of unity. It may be remarked that H-B problems of interpolation for real nodes has a considerable literature (cf. Lorentz and

Riemenschneider [8], Sharma [18], and van Rooij et al [19]), but when the nodes are on the unit circle, much less seems to be known (cf. [1]). However, as a consequence of the approach used in §4, we have been able to prove a general result on the unique solvability of a class of H-B problems in the roots of unity. More precisely, we have established the following result (cf. [1]):

THEOREM B. *For any nonnegative integer* q, *let* $\{m_i\}_{i=0}^{q}$ *be any nonnegative integers satisfying*

$$0=m_0<m_1<m_2<\cdots<m_q,$$

and let n *be any positive integer for which*

$$m_k\le kn \text{ for all } k=0,1,\ldots,q.$$

Then, the H-B problem $(0, m_1, m_2, \ldots, m_q)$ *in the* n*-th roots of unity is uniquely solvable for any given data.*

In this section, we examine the extension of Theorem 3 to the particular case $(0, m)$ of H-B interpolation in the n-th roots of unity, the case $q=1$ of Theorem B. (For this particular case, we remark that the unique solvability was first established by Kiš [6] for the case $m=2$, and then by Sharma [17] for the general case.) Because of unique solvability, for given positive integers m and n with $n\ge m$, and given any $f(z)\in A_\rho$, there is a unique polynomial $b_{2n-1}^{(0,m)}(z;f)$ in π_{2n-1} which interpolates f in the n-th roots of unity, and whose m-th derivative interpolates $f^{(m)}$ in the n-th roots of unity:

$$b_{2n-1}^{(0,m)}(\omega;f)=f(\omega) \quad\text{and}\quad (b_{2n-1}^{(0,m)}(\omega;f))^{(m)}=f^{(m)}(\omega), \tag{5.1}$$

where ω is any n-th root of unity. Of course, if $m=1$, then $b_{2n-1}^{(0,1)}(z;f)=h_{2n-1}(z;f)$, where $h_{2n-1}(z;f)$ is the Hermite interpolation polynomial of (3.1) with $r=2$. An integral representation for $b_{2n-1}^{(0,m)}(z;f)$ can be found by exhibiting a function $K_{2n-1}^{(0,m)}(t;z)$, which, as a function of z, is in π_{2n-1}, and which interpolates $(t-z)^{-1}$ and whose m-th derivative, as a function of z, interpolates $m!(t-z)^{-(m+1)}$, in the n-th roots of unity, for all $|t|>1$. Thus, by the Cauchy integral formula,

$$b_{2n-1}^{(0,m)}(z;f)=\frac{1}{2\pi i}\int_\Gamma f(t)K_{2n-1}^{(0,m)}(t,z)\,dt,$$

where Γ is any circle $|z|=R$ with $1<R<\rho$. For example, when $m=2$, a direct computation shows for $n\geq 2=m$ that

$$K^{(0,2)}_{2n-1}(t,z)=\frac{(t^n-z^n)(t^n+z^n-2)}{(t-z)(t^n-1)^2}+2\frac{(z^n-1)n}{(t^n-1)^3}\sum_{\nu=0}^{n-1}\frac{t^\nu z^{n-1-\nu}}{(3n-2\nu-3)}.$$

However, for $m\geq 2$, this method of finding $K^{(0,m)}_{2n-1}(t,z)$ becomes complicated and tedious. We shall circumvent this difficulty by using the method of §4.

For any positive integers m and n with $n\geq m$, set

$$b^{(0,m)}_{2n-1}(z;f)=\sum_{k=0}^{n-1}f(\omega^k)\alpha_{k,0}(z)+\sum_{k=0}^{n-1}f^{(m)}(\omega^k)\alpha_{k,m}(z), \tag{5.2}$$

where ω is a primitive n-th root of unity, and where $\{\alpha_{k,0}(z);\ \alpha_{k,m}(z)\}_{k=0}^{n-1}$ are the fundamental polynomials of this interpolation, i.e.,

$$\alpha_{k,0}(\omega^j)=\delta_{k,j}\quad\text{and}\quad\alpha^{(m)}_{k,0}(\omega^j)=0;\qquad \alpha_{k,m}(\omega^j)=0\quad\text{and}\quad\alpha^{(m)}_{k,m}(\omega^j)=\delta_{k,j},$$

for all $0\leq k,\ j\leq n-1$. If $f(z)=\sum_{k=0}^{\infty}a_kz^k$ is in A_ρ, set

$$B^{(0,m)}_{2n-1,0}(z;f):=\sum_{k=0}^{2n-1}a_kz^k, \tag{5.3}$$

and set

$$B^{(0,m)}_{2n-1,\nu}(z;f):=\sum_{j=0}^{n-1}a_{j+(\nu+1)n}z^jq_{j,\nu}(z),\qquad \nu=1,2,\ldots, \tag{5.4}$$

where $q_{j,\nu}(z)=q_{j,\nu}(z;n,m)\in\pi_n$ is defined by

$$q_{j,\nu}(z):=z^n+\frac{(\nu n+n+j)_m-(n+j)_m}{(n+j)_m-(j)_m}(z^n-1),\qquad j=0,1,\ldots,n-1, \tag{5.5}$$

where we use the standard notational convention that

$$\begin{cases}(j)_m=j(j-1)\cdots(j-m+1), & m\leq j,\quad\text{and}\\ (j)_m=0, & m>j.\end{cases}$$

Note that the denominator $(n+j)_m-(j)_m$ in (5.5) is positive for all $0\leq j\leq n-1$ since $n\geq m$.

We now prove

THEOREM 4. *Let $f(z) \in A_\rho$, and, for each fixed positive integer m and all positive integers n with $n \geq m$, let $b_{2n-1}^{(0,m)}(z;f)$, $B_{2n-1,0}^{(0,m)}(z;f)$, and $B_{2n-1,\nu}^{(0,m)}(z;f)$ $(\nu \geq 1)$ be the polynomials defined in* (5.2), (5.3), *and* (5.4), *respectively. Then, for each positive integer l,*

$$\lim_{n\to\infty} \left\{ b_{2n-1}^{(0,m)}(z;f) - \sum_{j=0}^{l-1} B_{2n-1,j}^{(0,m)}(z;f) \right\} = 0, \qquad \forall |z| < \rho^{1+l/2}, \tag{5.6}$$

the convergence being uniform and geometric for all $|z| \leq Z < \rho^{1+l/2}$. Moreover, the result of (5.6) *is best possible.*

We remark from (5.3) and (5.4) that each $B_{2n-1,\nu}^{(0,m)}(z;f)$ is in π_{2n-1}, which is formed in a systematic way, for $\nu \geq 1$, from the n consecutive terms $\sum_{j=0}^{n-1} a_{j+(\nu+1)n} z^{j+(\nu+1)n}$ of the power series expansion for f. We further remark that the bounds for the radii of convergence to zero for the Lagrange case (2.2) and the $(0, m)$ H-B case (5.6), on equating the number of coefficients of f used, are now respectively

$$\rho^{l+1} \quad \text{and} \quad \rho^{(l+1)/2}.$$

In order to prove Theorem 4, we shall need

LEMMA 3. *The following identity holds*:

$$b_{2n-1}^{(0,m)}(z;f) = B_{2n-1,0}^{(0,m)}(z;f) + \sum_{\nu=1}^{\infty} B_{2n-1,\nu}^{(0,m)}(z;f), \tag{5.7}$$

where the $B_{2n-1,\nu}^{(0,m)}(z;f)$ are defined in (5.3) *and* (5.4).

Proof. $b_{2n-1}^{(0,m)}$ is a linear projection mapping from A_ρ into π_{2n-1}. Thus, for any $f(z) = \sum_{j=0}^{\infty} a_j z^j$ in A_ρ, we have that

$$b_{2n-1}^{(0,m)}(z;f) = B_{2n-1,0}^{(0,m)}(z;f) + \sum_{s=2}^{\infty} \sum_{j=0}^{n-1} a_{j+sn} b_{2n-1}^{(0,m)}(z; z^{j+sn}). \tag{5.8}$$

On the other hand, from (5.2) we obtain

$$b_{2n-1}^{(0,m)}(z; z^{j+sn}) = \sum_{k=0}^{n-1} \omega^{kj} \alpha_{k,0}(z) + (j+sn)_m \sum_{k=0}^{n-1} \omega^{(n-m+j)k} \alpha_{k,m}(z). \tag{5.9}$$

Putting $f(z)=z^j$ and $f(z)=z^j(z^n-1)$ in succession in (5.2) easily yields, with $0\leq j\leq n-1$, that

$$\begin{cases} z^j = \sum_{k=0}^{n-1} \omega^{jk}\alpha_{k,0}(z), \\ z^j(z^n-1)=\{(n+j)_m-(j)_m\} \sum_{k=0}^{n-1} \omega^{(n-m+j)k}\alpha_{k,m}(z), \end{cases} \tag{5.10}$$

so that the sums $\sum_{k=0}^{n-1} \omega^{jk}\alpha_{k,0}(z)$ and $\sum_{k=0}^{n-1} \omega^{(n-m+j)k}\alpha_{k,m}(z)$ are each determined in terms of z^j and $z^j(z^n-1)$. Substituting for these sums in (5.9) and using (5.5) gives

$$b_{2n-1}^{(0,m)}(z; z^{j+sn}) = z^j q_{j,s-1}(z).$$

From this, using (5.4) and (5.8), (5.7) then follows. ■

Proof of Theorem 4. Denote for convenience the expression in braces in (5.6) by $\Delta_l^{(0,m)}(z)$. Then, from (5.7), we have for any $l\geq 1$ that

$$\Delta^{(0,m)}(z) = \sum_{\nu=l}^{\infty} B_{2n-1,\nu}^{(0,m)}(z; f).$$

Replacing a_v by its integral representation $\dfrac{1}{2\pi i}\displaystyle\int_\Gamma f(t)t^{-v-1}\,dt$, we see that

$$\Delta_l^{(0,m)}(z) = \frac{1}{2\pi i}\int_\Gamma f(t)K_{1,l}^{(0,m)}(t, z)\,dt, \tag{5.11}$$

where, from (5.4),

$$K_{1,l}^{(0,m)}(t, z) = \sum_{r=l}^{\infty}\sum_{j=0}^{n-1} \frac{z^j}{t^{j+1+(r+1)n}}\, q_{j,r}(z).$$

Using (5.5), however, it follows that

$$K_{1,l}^{(0,m)}(t, z) = \frac{z^n(t^n-z^n)}{(t-z)t^{(l+1)n}(t^n-1)} + K_{2,l}^{(0,m)}(t, z), \tag{5.12}$$

where

$$K_{2,l}^{(0,m)}(t,z)=(z^n-1)\sum_{r=l}^{\infty}\sum_{j=0}^{n-1}\frac{z^j}{t^{j+1+(r+1)n}}\left\{\frac{((r+1)n+j)_m-(n+j)_m}{(n+j)_m-(j)_m}\right\}. \tag{5.13}$$

Now, the difference $(n+j)_m-(j)_m$ strictly increases as j increases, so that

$$\max_{0\le j\le n-1}\left\{\frac{((r+1)n+j)_m-(n+j)_m}{(n+j)_m-(j)_m}\right\}<\frac{((r+2)n)_m}{(n)_m}<(r+2)^m. \tag{5.14}$$

Next, a short calculation using (5.14) shows that $|K_{2,l}^{(0,m)}(t,z)|$ can be bounded on the circle $|t|=R<|z|$ by

$$|K_{2,l}^{(0,m)}(t,z)|\le\frac{(|z^n|+1)(|z|^n+R^n)}{(|z|-R)R^{(l+2)n}}. \tag{5.15}$$

Thus, if $|f(t)|\le M$ on $|t|=R<|z|$, then from (5.11)–(5.15),

$$|\Delta_l^{(0,m)}(z)|\le\frac{|z|^n(R^n+|z|^n)MR}{(|z|-R)R^{(l+1)n}(R^n-1)}+\frac{(|z^n|+1)(|z|^n+R^n)MR}{(|z|-R)R^{(l+2)n}}.$$

Consequently,

$$\limsup_{n\to\infty}\{|\Delta_l^{(0,m)}(z)|:|z|\le\tau\rho^{1+l/r}\}^{1/2n}\le\tau<1,$$

where τ is any positive number with $\rho^{-l/2}\le\tau<1$, which gives the desired uniform and geometric convergence of (5.6). That (5.6) is best possible again follows from considering the special function $\hat{f}(z)=(\rho-z)^{-1}$ in A_ρ. ■

§6. Some Hermite-Birkhoff Interpolation Schemes: the (0, 2, 3) Case

In this section, we study the particular H-B interpolation problem (0, 2, 3) in the n-th roots of unity. In this case, it is known (cf. Theorem B of §5 and Sharma [15]) that this problem is uniquely solvable for all $n\ge2$, i.e., given any $f(z)\in A_\rho$ and any $n\ge2$, there is a unique polynomial $b_{3n-1}^{(0,2,3)}(z;f)$ in π_{3n-1} which interpolates f in the n-th roots of unity in the following sense:

$$(b_{3n-1}^{(0,2,3)}(\omega^k;f))^{(\nu)}=f^{(\nu)}(\omega^k),\quad \nu=0,2,3;\quad k=0,1,\dots,n-1, \tag{6.1}$$

where ω is any primitive n-th roof of unity. This polynomial can be expressed as

$$b_{3n-1}^{(0,2,3)}(z;f)=\sum_{k=0}^{n-1} f(\omega^k)\alpha_{k,0}(z)+\sum_{k=0}^{n-1} f''(\omega^k)\alpha_{k,2}(z)+\sum_{k=0}^{n-1} f'''(\omega^k)\alpha_{k,3}(z), \tag{6.2}$$

where $\alpha_{k,0}(z)$, $\alpha_{k,2}(z)$, and $\alpha_{k,3}(z)$ are the fundamental polynomials for this interpolation. These polynomials are explicitly given in Sharma [15], but their explicit form is not needed here. We do need, however, the result of

LEMMA 4. *For any integers r and j with $r\geq 2$ and $0\leq j\leq n-1$,*

$$\begin{cases} b_{3n-1}^{(0,2,3)}(z;z^{j+rn})=z^j\sum_{\nu=0}^{2}\binom{r}{\nu}(z^n-1)^\nu \\ +\binom{r}{3}\dfrac{6n^2(n+2j-1)Q_j(z)\cdot z^j}{(n+2j-1)(2n+j)_2-(3n+2j-1)(j)_2}, \end{cases} \tag{6.3}$$

where

$$Q_j(z):=(z^n-1)\left\{z^n-\frac{3n+2j-1}{n+2j-1}\right\}. \tag{6.4}$$

Proof. Expressing $z^{j+rn}=z^j\sum_{\nu=0}^{r}\binom{r}{\nu}(z^n-1)^\nu$ and noting from (6.1) that $b_{3n-1}^{(0,2,3)}(z;z^j(z^n-1)^\nu)\equiv 0$ for all $\nu\geq 4$, the linearity and reproducing character of the operator $b_{3n-1}^{(0,2,3)}$ gives

$$\begin{aligned} b_{3n-1}^{(0,2,3)}(z;z^{j+rn})&=\sum_{\nu=0}^{r}\binom{r}{\nu}b_{3n-1}^{(0,2,3)}(z;z^j(z^n-1)^\nu) \\ &=z^j\sum_{\nu=0}^{2}\binom{r}{\nu}(z^n-1)^\nu+\binom{r}{3}b_{3n-1}^{(0,2,3)}(z;z^j(z^n-1)^3). \end{aligned}$$

From (6.1) and (6.2), we easily deduce that

$$b_{3n-1}^{(0,2,3)}(z;z^j(z^n-1)^3)=6n^3\sum_{k=0}^{n-1}\omega^{k(n+j-3)}\alpha_{k,3}(z). \tag{6.6}$$

Next, with (6.4), simple manipulations show that the polynomial $z^jQ_j(z)$ satisfies the following conditions for $k=0,1,\ldots,n-1$:

$$z^jQ_j(z)\Big|_{z=\omega^k}=0, \qquad (z^jQ_j(z))''\Big|_{z=\omega^k}=0,$$

and

$$(z^jQ_j(z))'''\Big|_{z=\omega^k} = n\left[(2n+j)_2 - \frac{(3n+2j-1)}{(n+2j-1)}(j)_2\right]\omega^{k(n+j-3)},$$

so that from (6.2),

$$b_{3n-1}^{(0,2,3)}(z; z^jQ_j(z)) = n\left[(2n+j)_2 - \frac{(3n+2j-1)}{(n+2j-1)}(j)_2\right]\sum_{k=0}^{\infty}\omega^{k(n+j-3)}\alpha_{k,3}(z). \tag{6.7}$$

On the other hand, since $z^jQ_j(z) \in \pi_{3n-1}$, then $b_{3n-1}^{(0,2,3)}(z; z^jQ_j(z)) = z^jQ_j(z)$, so that combining (6.6) and (6.7) gives

$$b_{3n-1}^{(0,2,3)}(z; z^j(z^n-1)^3) = \frac{6n^2(n+2j-1)z^jQ_j(z)}{(n+2j-1)(2n+j)_2 - (3n+2j-1)(j)_2}.$$

Then, substituting the above in (6.5) gives the desired result of (6.3). ■

If $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is in A_ρ, set

$$B_{3n-1,0}^{(0,2,3)}(z; f) := \sum_{k=0}^{3n-1} a_k z^k, \tag{6.8}$$

and set

$$B_{3n-1,\nu}^{(0,2,3)}(z; f) := \sum_{j=0}^{n-1} a_{j+(\nu+2)n} b_{3n-1}^{(0,2,3)}(z; z^{j+(\nu+2)n}), \qquad \nu = 1, 2, \ldots. \tag{6.9}$$

With this notation, we come to

THEOREM 5. *Let $f(z) \in A_\rho$, and, for each integer $n \geq 2$, let $b_{3n-1}^{(0,2,3)}(z; f)$, $B_{3n-1,0}^{(0,2,3)}(z; f)$, and $B_{3n-1,\nu}^{(0,2,3)}(z; f)$ be the polynomials defined in* (6.1), (6.8), *and* (6.9), *respectively. Then, for each positive integer l,*

$$\lim_{n\to\infty}\left\{b_{3n-1}^{(0,2,3)}(z; f) - \sum_{j=0}^{l-1} B_{3n-1,j}^{(0,2,3)}(z; f)\right\} = 0, \qquad \forall |z| < \rho^{1+l/3}, \tag{6.10}$$

the convergence being uniform and geometric for all $|z| \leq Z < 1 + \rho^{1+l/3}$. Moreover, the result of (6.10) *is best possible.*

Proof. It follows, as in the reasoning in the proof of Theorem 4 (cf. (5.8)) that

$$b_{3n-1}^{(0,2,3)}(z;f)=B_{3n-1,0}^{(0,2,3)}(z;f)+\sum_{r=1}^{\infty}\sum_{j=0}^{n-1}a_{j+(r+2)n}b_{3n-1}^{(0,2,3)}(z;z^{j+(r+2)n}).$$

Replacing $a_{j+(r+2)n}$ by its integral representation, we have with Lemma 4 that

$$\begin{cases} b_{3n-1}^{(0,2,3)}(z;f)-\sum_{\nu=0}^{l-1}B_{3n-1,\nu}^{(0,2,3)}(z;f) \\ =\dfrac{1}{2\pi i}\int_{\Gamma}f(t)\left[\sum_{r=l}^{\infty}\sum_{j=0}^{n-1}\dfrac{z^j}{t^{j+1+(r+2)n}}\left\{A_r(z)+\binom{r+2}{3}C_j(z)\right\}\right]dt, \end{cases} \tag{6.11}$$

where

$$\begin{cases} A_r(z):=\sum_{\nu=0}^{2}\binom{r+2}{\nu}(z^n-1)^\nu, \quad \text{and} \\ C_j(z):=\dfrac{6n^2(n+2j-1)Q_j(z)}{(n+2j-1)(2n+j)_2-(3n+2j-1)(j)_2} \end{cases} \tag{6.12}$$

are polynomials of degree $2n$.

We first estimate the quantity in square brackets in (6.11) by splitting it into two parts, namely I_1 and I_2, where

$$I_1:=\sum_{r=l}^{\infty}\sum_{j=0}^{n-1}\frac{z^jA_r(z)}{t^{j+1+(r+2)n}},$$

and

$$I_2:=\sum_{r=l}^{\infty}\sum_{j=0}^{n-1}\binom{r+2}{3}\frac{z^jC_j(z)}{t^{j+1+(r+2)n}}.$$

With (6.12), we then have

$$\begin{aligned} I_1&=\sum_{\nu=0}^{2}(z^n-1)^\nu\sum_{r=0}^{\infty}\frac{\binom{r+l+2}{\nu}}{t^{(r+l+3)n}}\sum_{j=0}^{n-1}z^jt^{n-1-j} \\ &=\frac{(t^n-z^n)}{(t-z)}\frac{1}{t^{(l+3)n}}\sum_{\nu=0}^{2}(z^n-1)^\nu S_\nu(t), \end{aligned}$$

where

$$S_\nu(t) := \sum_{r=0}^{\infty} \frac{\binom{r+l+2}{\nu}}{t^{rn}}, \qquad \nu = 0, 1, 2.$$

For any t with $|t| > 1$, $S_0(t)$, $S_1(t)$, and $S_2(t)$ are bounded above from Lemma 2 by a fixed constant N_1, independent of n, so that

$$\left| \sum_{\nu=0}^{2} (z^n - 1)^\nu S_\nu(t) \right| \leq N_1 \sum_{\nu=0}^{2} (|z|^n + 1)^\nu,$$

from which it follows that

$$|I_1| \leq N_1 \left| \frac{t^n - z^n}{t - z} \right| \frac{\sum_{\nu=0}^{2} (|z|^n + 1)^\nu}{|t|^{(l+3)n}}.$$

In order to similarly bound $|I_2|$ above, we observe that the denominator in (6.12), namely

$$(n + 2j - 1)(2n + j)_2 - (3n + 2j - 1)(j)_2,$$

is an increasing function of j for all $0 \leq j \leq n - 1$, so that from (6.12),

$$|C_j(z)| \leq \frac{18 n^3 |Q_j(z)|}{2n(2n-1)(n-1)} \leq 18\, |Q_j(z)|. \tag{6.14}$$

Next, from (6.4),

$$Q_j(z) = z^{2n} - \left(1 + \frac{3n + 2j - 1}{n + 2j - 1}\right) z^n + \frac{3n + 2j - 1}{n + 2j - 1},$$

and as $(3n + 2j - 1)/(n + 2j - 1) \leq 5$ for all $0 \leq j \leq n - 1$, $n \geq 2$, we have that

$$|Q_j(z)| \leq |z|^{2n} + 6|z|^n + 5. \tag{6.15}$$

Combining the inequalities of (6.14) and (6.15), we obtain for $|z| > |t|$

$$|I_2| \leq 18 \frac{(|t|^n + |z|^n)(|z|^{sn} + 6|z|^n + 5)}{(|z| - |t|)|t|^{(l+3)n}} \sum_{r=0}^{\infty} \binom{r+l+2}{3} \frac{1}{|t|^{rn}}.$$

This last sum is again uniformly bounded above for all $n \geq 2$ from Lemma 2, so that for some constant N_2, indepedent of n,

$$|I_2| \leq N_2 \frac{(|t|^n + |z|^n)(|z|^{2n} + 6|z|^n + 5)}{(|z| - |t|)|t|^{(l+3)n}}. \tag{6.16}$$

Then, with (6.13) and (6.16), we can bound the integral in (6.11) for $|z| > \rho > R = |t|$ by

$$\left| b_{3n-1}^{(0,2,3)}(z; f) - \sum_{\nu=0}^{l-1} B_{3n-1,\nu}^{(0,2,3)}(z; f) \right|$$
$$\leq MR \frac{(R^n + |z|^n)}{(|z| - R)R^{(l+3)n}} \left\{ N_1 \sum_{\nu=0}^{2} (|z|^n + 1)^\nu + N_2(|z|^{2n} + 6|z|^n + 5) \right\},$$

which tends (uniformly and geometrically) to zero when $|z| < R^{1+l/3} < \rho^{1+l/3}$. This establishes the desired result of (6.10). Again, the sharpness of this result is provided by the function $\hat{f}(z) = (\rho - z)^{-1}$. ■

We remark that the bounds for the radii of convergence to zero for the Lagrange case (2.2) and the (0, 2, 3) H-B case (6.10), on equating the number of coefficients of f used, are now respectively

$$\rho^{l+1} \quad \text{and} \quad \rho^{(l+1)/3}.$$

§7. Some Hermite-Birkhoff Interpolation Schemes: the $(0, 1, \ldots, r-2, r+m-2)$ Case

In this section, we study the particular H-B interpolation problem $(0, 1, \ldots, r-2, r+m-2)$ in the n-th roots of unity, where r and m are arbitrary positive integers with $r \geq 2$ and $m \geq 1$. Note that when $m = 1$, this reduces to the Hermite interpolation problem $(0, 1, \ldots, r-1)$ in the n-th roots of unity which was treated in §3, while if $r = 2$, this reduces to the H-B interpolation problem $(0, m)$ in the n-th roots of unity which was treated in §5.

For this H-B problem $(0, 1, \ldots, r-2, r+m-2)$ in the n-th roots of unity, we know from Theorem B of §5 that this problem is uniquely solvable for any choice of $m \geq 1$ and $r \geq 2$, provided that $r+m-2 \leq (r-1)n$, i.e., given any $f(z) \in A_\rho$ and any n sufficiently large, there is a unique polynomial $b_{rn-1}^{(r-2,m)}(z; f)$ in π_{rn-1} which interpolates f in the n-th roots of unity in the following sense:

$$(b_{rn-1}^{(r-2,m)}(\omega^k; f))^{(j)} = f^{(j)}(\omega^k), \qquad j = 0, 1, \ldots, r-2, r+m-2; \quad k = 0, 1, \ldots, n-1, \tag{7.1}$$

where ω is any primitive solution of $\omega^n = 1$. In analogy with (5.2), we can express $b_{rn-1}^{(r-2,m)}(z; f)$ as (assuming n sufficiently large)

$$b_{rn-1}^{(r-2,m)}(z; f) = \sum_{\nu=0}^{r-2} \sum_{k=0}^{n-1} f^{(\nu)}(\omega^k)\alpha_{k,\nu}(z) + \sum_{k=0}^{n-1} f^{(r+m-2)}(\omega^k)\alpha_{k,r+m-2}(z), \tag{7.2}$$

where $\{\alpha_{k,\nu}(z)\}_{k=0,\,\nu=0}^{n-1,\,r-2}$ and $\{\alpha_{k,r+m-2}(z)\}_{k=0}^{n-1}$ are the associated fundamental polynomials for this interpolation. Since $b_{rn-1}^{(r-2,m)}$ reproduces polynomials in π_{rn-1}, we have

$$b_{rn-1}^{(r-2,m)}(z; z^{sn+j}) = z^{sn+j} \quad \text{for all} \quad 0 \le s \le r-1, 0 \le j \le n-1. \tag{7.3}$$

Next, for $s \ge r$, we use the identity

$$z^{sn+j} = z^j \sum_{\nu=0}^{s} \binom{s}{\nu}(z^n - 1)^\nu,$$

and observe, from the linearity and reproducing character of $b_{rn-1}^{(r-2,m)}$ and also (7.1), that

$$b_{rn-1}^{(r-2,m)}(z; z^{sn+j}) = z^j \sum_{\nu=0}^{r-1} \binom{s}{\nu}(z^n - 1)^\nu + \sum_{\nu=0}^{m-2} \binom{s}{\nu+r} b_{rn-1}^{(r-2,m)}(z; z^j(z^n - 1)^{\nu+r}). \tag{7.4}$$

Using (7.2) gives that

$$\begin{aligned} b_{rn-1}^{(r-2,m)}(z; z^j(z^n - 1)^{\nu+r}) &= \sum_{k=0}^{n-1} \left[\frac{d^{r+m-2}}{dz^{r+m-2}}\{z^j(z^n - 1)^{\nu+r}\}\right]_{z=\omega^k} \cdot \alpha_{k,r+m-2}(z) \\ &= N_{\nu+r,j} \sum_{k=0}^{n-1} \omega^{(n+j-r-m+2)k}\alpha_{k,r+m-2}(z), \end{aligned} \tag{7.5}$$

where $N_{\nu+r,j} = N_{\nu+r,j}(n)$ is defined by

$$N_{\nu+r,j} := \sum_{\mu=0}^{\nu+r} \binom{\nu+r}{\mu}(-1)^{\nu+r-\mu}(n\mu + j)_{r+m-2}, \tag{7.6}$$

using the convention $(j)_m$ of §5. Now, if $\nu=-1$ in (7.5), the polynomial $z^j(z^n-1)^{r-1}$ is reproduced by the interpolation operator $b_{rn-1}^{(r-2,m)}$ of (7.1), so that in this case, (7.5) reduces to

$$z^j(z^n-1)^{r-1}=N_{r-1,j}\sum_{k=0}^{n-1}\omega^{(n+j-r-m+2)k}\alpha_{k,r+m-2}(z). \tag{7.7}$$

Combining (7.4)–(7.7) then yields

$$b_{rn-1}^{(r-2,m)}(z;z^{sn+j})=z^j\sum_{\nu=0}^{r-1}\binom{s}{\nu}(z^n-1)^\nu+z^j(z^n-1)^{r-1}\sum_{\nu=0}^{s-r}\binom{s}{\nu+r}\frac{N_{\nu+r,j}}{N_{r-1,j}}. \tag{7.8}$$

Now, for any $f(z)=\sum_{\nu=0}^{\infty}a_\nu z^\nu$ in A_ρ, we define the polynomials

$$B_{rn-1,0}^{(r-2,m)}(z;f):=\sum_{\nu=0}^{rn-1}a_\nu z^\nu, \tag{7.9}$$

and

$$B_{rn-1,\nu}^{(r-2,m)}(z;f):=\sum_{j=0}^{n-1}a_{j+(\nu+r-1)n}b_{rn-1}^{(r-2,m)}(z;z^{j+(\nu+r-1)n}), \qquad \nu=1,2,\ldots. \tag{7.10}$$

With these polynomials, we state

THEOREM 6. *Let* $f(z)\in A_\rho$ *and, for any fixed positive integers* $r\geq 2$ *and* $m\geq 1$, *let* $b_{rn-1}^{(r-2,m)}(z;f)$, $B_{rn-1,0}^{(r-2,m)}(z;f)$, *and* $B_{rn-1,\nu}^{(r-2,m)}(z;f)(\nu\geq 1)$ *be the polynomials defined in* (7.2), (7.9), *and* (7.10), *respectively. Then, for each positive integer* l,

$$\lim_{n\to\infty}\left\{b_{rn-1}^{(r-2,m)}(z;f)-\sum_{j=0}^{l-1}B_{rn-1,j}^{(r-2,m)}(z;f)\right\}=0, \qquad \forall |z|<\rho^{1+l/r}, \tag{7.11}$$

the convergence being uniform and geometric for all $|z|\leq Z<\rho^{1+l/r}$. *Moreover, the result of* (7.11) *is best possible.*

We omit the proof of this theorem which follows along lines similar to the proofs of the previous sections. In particular, as in (5.14), we need upper bounds for the quantity

$$\max_{0\leq j\leq n-1}\frac{N_{\nu+r,j}}{N_{\nu-1,j}}$$

in order to bound $B_{m-1,s}^{(r-2,m)}(z;f)$ for all $s \geq l$. These estimates follow easily as $N_{\nu+r,j}$ is, for sufficiently large n, a *positive* increasing function of j (which is analogous to the statement preceeding (5.14)). This last property of $N_{\nu+r,j}$ is proved by treating the sums of (7.6) defining $N_{\nu+r,j}$, as divided differences and using results of Kloosterman [7].

We remark that the bounds for the radii of convergence to zero for the Lagrange case (2.2) and the $(0, 1, \ldots, r-2, r+m-2)$ H-B case (7.11) on equating the number of coefficients of f used, are now respectively

$$\rho^{l+1} \quad \text{and} \quad \rho^{(l+1)/r}. \tag{7.12}$$

Thus, in using either (7.11) or (7.12), we see that the $(0, 1, \ldots, r-2, r+m-2)$ H-B case behaves *precisely* like the Hermite case $(0, 1, \ldots, r-1)$, in comparison with the Lagrange case, with respect to the domain of convergence to zero of the associated differences of interpolatory polynomials. In the same manner, we see that the $(0, m)$ H-B case and the $(0, 2, 3)$ H-B case behave respectively like the Hermite cases $r=2$ and $r=3$. Similar analogies can be drawn from the results to be derived in §§8–9.

It may be thought that the statements of Theorems 4–6 are quite similar, although the proofs differ considerably in detail. In view of this, Theorem B of §5 suggests a unification of the three special cases of the H-B problem treated in Theorems 4–6. However, we were not able to achieve this. Instead, we offer the following conjecture.

CONJECTURE. *Consider the H-B interpolation problem* $(0, m_1, m_2, \ldots, m_q)$, *where* $m_{k_\infty} \leq kn$ *for all* $k=0, 1, \ldots, q$ *(cf. Theorem B of* §5*). For any* $f(z)=\sum_{j=0}^{\infty} a_j z^j$ *in* A_ρ, *let* $b_{(q+1)n-1}(z;f)$ *be the associated unique H-B polynomial interpolant in* $\pi_{(q+1)m-1}$, *and let*

$$B_{(q+1)n-1,0}(z;f) := \sum_{\nu=0}^{(q+1)n-1} a_\nu z^\nu, \tag{7.13}$$

and

$$B_{(q+1)n-1,\nu}(z;f) := \sum_{j=0}^{n-1} a_{j+(\nu+q)n} b_{(q+1)n-1}(z; z^{j+(\nu+q)n}), \qquad \nu = 1, 2, \ldots. \tag{7.14}$$

Then, for each positive integer l

$$\lim_{n\to\infty} \left\{ b_{(q+1)n-1}(z;f) - \sum_{j=0}^{l-1} B_{(q+1)n-1,j}(z;f) \right\} = 0, \qquad \forall |z| < \rho^{1+l/q+1}, \tag{7.15}$$

the convergence being uniform and geometric for all $|z| \le Z < \rho^{1+l/(q+1)}$. *Moreover, the result of* (7.15) *is best possible.*

§8. Next-to-Interpolatory Polynomials

We shall now show that Theorem A of Walsh can be extended to polynomial operators which are not necessarily interpolatory. Given $f(z) \in A_\rho$, and given any $n \ge 2$, let $s_{n-2}(z; f)$ denote the polynomial in π_{n-2} which minimizes

$$\max_{0 \le k \le n} |p_{n-2}(\omega^k) - f(\omega^k)|,$$

over all polynomials in π_{n-2}, where ω is any primitive n-th root of unity. Based on a result of Rivlin and Shapiro [14] which involves the concept of extremal signature, it can be verified that $s_{n-2}(z; f)$ in π_{n-2} is given by

$$s_{n-2}(z; f) = \frac{1}{n} \sum_{k=0}^{n-1} \frac{(z^n - 1)(f(\omega^k) - \lambda\omega^{-k})}{(z - \omega^k)} \omega^k, \tag{8.1}$$

where

$$\lambda := \frac{1}{n} \sum_{k=0}^{n-1} \omega^k f(\omega^k).$$

From (8.1), it then follows that $s_{n-2}(z; f)$ has the integral representation

$$s_{n-2}(z; f) = \frac{1}{2\pi i} \int_\Gamma \frac{f(t)t(t^{n-1} - z^{n-1})}{(t - z)(t^n - 1)} dt. \tag{8.2}$$

On setting, as in (2.1),

$$P_{n-2,j}(z; f) := \sum_{k=0}^{n-2} a_{k+j(n-1)} z^k, \qquad j = 0, 1, 2, \ldots, \tag{8.3}$$

we next establish

THEOREM 7. *Let* $f(z) \in A_\rho$. *Then, for each positive integer* l,

$$\lim_{n\to\infty} \{s_{n-2}(z; f) - P_{n-2,0}(z; f)\} = 0, \qquad \forall |z| < \rho^2, \tag{8.4}$$

the convergence being uniform and geometric for all $|z| \le Z < \rho^2$. *Moreover, the result of* (8.4) *is best possible.*

Proof. The proof follows along the lines of that of Theorem 1, on observing that

$$s_{n-2}(z;f) - P_{n-2,0}(z;f) = \frac{1}{2\pi i}\int_\Gamma \frac{f(t)(t^{n-1}-z^{n-1})\,dt}{(t-z)(t^n-1)t^{n-1}},$$

from which (8.4) easily follows. ∎

The integral formula (8.2) for $s_{n-2}(z;f)$ can be iterated, leading us to define

$$s_{n-1-r,r}(z;f) := \frac{1}{2\pi i}\int_\Gamma \frac{f(t)t^r(t^{n-r}-z^{n-r})}{(t-z)(t^n-1)}\,dt, \qquad r = 1, 2, \ldots, n-1, \tag{8.5}$$

a polynomial in π_{n-1-r}. This polynomial does not possess any approximation-theoretic interpretation for $r \ge 2$, but does give rise to the following analog of Theorem 7, which for convenience is stated without proof.

THEOREM 8. *Let* $f(z) \in A_\rho$, *and for each positive integer r set*

$$S_{n-1-r,\nu}(z;f) := \sum_{j=0}^{n-1-r} a_{j+\nu(n-r)} z^j, \qquad \nu = 1, 2, \ldots; n > r. \tag{8.6}$$

Then, for each positive integer l,

$$\lim_{n\to\infty}\left\{s_{n-1-r,r}(z;f) - \sum_{k=0}^{l-1} S_{n-1-r,k}(z;f)\right\} = 0, \qquad \forall |z| < \rho^{1+l}, \tag{8.7}$$

the convergence being uniform and geometric on all $|z| \le Z < \rho^{1+l}$. *Moreover, the result of* (8.7) *is best possible.*

We remark that the polynomial $s_{n-1-r,r}(z;f)$ of (8.5) also arises from a different derivation in (9.7) of §9.

Motzkin and Sharma [11] have considered next-to-interpolatory polynomials on sets with multiplicities. We shall consider, for the sake of brevity, only the case of multiplicity $r = 2$, though higher-order multiplicities can similarly be treated.

To begin, let ω denote a principal n-th root of unity, and denote by $\hat{h}_{2n-2}$ the polynomial in π_{2n-2} which interpolates f in the n-th roots of unity, and which simultaneously minimizes

$$\max_k |\hat{h}'_{2n-2}(\omega^k; f) - f'(\omega^k)|,$$

among all polynomials in π_{2n-2} which interpolate f in the n-th roots of unity. Again, from Rivlin and Shapiro [14], it can be verified that

$$\hat{h}_{2n-2}(z; f) = \sum_{k=0}^{n} f(\omega^k)\{1-(z-\omega^k)(n-1)\omega^{-2k}\}l_k^2(z)$$
$$+ \sum_{k=0}^{n} \{f'(\omega^k) - \lambda\omega^{-2k}\}(z-\omega^k)l_k^2(z),$$

where

$$l_k(z) := \frac{1}{n}\left(\frac{z^n-1}{z-\omega^k}\right)\omega^k,$$

and where

$$\lambda := \frac{1}{n}\left[(n-1)\sum_{k=0}^{n} \omega^k f(\omega^k) - \sum_{k=0}^{n} \omega^{2k} f'(\omega^k)\right].$$

By elementary calculations, it turns out that

$$\lambda = \frac{n}{2\pi i}\int_\Gamma \frac{f(t)}{(t^n-1)^2}\,dt.$$

It is also easy to verify that

$$\begin{cases} \hat{h}_{2n-2}(z; f) = \dfrac{1}{2\pi i}\displaystyle\int_\Gamma \frac{f(t)[(t^n-1)^2-(z^n-1)^2]}{(t-z)(t^n-1)^2}\,dt \\ \qquad - \dfrac{z^{n-1}(z^n-1)}{2\pi i}\displaystyle\int_\Gamma \frac{f(t)\,dt}{(t^n-1)^2}. \end{cases} \tag{8.9}$$

Since $\hat{h}_{2n-2}(z; f) \in \pi_{2n-2}$, we can rewrite (8.9) as

$$\hat{h}_{2n-2}(z; f) = \frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{(t-z)}\left[\frac{t(t^{2n-1}-z^{2n-1})}{(t^n-1)^2} - \frac{2(t^n-z^n)}{(t^n-1)^2}\right]dt$$
$$+ \frac{z^{n-1}}{2\pi i}\int_\Gamma \frac{f(t)\,dt}{(t^n-1)^2}.$$

On setting

$$\hat{H}_{2n-2,0}(z;f):=P_{2n-2,0}(z;f)=\frac{1}{2\pi i}\int_\Gamma \frac{f(t)(t^{2n-1}-z^{2n-1})}{(t-z)t^{2n-1}}\,dt,$$

we easily see that

$$\lim_{n\to\infty}\{\hat{h}_{2n-2}(z;f)-\hat{H}_{2n-2,0}(z;f)\}=0, \qquad \forall |z|<\rho^2.$$

Similarly, if we set for $k>1$,

$$\hat{H}_{2n-2,k}(z;f):=(1+k)\sum_{\nu=n}^{2n-2} a_{\nu+kn}z^\nu + a_{n-1+kn}z^{n-1} + (1-k)\sum_{\nu=0}^{n-2} a_{\nu+kn}z^\nu, \tag{8.10}$$

then we can analogously establish the result of

THEOREM 9. *Let $f(z)\in A_\rho$. Then, for each positive integer l,*

$$\lim_{n\to\infty}\left\{\hat{h}_{2n-2}(z;f)-\sum_{k=0}^{l-1}\hat{H}_{2n-2,k}(z;f)\right\}=0, \qquad \forall |z|<\rho^{1+l}, \tag{8.11}$$

the convergence being uniform and geometric for all $|z|\le Z<\rho^{1+l}$. Moreover, the result of (8.11) *is best possible.*

§9. Interpolation by Polynomials in z and z^{-1}

For each ordered pair (m, n) of nonnegative integers, and for any $f(z)=\sum_{j=0}^\infty a_j z^j$ in A_ρ, let $q^{(m,n)}(z;f)$ be the Lagrange interpolant of $z^n f(z)$ in the $(m+n+1)$-th roots of unity. Then, $z^{-n}q^{(m,n)}(z;f)$ can be uniquely expressed as the sum of a polynomial in π_m in the variable z and a polynomial in π_n in the variable z^{-1}, i.e., if $q^{(m,n)}(z;f)=\sum_{j=0}^{m+n}\alpha_j z^j$, then

$$z^{-n}q^{(m,n)}(z;f):=r_m^{(m,n)}(z;f)+s_n^{(m,n)}(z^{-1};f), \tag{9.1}$$

where $r_m^{(m,n)}(z;f):=\sum_{j=0}^m \alpha_{j+n}z^j$ and where $s_n^{(m,n)}(z^{-1};f):=\sum_{j=0}^{n-1}\alpha_j z^{j-n}$. Now, $r_m^{(m,n)}$ can, in the spirit of Walsh's Theorem A, be compared with the m-th partial sum of the power series expansion of f. Thus, we define

$$P_{m,n,0}(z;f):=\sum_{j=0}^{m} a_j z^j;\ P_{m,n,k}(z;f):=\sum_{j=0}^{m} a_{k(m+n+1)+j} z^j, \qquad k\geq 1, \tag{9.2}$$

and

$$Q_{m,n,k}(z^{-1};f):=\sum_{j=0}^{n-1} a_{k(m+n+1)-n+j-1} z^{j-n}, \qquad k\geq 1, \tag{9.3}$$

and establish the following analog of Theorem 1.

THEOREM 10. *Let $f(z)\in A_\rho$ and let $\{(m_i, n_i)\}_{i=1}^{\infty}$ be any sequence of ordered pairs of nonnegative integers for which there exists an α with $0\leq\alpha<\infty$ such that*

$$\lim_{i\to\infty} m_i = +\infty \quad \textit{and} \quad \lim_{i\to\infty} (n_i/m_i)=\alpha. \tag{9.4}$$

With the definitions of (9.1) *and* (9.2), *then for each positive integer l,*

$$\lim_{i\to\infty}\left[r_{m_i}^{(m_i,n_i)}(z;f)-P_{m_i,n_i,0}(z;f)-\sum_{k=1}^{l-1} P_{m_i,n_i,k}(z;f)\right]=0 \tag{9.5}$$

for $|z|<\rho^{1+l+l\alpha}$, where the convergence is uniform and geometric for $|z|\leq Z<\rho^{1+l+l\alpha}$. Moreover, the result of (9.5) *is best possible (in the sense that* (9.5) *is not in general valid on $|z|=\rho^{1+l+l\alpha}$ for all $f\in A_\rho$ and all sequences satisfying* (9.4)). *Finally, if $\alpha>0$, then for each positive integer l,*

$$\lim_{i\to\infty}\left[s_{m_i}^{(m_i,n_i)}(z^{-1};f)-\sum_{k=1}^{l-1} Q_{m_i,n_i,k}(z^{-1};f)\right]=0 \tag{9.6}$$

for all $|z|>\rho^{-l/\alpha-(l-1)}$, where the convergence is uniform and geometric for $|z|\geq Z>\rho^{-l/\alpha-(l-1)}$, and the result of (9.6) *is best possible.*

Remark. For the case $l=1$, Theorem 10 was given by Sharma [16]. For $l=1$ and for $n_i=0$ for all i, we note that Theorem 10 again gives Walsh's Theorem A.

Proof. The Lagrange interpolant $\hat{q}(z;f)$ of $z^n f(z)$ in the $(m+n+1)$-th roots of unity can be expressed from (2.3) as

$$\hat{q}(z;f)=\frac{1}{2\pi i}\int_\Gamma \frac{t^n f(t)[t^{m+n+1}-z^{m+n+1}]}{(t-z)(t^{m+n+1}-1)}\,dt,$$

of the power series expansion of f. Thus, we define

$$P_{m,n,0}(z;f):=\sum_{j=0}^{m} a_j z^j;\ P_{m,n,k}(z;f):=\sum_{j=0}^{m} a_{k(m+n+1)+j} z^j, \qquad k\geq 1, \tag{9.2}$$

and

$$Q_{m,n,k}(z^{-1};f):=\sum_{j=0}^{n-1} a_{k(m+n+1)-n+j-1} z^{j-n}, \qquad k\geq 1, \tag{9.3}$$

and establish the following analog of Theorem 1.

THEOREM 10. *Let $f(z)\in A_\rho$ and let $\{(m_i, n_i)\}_{i=1}^{\infty}$ be any sequence of ordered pairs of nonnegative integers for which there exists an α with $0\leq\alpha<\infty$ such that*

$$\lim_{i\to\infty} m_i=+\infty \quad \textit{and} \quad \lim_{i\to\infty} (n_i/m_i)=\alpha. \tag{9.4}$$

With the definitions of (9.1) and (9.2), then for each positive integer l,

$$\lim_{i\to\infty}\left[r_{m_i}^{(m_i,n_i)}(z;f)-P_{m_i,n_i,0}(z;f)-\sum_{k=1}^{l-1} P_{m_i,n_i,k}(z;f)\right]=0 \tag{9.5}$$

for $|z|<\rho^{1+l+l\alpha}$, where the convergence is uniform and geometric for $|z|\leq Z<\rho^{1+l+l\alpha}$. Moreover, the result of (9.5) is best possible (in the sense that (9.5) is not in general valid on $|z|=\rho^{1+l+l\alpha}$ for all $f\in A_\rho$ and all sequences satisfying (9.4)). Finally, if $\alpha>0$, then for each positive integer l,

$$\lim_{i\to\infty}\left[s_{m_i}^{(m_i,n_i)}(z^{-1};f)-\sum_{k=1}^{l-1} Q_{m_i,n_ik}(z^{-1};f)\right]=0 \tag{9.6}$$

for all $|z|>\rho^{-l/\alpha-(l-1)}$, where the convergence is uniform and geometric for $|z|\geq Z>\rho^{-l/\alpha-(l-1)}$, and the result of (9.6) is best possible.

Remark. For the case $l=1$, Theorem 10 was given by Sharma [16]. For $l=1$ and for $n_i=0$ for all i, we note that Theorem 10 again gives Walsh's Theorem A.

Proof. The Lagrange interpolant $\hat{q}(z;f)$ of $z^n f(z)$ in the $(m+n+1)$-th roots of unity can be expressed from (2.3) as

$$\hat{q}(z;f)=\frac{1}{2\pi i}\int_{\Gamma}\frac{t^n f(t)[t^{m+n+1}-z^{m+n+1}]}{(t-z)(t^{m+n+1}-1)}\,dt,$$

and on writing $[t^{m+n+1}-z^{m+n+1}]=(t^{m+1}-z^{m+1})z^n+t^{m+1}(t^n-z^n)$, it follows from (9.1) that

$$r_m^{(m,n)}(z;f)=\frac{1}{2\pi i}\int_\Gamma \frac{t^n f(t)(t^{m+1}-z^{m+1})}{(t-z)(t^{m+n+1}-1)}\,dt,$$

and that

$$s_n^{(m,n)}(z^{-1};f)=\frac{1}{2\pi i}\int_\Gamma \frac{t^{m+n+1}f(t)(t^n-z^n)}{(t-z)(t^{m+n+1}-1)z^n}\,dt. \tag{9.8}$$

With (9.7) and the definitions of (9.2), it is readily seen that

$$r_m^{(m,n)}(z;f)-P_{m,n,0}(z;f)-\sum_{k=1}^{l-1}P_{m,n,k}(z;f)$$
$$=\frac{1}{2\pi i}\int_\Gamma \frac{f(t)(t^m-z^m)\,dt}{(t-z)(t^{m+n+1}-1)t^{m+1+(l-1)(m+n+1)}},$$

from which the result of (9.5) easily follows for any sequence of pairs of integers satisfying (9.4).

To show that the result of (9.5) is best possible, choose $\hat{f}(z)=(\rho-z)^{-1}$ in A_ρ, and let $\{m_i\}_{i=1}^\infty$ be any sequence of nonnegative integers with $\lim_{i\to\infty}m_i=+\infty$. For any real $\alpha\geq 0$, set $n_i:=[\alpha m_i]$, the integer part of αm_i, for all $\alpha\geq 1$, so that (9.4) is valid. In this case, it can be verified that the result of (9.5) fails to be valid for $z=\rho^{1+l+l\alpha}$.

Continuing, we can equivalently express $s_n^{(m,n)}(z^{-1};f)$ of (9.8) as

$$s_n^{(m,n)}(z^{-1};f)=\frac{1}{2\pi i}\int_\Gamma \frac{f(t)(t^n-z^n)\,dt}{(t-z)z^n}+\frac{1}{2\pi i}\int_\Gamma \frac{f(t)(t^n-z^n)\,dt}{(t-z)(t^{m+n+1}-1)z^n}.$$

But, for $|z|>\rho$, the first integral above vanishes by virtue of analyticity, whence

$$s_n^{(m,n)}(z^{-1};f)=\frac{1}{2\pi i}\int_\Gamma \frac{f(t)(t^n-z^n)\,dt}{(t-z)(t^{m+n+1}-1)z^n}. \tag{9.9}$$

Then, with (9.9) and the definitions of (9.3), it is readily seen that

$$s_n^{(m,n)}(z^{-1};f)-\sum_{k=1}^{l-1}Q_{m,n,k}(z^{-1};f)=\frac{1}{2\pi i}\int_\Gamma \frac{f(t)(t^n-z^n)\,dt}{(t-z)t^{(l-1)(m+n+1)}(t^{m+n+1}-1)z^n}, \tag{9.10}$$

from which the result of (9.6) easily follows for the case $\alpha>0$. That the result of (9.6) is best possible follows from a construction similar to that above where it was shown that (9.5) is best possible. Finally, we remark that, in the case that $\alpha=0$, (9.6) is valid for all $|z|>\varepsilon$, where $\varepsilon>0$ is arbitrary. ■

§10. Concluding Remarks

In previous sections, the emphasis has been on generalizations of Walsh's Theorem A, involving interpolation only in the n-th roots of unity. There is considerable literature, however, on interpolation (and/or approximation) in points other than the n-th roots of unity (cf. Curtiss [2], Gaier [3], Kakahashi [5], Okada [12], Walsh and Sharma [21] and Baishanski [22]) which give the mechanism of extending our previous results beyond simply the n-th roots of unity.

To indicate one such extension, consider any sequence $\{\omega_{n_i}(z)\}_{i=1}^{\infty}$ of polynomials $\omega_{n_i}(z)$ with $\lim_{i\to\infty} n_i=\infty$ such that

$$\omega_{n_i}(z):=z^{n_i}+\sum_{j=0}^{n_i-1}\gamma_{j,n_i}z^j, \qquad \forall i\geq 1. \tag{10.1}$$

If $\omega_{n_i}(z)$ has all its zeros in $|z|\leq 1+\varepsilon$, then for any $f(z)\in A_\rho$, set

$$\tilde{p}_{n_i-1}(z;f):=\frac{1}{2\pi i}\int_\Gamma f(t)\frac{\omega_{n_i}(t)-\omega_{n_i}(z)}{\omega_{n_i}(t)(t-z)}\,dt, \tag{10.2}$$

where Γ is any circle $|t|=R$ with $1+\varepsilon<R<\rho$. Clearly, $\tilde{p}_{n_i-1}(z;f)$ is the polynomial interpolant of f in π_{n_i-1} which interpolates f in the zeros of ω_{n_i}. Note that the particular sequences $\{z^n-1\}_{n=1}^{\infty}$ and $\{z^n\}_{n=1}^{\infty}$ respectively generate, from (10.2), the polynomial interpolants $p_{n-1}(z;f)$ and $P_{n-1}(z;f)$ of $f(z)\in A_\rho$ in (1.3) of Walsh's Theorem A, and note, moreover, that these sequences have correspondingly many zero terms $\gamma_{j,n}$ in (10.1). As suggested by Professor E. B. Saff, suitable restrictions on the magnitudes of the coefficients γ_{j,n_i} in (10.1) would lead to generalizations of Walsh's Theorem A. Indeed, the condition that we impose on (10.1) is that there is a number α with $-\infty\leq\alpha<1$ for which

$$\limsup_{i\to\infty}\left(\sum_{j=0}^{n_i-1}|\gamma_{j,n_i}|\rho^j\right)^{1/n_i}=\rho^\alpha(<\rho), \tag{10.3}$$

where ρ, satisfying $1<\rho<\infty$, is the parameter associated with the set A_ρ. The collection of all sequences (10.1) satisfying (10.3) for a fixed value of α with $-\infty \leq \alpha < 1$ will be denoted by Ω_α.

We next establish

LEMMA 5. *If $\{\omega_{n_i}(z)\}_{i=1}^{\infty} \in \Omega_\alpha$, then, given any $\varepsilon > 0$ with $\rho^\alpha + \varepsilon < \rho$, there is an $i_0(\varepsilon)$ such that $\omega_{n_i}(z)$ has all its zeros in $|z| < \rho^\alpha + \varepsilon$ for all $i \geq i_0(\varepsilon)$.*

Proof. Since $\rho^\alpha + \varepsilon < \rho$, it follows that

$$\sum_{j=0}^{n_i - 1} |\gamma_{j,n_i}| (\rho^\alpha + \varepsilon)^j \leq \sum_{j=0}^{n_i - 1} |\gamma_{j,n_i}| \rho^j,$$

so that from (10.3),

$$\limsup_{i \to \infty} \left(\sum_{j=0}^{n_i - 1} |\gamma_{j,n_i}| (\rho^\alpha + \varepsilon)^j \right)^{1/n_i} \leq \rho^\alpha.$$

Hence, there exists an $i_0(\varepsilon)$ such that

$$\sum_{j=0}^{n_i - 1} |\gamma_{j,n_i}| (\rho^\alpha + \varepsilon)^j - (\rho^\alpha + \varepsilon)^{n_i} < 0, \qquad \forall i \geq i_0(\varepsilon).$$

By a classical result of Cauchy (cf. Marden [9, p. 122]), $\omega_{n_i}(z)$ then has all its zeros in $|z| < \rho^\alpha + \varepsilon$ for all $i \geq i_0(\varepsilon)$. ∎

From the sequence $\{z^n - 1\}_{n=1}^{\infty}$, which is an element of Ω_0, let us form the sequence $\{(z^n - 1)^r\}_{n=1}^{\infty}$ where r is any fixed positive integer. This latter sequence, when used with (10.2), generates the Hermite polynomial interpolants of order r of f in the n-th roots of unity. We note, however, from (10.3) that this latter sequence is contained in $\Omega_{1-1/r}$, but as the zeros of the polynomials of this sequence all evidently satisfy $|z| = 1$, Lemma 5 gives a weaker result concerning the zeros of $(z^n - 1)^r$ when $r > 1$. This suggests the following normalization to conform to Walsh's Theorem A. Define $\hat{\Omega}_\alpha$ to be the subset of all sequences $\{\omega_{n_i}(z)\}_{i=1}^{\infty}$ of Ω_α for which, given any $\varepsilon > 0$, there is an $i_1(\varepsilon)$ such that $\omega_{n_i}(z)$ has all its zeros in $|z| < 1 + \varepsilon$ for all $i \geq i_1(\varepsilon)$. Thus, $\{(z^n - 1)^r\}_{n=1}^{\infty} \in \hat{\Omega}_{1-1/r}$, and we note from Lemma 5 that $\hat{\Omega}_\alpha = \Omega_\alpha$ for all $\alpha \leq 0$.

With this notation, we establish

THEOREM 11. *Let* $\{\omega_{n_i}^{(1)}(z)\}_{i=1}^{\infty}$ *be in* $\hat{\Omega}_{\alpha_1}$ *and* $\{\omega_{n_i}^{(2)}(z)\}_{i=1}^{\infty}$ *be in* $\hat{\Omega}_{\alpha_2}$, *both having the same counting set* $\{n_i\}_{i=1}^{\infty}$, *and let* $f(z)\in A_\rho$. *If* $\{p_{n_i-1}^{(j)}(z;f)\}_{i=1}^{\infty}$, $j=1,2$. *are the associated interpolants of* f *of* (10.2), *then*

$$\lim_{i\to\infty}\{p_{n_i-1}^{(1)}(z;f)-p_{n_i-1}^{(2)}(z;f)\}=0 \qquad \forall |z|<\rho^{2-\max(\alpha_1;\alpha_2)}, \tag{10.4}$$

the convergence being uniform and geometric in $|z|\le Z<\rho^{2\text{-}\max(\alpha_1;\alpha_2)}$. *Moreover, the result of* (10.4) *is best possible.*

We remark that since $\{z^n-1\}_{n=1}^{\infty}\in\Omega_0$ and since $\{z^n\}_{n=1}^{\infty}\in\Omega_{-\infty}$, then Theorem 11 generalizes Walsh's Theorem A.

Proof. The sequence $\{z^{n_i}\}_{i=1}^{\infty}$ is, be definition, in $\hat{\Omega}_{-\infty}$, and its associated polynomial interpolants of $f(z)=\sum_{j=0}^{\infty}a_jz^j$ in A_ρ are from (10.2), just $P_{n_i-1,0}(z;f)=\sum_{j=0}^{n_i-1}a_jz^j$ (cf. (2.4)). Thus, from (10.2),

$$\tilde{p}_{n_i-1}^{(k)}(z;f)-P_{n_i-1,0}(z;f)=\frac{1}{2\pi i}\int_\Gamma\frac{f(t)}{(t-z)}\left[\frac{z^{n_i}\omega_{n_i}^{(k)}(t)-t^{n_i}\omega_{n_i}^{(k)}(z)}{\omega_{n_i}^{(k)}(t)\cdot t^{n_i}}\right]dt, \tag{10.5}$$

for $k=1,2$. Thus, for i sufficiently large, and for $1<R<\rho<\mu=|z|$, the quantity in brackets in the integral of (10.5) is bounded above (cf. (10.1)) by

$$\frac{\sum_{j=0}^{n_i-1}|\gamma_{j,n_i}^{(k)}|(\mu^{n_i}R^j+R^{n_i}\mu^j)}{\{R^{n_i}-\sum_{j=1}^{n_i-1}|\gamma_{j,n_i}^{(k)}|R^j\}R^{n_i}}.$$

Since $1<R<\mu$ implies that $\mu^{n_i}R^j>R^{n_i}\mu^j$ for all $0\le j\le n_i-1$, the n_i-th root of the above quotient is bounded above by

$$\frac{2^{1/n_i}\mu\left(\sum_{j=0}^{n_i-1}|\gamma_{j,n_i}^{(k)}|R^j\right)^{1/n_i}}{\{R^{n_i}-\sum_{j=0}^{n_i-1}|\gamma_{j,n_i}^{(k)}|R^j\}^{1/n_i}\cdot R}.$$

But as $R<\rho$, the above expression, coupled with (10.3) and (10.5), gives us for $\rho^{-1}<\tau<1$ that

$$\limsup_{i\to\infty}\{\max|\tilde{p}_{n_i-1}^{(k)}(z;f)-P_{n_i-1,0}(z;f)|:|z|\le\tau\rho^{2-\alpha_k}\}^{1/n_i}\le\tau<1, \tag{10.6}$$

for each $k=1,2$. Thus, with the triangle inequality applied to (10.6), (10.4) follows.

Finally, direct computation with the function $\hat{f}(z) := (\rho - z)^{-1}$ with particular sequences of polynomials $\{\omega_{n_i}^{(k)}(z)\}_{i=1}^{\infty}$ in $\hat{\Omega}_{\alpha_k}$, $k = 1, 2$, again shows that the result of (10.4) is best possible. ■

We further note that since $\{(z^n - 1)^r\}_{n=1}^{\infty} \in \hat{\Omega}_{1-1/r}$, Theorem 11 also generalizes the case $l = 1$ of Hermite interpolation in Theorem 3. As an open question, we can ask if assumptions analogous to that of (10.3) can be formulated to similarly generalize previous results on Hermite-Birkhoff interpolation, etc.

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