THEOREMS OF STEIN-ROSENBERG TYPE II. OPTIMAL PATHS OF RELAXATION IN THE COMPLEX PLANE

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1. INTRODUCTION

In [2], results in ways similar to the classical Stein-Rosenberg Theorem (cf. [1], [4], and [5]) have been obtained for arbitrary splittings, and without the usual positivity assumptions. The main purpose of this note is to extend the results of [2] by obtaining <u>paths of optimal relaxation</u>, for small complex relaxation factors $\omega = re^{i\theta}$, for the extrapolated Jacobi iterative method (JOR) and for the successive overrelaxation iterative method (SOR).

By way of notation, let $\mathbb{C}^{n,n}$ denote the collection of all $n \times n$ complex matrices $B = [b_{i,j}]$. If $B \in \mathbb{C}^{n,n}$, its spectrum, $\sigma(B)$, is defined as usual by

$$\sigma(B) := \{\lambda : \det(\lambda I - B) = 0\},$$

and its spectral radius, $\rho(B)$, is defined as

$$\rho(B) := \max\{|\lambda| : \lambda \in \sigma(B)\}.$$

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To review the relevant parts of [2], suppose that $A \in \mathbb{C}^{n,n}$ admits the splitting

$$A = D - L - U. \tag{1.1}$$

Here, it is assumed only that D is nonsingular, with D, L, and U in $\mathbb{C}^{n,n}$ satisfying (1.1). Associated with this splitting are the <u>extrapolated Jacobi matrix</u>, J_ω, defined by

$$J_\omega := I - \omega D^{-1}A, \tag{1.2}$$

and the <u>successive overrelaxation matrix</u>, $\mathcal{L}_\omega$, defined by

$$\mathcal{L}_\omega := (D - \omega L)^{-1}\{(1-\omega)D + \omega U\}. \tag{1.3}$$

Note that J_ω is defined for all complex numbers ω, while $\mathcal{L}_\omega$ is defined for all complex ω for which $D - \omega L$ is nonsingular. We further define

$$\Omega_J := \{\omega \in \mathbb{C}: \rho(J_\omega) < 1\}, \tag{1.4}$$

$$\Omega_{\mathcal{L}} := \{\omega \in \mathbb{C}: \rho(\mathcal{L}_\omega) < 1\}, \tag{1.5}$$

and

$$K(D^{-1}A) := \text{closed convex hull of } \sigma(D^{-1}A). \tag{1.6}$$

We next state some of the conclusions of [2, Thms. 3.1 and 3.4].

<u>Theorem A</u>. <u>For the splitting of</u> A <u>in</u> (1.1), <u>assume that</u> D <u>is nonsingular</u>. <u>Then</u>,

$$\Omega_J \cap \Omega_{\mathcal{L}} \neq \emptyset \quad \text{iff } 0 \notin K(D^{-1}A). \tag{1.7}$$

<u>Moreover</u>, <u>if</u> $0 \notin K(D^{-1}A)$, <u>then there exist a real</u> $\hat{\theta}$ <u>and an</u> $r_0 > 0$ <u>for which</u>

$$\Omega_J \cap \Omega_{\mathcal{L}} \supset \{\omega = re^{i\hat{\theta}}: \text{ for all } 0 < r < r_0\}. \tag{1.8}$$

Note that the results of (1.7) and (1.8) give the <u>simultaneous convergence</u> of J_ω and $\mathcal{L}_\omega$ (i.e., $\rho(J_\omega) < 1$ and $\rho(\mathcal{L}_\omega) < 1$) for all $\omega = re^{i\hat{\theta}}$ with $0 < r < r_0$, which is reminiscent of the well-known Stein-Rosenberg Theorem [4].

2. OPTIMAL PATHS OF RELAXATION FOR J_ω

Assuming $0 \notin K(D^{-1}A)$, we see from (1.8) that

$$\min_{0 \le \theta \le 2\pi} \rho(J_{re^{i\theta}}) < 1, \text{ and } \min_{0 \le \theta \le 2\pi} \rho(\mathcal{L}_{re^{i\theta}}) < 1, \tag{2.1}$$

for each $r > 0$ sufficiently small. The first goal of this section is i) to show that there exists a unique $\tilde{\theta}(r)$, for each r sufficiently small, such that

$$\rho(J_{re^{i\tilde{\theta}(r)}}) = \min_{0 \le \theta \le 2\pi} \rho(J_{re^{i\theta}}), \tag{2.2}$$

and ii) to explicitly determine $\tilde{\theta}(r)$ asymptotically, as $r \to 0$, solely from the geometrical description of $K(D^{-1}A)$. To our knowledge, the existence of such an optimal path of relaxation and its geometrical description has not been discussed in this generality in the literature.

Since $0 \notin K(D^{-1}A)$, then

$$\tau := \min\{|\xi| : \xi \in K(D^{-1}A)\} > 0, \tag{2.3}$$

and there exists a unique point $\hat{z} \in \partial K(D^{-1}A)$ for which $\hat{z} = \tau e^{i\psi}$ (where $\partial K(D^{-1}A)$ denotes the boundary of $K(D^{-1}A)$). Two cases arise:

Case 1. The circle $|z| = \tau$ intersects $\partial K(D^{-1}A)$ in a vertex of $K(D^{-1}A)$, which implies that $\tau e^{i\psi}$ is an eigenvalue of $D^{-1}A$, and that all other eigenvalues ξ of $D^{-1}A$ satisfy

$$\min[\operatorname{Re} \xi e^{-i\psi} : \xi \in \sigma(D^{-1}A)\setminus\{\tau e^{i\psi}\}] > \tau, \tag{2.4}$$

as pictured in Figure 1 below.

Case 2. The circle $|z| = \tau$ intersects $\partial K(D^{-1}A)$ in a point of a line segment of $\partial K(D^{-1}A)$ which is perpendicular to the ray $\{z = re^{i\psi} : r \ge 0\}$, as shown in Figure 2 below. In this case, if we set

$$E := \{\xi \in \sigma(D^{-1}A) : \operatorname{Re} \xi e^{-i\psi} = \tau\}, \tag{2.5}$$

then E contains at least two eigenvalues of $D^{-1}A$. Moreover, we set

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p. 234. In "Figure 1: Case 1", read "Figure 1: Case 2";

in "Figure 2: Case 2", read "Figure 2: Case 1"

line -2, read "(cf. (2.7))" for "(cf. (2.6))".

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$$\mu_1 := \max\{\mathrm{Im}\ \xi e^{-i\psi} : \xi \in E\}; \quad \mu_2 := \min\{\mathrm{Im}\ \xi e^{-i\psi} : \xi \in E\}, \tag{2.6}$$

and it is evident in this case that $\mu_1 \geq 0$, $\mu_2 \leq 0$, with $|\mu_1| + |\mu_2| > 0$, and that $\xi_j := (\tau + i\mu_j)e^{i\psi}$ are in $\sigma(D^{-1}A)$ for $j = 1, 2$.

Next, it is convenient to define $E := \{\tau e^{i\psi}\}$ for Case 1, so that $\mu_1 = \mu_2 = 0$ for Case 1. In either case, we further set

$$\tilde{\theta}(r) := -\psi + \arcsin(-r(\mu_1 + \mu_2)/2) \text{ for all } 0 < r < 2/(|\mu_1| + |\mu_2|), \tag{2.7}$$

where $-\frac{\pi}{2} < \tilde{\theta}(r) + \psi < \frac{\pi}{2}$, and where the upper bound on r is infinite if $|\mu_1| + |\mu_2| = 0$.

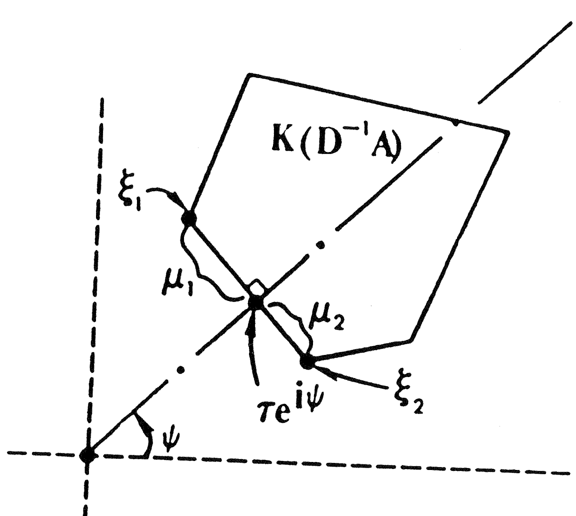

Figure 1: Case 1.

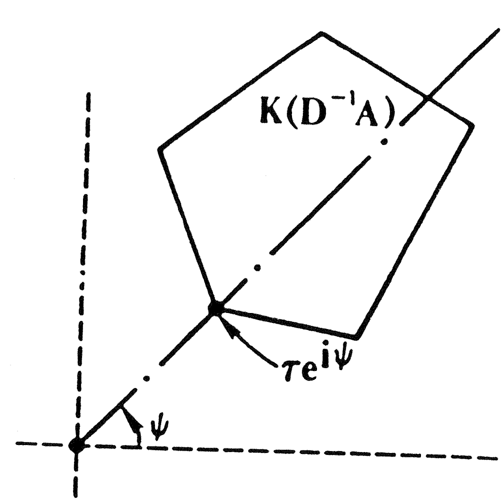

Figure 2: Case 2.

With these definitions, we establish

<u>Theorem 2.1</u>. <u>Assume that</u> $A = D - L - U$ <u>with</u> D <u>nonsingular</u>, <u>that</u> $0 \notin K(D^{-1}A)$, <u>and that</u> $\tau e^{i\psi}$ <u>is the point of</u> $K(D^{-1}A)$ <u>closest to the origin</u>. <u>Then</u>, <u>there exists a positive constant</u> m <u>such that</u>, <u>on each circle</u> $|\omega| = r$ <u>with</u> $0 < r < m$, <u>there is a unique</u> $\tilde{\theta}(r)$, <u>given by</u> (2.7), <u>for which</u>

$$\rho(J_{re^{i\tilde{\theta}(r)}}) = \min_{0 \leq \theta \leq 2\pi} \rho(J_{re^{i\theta}}) < 1. \tag{2.8}$$

<u>Moreover</u>, (cf. (2.6)),

$$\tilde{\theta}(r) = -\psi - r(\mu_1 + \mu_2)/2 - r^3(\mu_1 + \mu_2)^3/48 + \mathcal{O}(r^5), \text{ as } r \to 0, \tag{2.9}$$

so that

$$\lim_{r \to 0} \tilde{\theta}(r) = -\psi, \tag{2.10}$$

and

$$\rho(J_{re^{i\tilde{\theta}(r)}}) = 1 - r\tau - \frac{r^2}{2}(\mu_1\mu_2) + \frac{r^3\tau}{4}(\mu_1^2 + \mu_2^2) + \mathcal{O}(r^4), \text{ as } r \to 0. \tag{2.11}$$

Remark. Note that (2.10) gives us that the uniquely defined optimal path of relaxation, for J_ω with $|\omega|$ small, is tangential to the ray $\{re^{-i\psi} : r > 0\}$, as $|\omega| \to 0$.

Proof. Because of rotations, there is no loss of generality in assuming in the proof that $\psi = 0$. On considering Case 1 with $\psi = 0$, then τ is an eigenvalue of $D^{-1}A$, and

$$|1 - re^{i\theta}\tau|^2 = 1 - 2r\tau\cos\theta + r^2\tau^2 > (1 - r\tau)^2 \text{ for all } 0 < \theta < 2\pi. \tag{2.12}$$

On defining $m_1 := 2/M_1$ where

$$M_1 := \max\{\text{Re}\,\xi + \tau + \frac{(\text{Im}\,\xi)^2}{\text{Re}\,\xi - \tau} : \xi \in \sigma(D^{-1}A)\setminus\{\tau\}\},$$

it can be verified that if $0 < r < m_1$, then $(1 - r\tau)^2 \geq |1 - r\xi|^2$ for all $\xi \in \sigma(D^{-1}A)$. Thus, with (2.12), $\rho(J_{re^{i\theta}}) > 1 - r\tau$ for all $0 < \theta < 2\pi$ and for all $0 < r < m_1$, while $\rho(J_r) = 1 - r\tau$. Thus, $\tilde{\theta}(r) = 0$ is the unique value of θ for which (2.2) holds, for all $0 < r < m_1$, under our normalization $\psi = 0$. Further, as a result of our convention that $\mu_1 = \mu_2 = 0$ for Case 1, $\tilde{\theta}(r)$, as defined in (2.7), is zero, and (2.9)-(2.11) follow immediately.

Assuming Case 2 with $\psi = 0$, it can be verified from the definitions of (2.5) and (2.7) that

$$\max\{|1 - re^{i\theta}\xi| : \xi \in E\} > |1 - re^{i\tilde{\theta}(r)}\xi_1| = |1 - re^{i\tilde{\theta}(r)}\xi_2| \tag{2.13}$$

for all $\theta \neq \tilde{\theta}(r)$ (mod 2π), and that there exists an $m_2 > 0$ such that

$$|1 - re^{i\tilde{\theta}(r)}\xi_1| = |1 - re^{i\tilde{\theta}(r)}\xi_2| \geq |1 - re^{i\tilde{\theta}(r)}\xi| \text{ for all } \xi \in \sigma(D^{-1}A)\setminus E, \tag{2.14}$$

for all $0 < r < m_2$. From the above two inequalities, (2.8) follows. Then, from (2.7), (2.9)-(2.11) follow by direct calculation. ■

If we assume that $D^{-1}A$ is a <u>real</u> matrix with $0 \notin K(D^{-1}A)$, then, from the fact that the nonreal eigenvalues of $D^{-1}A$ necessarily occur in conjugate complex pairs, it is evident that either $\psi = 0$ or $\psi = \pi$, and that (cf. (2.6)) $\mu_1 + \mu_2 = 0$. Consequently, from (2.7), $\tilde{\theta}(r) = 0$ or $\tilde{\theta}(r) = +\pi$ for all $r > 0$, and we immediately have from Theorem 2.1 the result of

<u>Corollary 2.2</u>. <u>Assume that</u> $A = D - L - U$ <u>with</u> D <u>nonsingular, and that</u> $D^{-1}A$ <u>is a real matrix</u>. <u>If</u> $\mathrm{Re}\, \xi > 0$ <u>for all</u> $\xi \in \sigma(D^{-1}A)$, <u>then</u>

$$\min_{0 \le \theta \le 2\pi} \rho(J_{re^{i\theta}}) = \rho(J_r) \tag{2.15}$$

<u>for all</u> $r > 0$, <u>while if</u> $\mathrm{Re}\, \xi < 0$ <u>for all</u> $\xi \in \sigma(D^{-1}A)$, <u>then</u>

$$\min_{0 \le \theta \le 2\pi} \rho(J_{re^{i\theta}}) = \rho(J_{-r}) \tag{2.16}$$

<u>for all</u> $r > 0$. <u>In either case</u>,

$$\min_{0 \le \theta \le 2\pi} \rho(J_{re^{i\theta}}) = \{(1 - r\tau)^2 + r^2\mu_1^2\}^{1/2} = 1 - r\tau + \frac{r^2\mu_1^2}{2} + \frac{r^3\tau\mu_1^2}{2} + \mathcal{O}(r^4), \text{ as } r \to 0. \tag{2.17}$$

3. OPTIMAL PATHS OF RELAXATION FOR $\mathcal{L}_\omega$

We next obtain an analogue of Theorem 2.1 for the matrix $\mathcal{L}_\omega$ when $|\omega|$ is small. On defining

$$Q(\omega) := D^{-1}A + \omega D^{-1}L(I - \omega D^{-1}L)^{-1}D^{-1}A, \text{ for all } |\omega| \text{ small}, \tag{3.1}$$

we see from (3.1) that

$$\mathcal{L}_\omega = I - \omega Q(\omega), \quad \text{for all } |\omega| \text{ small}. \tag{3.2}$$

Next, if we set

$$K(Q(\omega)) := \text{closed convex hull of } \sigma(Q(\omega)), \tag{3.3}$$

then, since $Q(\omega) \to D^{-1}A$ as $|\omega| \to 0$ from (3.1), it follows from the continuity of the eigenvalues involved that

$$K(Q(\omega)) \to K(D^{-1}A), \quad \text{as } |\omega| \to 0. \tag{3.4}$$

Now, because the optimal path of relaxation for $|\omega|$ small for $J_\omega = I - \omega D^{-1}A$, from Theorem 2.1, depends only on the geometry of $K(D^{-1}A)$, it is not surprising from (3.4) that the same might be true for $\mathcal{L}_\omega$. More precisely, we have

Theorem 3.1. Assume that $A = D - L - U$ with D nonsingular, that $0 \notin K(D^{-1}A)$, and that $\tau e^{i\psi}$ is the point of $K(D^{-1}A)$ closest to the origin. Then, there exists a positive constant m' such that, on each circle $|\omega| = r$ with $0 < r < m'$, there is a $\hat{\theta}(r)$ for which

$$\rho(\mathcal{L}_{re^{i\hat{\theta}(r)}}) = \min_{0 \le \theta \le 2\pi} \rho(\mathcal{L}_{re^{i\theta}}) < 1. \tag{3.5}$$

Moreover,

$$\lim_{r \to 0} \hat{\theta}(r) = -\psi, \tag{3.6}$$

and

$$\rho(\mathcal{L}_{re^{i\hat{\theta}(r)}}) = 1 - r\tau + \mathcal{O}(r^{1+1/n}), \text{ as } r \to 0. \tag{3.7}$$

Remark. As in Theorem 2.1, an optimal path of relaxation for $\mathcal{L}_\omega$, with $|\omega|$ small, is tangential to the ray $\{re^{-i\psi} : r > 0\}$ as $|\omega| \to 0$.

Proof. We shall establish this result only for Case 1, since Case 2 is similarly treated. That a $\hat{\theta}(r)$ exists on each circle $|\omega| = r$ for which (3.5) holds, follows of course by continuity. Now, in Case 1, we know that $\tau e^{i\psi} \in \sigma(D^{-1}A)$. Then, applying Ostrowski's classical result [3, p. 334] on the perturbation of eigenvalues of a matrix, there is at least one eigenvalue $\xi(\omega) \in \sigma(Q(\omega))$ for which

$$|\xi(\omega) - \tau e^{i\psi}| = \mathcal{O}(|\omega|^{1/n}), \text{ as } |\omega| \to 0, \tag{3.8}$$

where n is the order of A. Thus,

$$\eta(r, \theta) := 1 - re^{i\theta}\xi(re^{i\theta}) \tag{3.9}$$

is an eigenvalue of $\mathcal{L}_{re^{i\theta}}$ from (3.2) for each such $\xi(\omega)$. The modulus of each such $\eta(r, \theta)$ is, from (3.8), just

$$|\eta(r, \theta)| = 1 - r\tau \cos(\theta + \psi) + \mathcal{O}(r^{1+1/n}), \text{ as } r \to 0, \tag{3.10}$$

uniformly in θ. For any fixed ϵ with $0 < \epsilon < 1$, suppose that $\cos(\theta + \psi) \le 1 - \epsilon$, so that from (3.10),

$$|\eta(r, \theta)| \ge 1 - r\tau + \epsilon r\tau + \mathcal{O}(r^{1+1/n}), \text{ as } r \to 0.$$

On the other hand, the choice of $\theta = -\psi$ in (3.10) gives the asymptotically smaller quantity $1 - r\tau + \mathcal{O}(r^{1+1/n})$ as $r \to 0$. Thus, we see that, for each fixed $r > 0$ sufficiently small, if $\tilde{\theta}(r)$ minimizes $|\eta(r, \theta)|$ over $0 \le \theta \le 2\pi$ and over all such eigenvalues $\eta(r, \theta)$, then $\tilde{\theta}(r)$ cannot satisfy $\cos(\tilde{\theta}(r) + \psi) \le 1 - \epsilon$ for all $r \to 0$. Next, as in the proof of Theorem 2.1 in Case 1, it turns out that $\min_{0\le\theta\le 2\pi} \rho(\mathcal{L}_{re^{i\theta}})$, for r small, is governed by the behavior of such eigenvalues $\eta(r, \theta)$, so that

$$\min_{0\le\theta\le 2\pi} \rho(\mathcal{L}_{re^{i\theta}}) = 1 - r\tau + \mathcal{O}(r^{1+1/n}), \text{ as } r \to 0, \tag{3.11}$$

which gives the desired results of (3.5) and (3.7). The previous discussion concerning $\cos(\theta + \psi) \le 1 - \epsilon$ also shows that $\lim_{r\to 0} \tilde{\theta}(r) = -\psi$. ∎

On comparing (2.11) of Theorem 2.1 and (3.7) of Theorem 3.1, we see that the spectral radii of the optimized (in θ) extrapolated Jacobi matrix and optimized successive overrelaxation matrix have the same asymptotic expansions through first order terms in r, as $r \to 0$. Because of the crude estimation of perturbation effects in $Q(\omega)$ for $|\omega|$ small, which resulted in the term $\mathcal{O}(r^{1+1/n})$ in (3.7), it is not possible to say which of these optimized procedures is <u>faster</u> as $r \to 0$, unlike the classical Stein-Rosenberg Theorem. Nevertheless, the following interesting observations can be deduced.

If the term $\mathcal{O}(r^{1+1/n})$ in (3.7) actually behaves asymptotically as

μr^{σ}, as $r \to 0$, where $|\mu| > 0$ and where $1 < \sigma < 2$, then it is evident from (2.11) and (3.7) that the optimized extrapolated Jacobi procedure is asymptotically faster than the optimized successive overrelaxation procedure for all $r > 0$, sufficiently small. This means, for example, that if the Jordan normal form, corresponding to some eigenvalue in the subset E of $D^{-1}A$ of (2.5), is not diagonal, and if the perturbation $Q(\omega)$ of $D^{-1}A$, for $\omega = re^{-i\psi}$ with $r > 0$ small, separates the eigenvalues associated with this Jordan block, then the optimized extrapolated Jacobi method will be asymptotically faster than the optimized successive overrelaxation method for all $r > 0$ sufficiently small.

Now, the above deduction is precisely the opposite of what one expects in the usual Stein-Rosenberg Theorem. Why? The assumptions in the usual Stein-Rosenberg Theorem require that $B: = D^{-1}L + D^{-1}U$ be a nonnegative matrix, but this, by the Perron-Frobenius Theorem, insures that the Jordan blocks for B, corresponding to the subset E of $D^{-1}A = I-B$, are all 1×1 matrices and hence diagonal. Thus, the assumptions of the previous paragraph are not met!

In a later paper, we will give a more precise form of Theorem 3.1, based on perturbation theory, for optimal paths of relaxation for the matrix $\mathcal{L}_\omega$ in the complex plane. Included there will be results, both numerical and graphical, illustrating these comparisons of the iteration matrices J_ω and $\mathcal{L}_\omega$.

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