Numer. Math. 38, 179–192 (1981)

Numerische Mathematik

On the LU Factorization of M-Matrices*

Richard S. Varga** and Da-Yong Cai***

Institute for Computational Mathematics, Kent State University, Kent, Ohio 44242, USA

Summary. In this paper, we give in Theorem 1 a characterization, based on graph theory, of when an M-matrix A admits an LU factorization into M-matrices, where L is a nonsingular lower triangular M-matrix and U is an upper triangular M-matrix. This result generalizes earlier factorization results of Fiedler and Pták (1962) and Kuo (1977). As a consequence of Theorem 1, we show in Theorem 3 that the condition $\mathbf{x}^T A \geqq \mathbf{0}^T$ for some $\mathbf{x} > \mathbf{0}$, for an M-matrix A, is both necessary and sufficient for PAP^T to admit such an LU factorization for every $n \times n$ permutation matrix P. This latter result extends recent work of Funderlic and Plemmons (1981). Finally, Theorem 1 is extended in Theorem 5 to give a characterization, similarly based on graph theory, of when an M-matrix A admits an LU factorization into M-matrices.

Subject Classifications: AMS(MOS) 15 A 23; CR: 5.14.

1. Introduction

An $n \times n$ M-matrix $A = [a_{i,j}]$ is said to *admit an LU factorization into $n \times n$ M-matrices* if A can be expressed as

$$A = LU, \tag{1.1}$$

where $L := [\ell_{i,j}]$ is an $n \times n$ *lower triangular M-matrix* (i.e., $\ell_{i,i} \geqq 0$, $\ell_{i,j} \leqq 0$ for all $i > j$ and $\ell_{i,j} = 0$ for all $j > i$, where $1 \leqq i,j \leqq n$), and where $U := [u_{i,j}]$ is an $n \times n$ *upper triangular M-matrix* (i.e., $u_{i,i} \geqq 0$, $u_{i,j} \leqq 0$ for all $j > i$ and $u_{i,j} = 0$ for all

* Dedicated to Professor Ky Fan on his sixty-seventh birthday, September 19, 1981.

** Research supported in part by the Air Force Office of Scientific Research, and by the Department of Energy

*** Visiting scholar from the Department of Applied Mathematics, Qing-Hua University, Beijing, People's Republic of China

0029-599X/81/0038/0179/$02.80

$i>j$, where $1 \leqq i, j \leqq n$). A well-known result of Fiedler and Pták in 1962 (cf. [4, Theorems 3.1 and 3.3]) gives that any nonsingular M-matrix admits such an LU factorization (1.1) into M-matrices, with L and U both nonsingular.

There has been revived interest in this factorization question. In 1977, Kuo [7] extended this earlier result of Fiedler and Pták by showing that any $n \times n$ *irreducible* M-matrix (singular or not) admits an LU factorization (1.1) into M-matrices, with, say, L nonsingular. (The analogous result is also true with U nonsingular since an M-matrix A admits an LU factorization in M-matrices with L nonsingular iff A^T admits an LU factorization into M-matrices with U nonsingular). Thus, the above results of Fiedler and Pták, and Kuo, can be seen as contributing to the following:

Problem 1. Characterize those M-matrices which Admit an LU Factorization into M-matrices with L Nonsingular

Obviously, to completely settle Problem 1, it remains only to determine which singular and reducible M-matrices admit an LU factorization into M-matrices with L nonsingular. First of all, not every singular and reducible M-matrix has such a factorization, as an examination of the particular matrix

$$A_1 := \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} \tag{1.2}$$

directly shows. On the other hand, because of connections with compartmental problems (cf. [5]), Funderlic and Plemmons [6] have recently extended Kuo's result by showing that if an $n \times n$ M-matrix A satisfies

$$\mathbf{x}^T A \geqq \mathbf{0}^T \quad \text{for some } \mathbf{x} > \mathbf{0}, \tag{1.3}$$

then A admits an LU factorization into M-matrices with nonsingular L. (Here, we use the notation that $\mathbf{x} := [x_1, \ldots, x_n]^T \geqq \mathbf{0}$ or $\mathbf{x} > \mathbf{0}$ means respectively that $x_i \geqq 0$ or $x_i > 0$ for all $1 \leqq i \leqq n$.) This result, however, does not completely settle Problem 1. To see this, consider the singular and reducible M-matrix A_2, where

$$A_2 := \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \tag{1.4}$$

which has a trivial LU factorization into M-matrices with nonsingular L, as shown above. It is immediate that (1.3) fails for A_2.

Condition (1.3) does, however, carry further implications. As was observed in [6], for any $n \times n$ permutation P, it is evident that if (1.3) holds, then

$$\mathbf{z}^T (PAP^T) \geqq \mathbf{0}^T, \quad \text{where } \mathbf{z} := P\mathbf{x} > \mathbf{0}.$$

In other words, if $\mathscr{P}_n$ denotes the collection of all $n \times n$ permutation matrices, then the $n \times n$ M-matrix A satisfies (1.3) iff PAP^T satisfies (1.3) for all $P \in \mathscr{P}_n$. Consequently, the result of Funderlic and Plemmons [6] gives that (1.3) is a *sufficient* condition that PAP^T admits an LU factorization into M-matrices with nonsingular L for *every* $P \in \mathscr{P}_n$. One of our main results, stated as Theorem 3 below, is that (1.3) is *necessary* as well.

To state our main results, additional notation is required. Given any $n\times n$ complex matrix $A=[a_{i,j}]$, let $G_n(A)$ denote its *directed graph* (cf. [9, p. 19]) on n given distinct vertices $v_1, v_2, \ldots, v_n$, where $a_{i,j}\neq 0$ is interpreted as an arc from v_i to v_j. More generally, a *path* from vertex v_i to vertex v_j is a sequence of arcs,

$$\{a_{k_r,k_{r+1}}\}_{r=1}^{\ell} \quad \text{with } \ell\geqq 1, a_{k_r,k_{r+1}}\neq 0, \quad \text{and with } k_1=i, k_{\ell+1}=j. \tag{1.5}$$

Next, with $\langle n\rangle := \{1,2,\ldots,n\}$, let $\alpha=\{\alpha_1,\alpha_2,\ldots,\alpha_k\}$ be a nonempty subset of $\langle n\rangle$ where, for convenience, we order the elements of α as $1\leqq\alpha_1<\alpha_2<\ldots<\alpha_k\leqq n$. Then, $A[\alpha]$ denotes the induced principal submatrix of A determined by α, i.e.,

$$A[\alpha]=[a_{i,j}], \qquad \text{where } i,j\in\alpha. \tag{1.6}$$

We shall say that α is a *proper* subset of $\langle n\rangle$ if $\varnothing\neq\alpha\subsetneqq\langle n\rangle$.

With the above notation, we state our main results, and their corollaries. Proofs of these basic assertions will be given in § 3. Our first result, Theorem 1, gives a solution to Problem 1.

Theorem 1. *Let A be an $n\times n$ M-matrix. Then, the following are equivalent:*

i) A admits an LU factorization into M-matrices with nonsingular L;

ii) for every proper subset $\alpha=\{\alpha_1,\alpha_2,\ldots,\alpha_k\}$ of $\langle n\rangle$ for which $A[\alpha]$ is singular and irreducible, there is no path in the directed graph $G_n(A)$ of A from vertex v_t to vertex v_{α_j} for any $t>\alpha_k$ and any $1\leqq j\leqq k$.

We remark that the previous results of Fiedler and Pták [4] and Kuo [7], on factoring M-matrices, are both special cases of Theorem 1. To see this, it is impossible to find (cf. Lemma 3 of § 2) a proper subset α of $\langle n\rangle$ for which $A[\alpha]$ is singular and irreducible if A is either a nonsingular M-matrix or an irreducible singular M-matrix. Thus, *ii)* of Theorem 1 holds vacuously, whence A admits such a factorization.

As an immediate consequence of Theorem 1, we have

Corollary 2. *Let A be an $n\times n$ M-matrix, and let $\alpha=\{\alpha_1,\alpha_2,\ldots,\alpha_k\}$ be any proper subset of $\langle n\rangle$ for which $A[\alpha]$ is singular and irreducible. Then*

$$a_{t,p}=0 \qquad \text{for all } t>\alpha_k \quad \text{and all } p\in\alpha \tag{1.7}$$

is a necessary condition that A admits an LU factorization into M-matrices with nonsingular L.

To illustrate Theorem 1, consider the matrix A_1 of (1.2), which is a singular reducible M-matrix. On choosing $\alpha=\{1\}$, then $A_1[\alpha]=[0]$ is evidently singular. As we *define* all 1×1 matrices in this paper to be irreducible, then $A_1[\alpha]$ is also irreducible. Since the directed graph $G_2(A_1)$ of A_1 from (1.2) has a path from vertex v_2 to vertex v_1, this shows that *ii)* of Theorem 1 fails. As we have seen, A_1 does not admit an LU factorization into M-matrices with nonsingular L.

As a less trivial example, consider the singular reducible M-matrix

$$A_3 = \left[\begin{array}{cc|cc|cc} 6 & -1 & 0 & 0 & 0 & 0 \\ -1 & 6 & 0 & -1 & 0 & -1 \\ \hline 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 6 & -1 \\ -1 & 0 & 0 & 0 & -1 & 6 \end{array}\right]. \tag{1.8}$$

On choosing $\alpha = \{3, 4\}$, then

$$A_3[\alpha] = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

is singular and irreducible. On examining the directed graph $G_6(A_3)$, we see that there is a path from v_6 to v_4, so that, from Theorem 2, A does *not* admit an LU factorization into M-matrices with nonsingular L. The point of this example is that (1.7) of Corollary 2 *is* satisfied for A_3, which shows that (1.7) of Corollary 2 is *not* sufficient in general for A to admit an LU factorization into M-matrices with nonsingular L.

Our next result gives equivalent characterization of (1.3) for M-matrices.

Theorem 3. *Let A be an $n \times n$ M-matrix. Then, the following are equivalent:*

i) A satisfies (1.3);

ii) PAP^T admits an LU factorization into M-matrices with nonsingular L for each $P \in \mathcal{P}_n$.

iii) for every proper subset $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ of $\langle n \rangle$ for which $A[\alpha]$ is singular and irreducible, then $a_{t,p} = 0$ for all $t \notin \alpha$ and all $p \in \alpha$.

To illustrate the result of Theorem 3, consider the matrix A_3 of (1.8). Now, $\alpha = \{3, 4\}$ is the only proper subset of $\langle n \rangle$ for which $A_3[\alpha]$ is singular and irreducible, and as *iii)* of Theorem 3 fails for $A_3[\alpha]$, then A_3 does not satisfy (1.3). On the other hand, note that the transpose, A_3^T, of A_3 *does*, by inspection, satisfy *iii)* of Theorem 3, so that A_3^T satisfies (1.3). It is evident that Theorem 3 can be used to give a necessary and sufficient condition for an $n \times n$ M-matrix A to be such that either A or A^T satisfies (1.3).

On reconsidering the matrix A_1 of (1.2), we know that A_1 does *not* admit an LU factorization into M-matrices with nonsingular L, while for the permutation matrix $\hat{P} := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, we see from (1.4) that $\hat{P} A_1 \hat{P}^T = A_2$ *does* admit such a factorization. This suggests the following decomposition of $\mathcal{P}_n$. For a given $n \times n$ M-matrix A, set

$$\mathcal{P}_n^g(A) := \{P \in \mathcal{P}_n : PAP^T \text{ admits an } LU \text{ factorization into } M\text{-matrices with nonsingular } L\}, \tag{1.9}$$

and set

$$\mathcal{P}_n^b(A) := \mathcal{P}_n \setminus \mathcal{P}_n^g(A). \tag{1.10}$$

In the case of A_1 of (1.2), then $\mathscr{P}_2^g(A_1)=\{\hat{P}\}$, while $\mathscr{P}_2^b(A_1)=\{I\}$. Obviously, Theorem 3 asserts that

$$\mathscr{P}_n^g(A)=\mathscr{P}_n \text{ (so that } \mathscr{P}_n^b(A)=\varnothing) \text{ iff } A \text{ satisfies (1.3).} \tag{1.11}$$

Moreover, Kuo [7] has shown that $\mathscr{P}_n^g(A)$ is *never* empty for any M-matrix A. For a constructive proof of this, simply apply Theorem 1 to the reduced normal form (cf. (2.15)) of any $n \times n$ M-matrix.

Next, given any $n \times n$ M-matrix A, we know from the previous discussion that $\mathscr{P}_n^g(A) \neq \varnothing$, and it is of interest to determine the *cardinality* $|\mathscr{P}_n^g(A)|$ of $\mathscr{P}_n^g(A)$ (i.e., the exact number of its elements). Now, the *general* determination of $|\mathscr{P}_n^g(A)|$ for a given $n \times n$ singular and reducible M-matrix is a rather complicated combinatorial problem, but to give the flavor of this problem, we include the special combinatorial result of Theorem 4, whose proof will also given in § 3.

Theorem 4. *Let A be an $n \times n$ singular and reducible M-matrix such that there is a $P \in \mathscr{P}_n$ for which* (cf. (2.15))

$$PAP^T = \tilde{A} := \begin{bmatrix} \tilde{A}_{1,1} & \tilde{A}_{1,2} \\ 0 & \tilde{A}_{2,2} \end{bmatrix}, \tag{1.12}$$

where $\tilde{A}_{1,1}$ is an $m_1 \times m_1$ nonsingular irreducible M-matrix, where $\tilde{A}_{2,2}$ is an $m_2 \times m_2$ singular irreducible M-matrix, and where $\tilde{A}_{1,2} \neq \mathcal{O}$. Then,

$$|\mathscr{P}_n^g(A)| = m_2(n-1)!, \quad \textit{and} \quad |\mathscr{P}_n^b(A)| = m_1(n-1)!. \tag{1.13}$$

As an application of Theorem 4, it can be verified that the matrix A_3 of (1.8) satisfies the hypotheses of Theorem 4 with $n=6$, $m_1=4$, and $m_2=2$. As such, it follows from (1.13) that

$$|\mathscr{P}_6^g(A_3)| = 240, \quad \text{and} \quad |\mathscr{P}_6^b(A_3)| = 480.$$

We remark that if the roles of $\tilde{A}_{1,1}$ and $\tilde{A}_{2,2}$ in (1.12) are *interchanged*, i.e., $\tilde{A}_{1,1}$ is an $m_1 \times m_1$ singular irreducible M-matrix and $\tilde{A}_{2,2}$ is an $m_2 \times m_2$ nonsingular irreducible M-matrix, then we have, in contrast with (1.13), that

$$|\mathscr{P}_n^g(A)| = n!, \quad \text{and} \quad |\mathscr{P}_n^b(A)| = 0. \tag{1.14}$$

As previously noted, the LU factorization (1.1) of an M-matrix A where the upper-triangular matrix U is now nonsingular (instead of L) amounts simply to an LU factorization of A^T with nonsingular L. Thus, since the directed graph $G_n(A^T)$ of A^T can be immediately obtained by simply *reversing* the direction of all arcs in the directed graph of $G_n(A)$, it is evident that Theorem 1 can be used to give a necessary and sufficient condition for an $n \times n$ M-matrix A to be such that A admits an LU factorization into M-matrices with either nonsingular L or with nonsingular U. To illustrate this, consider the singular reducible M-matrix

$$A_4 = \begin{bmatrix} 1 & -1 & -1 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & 1 \end{bmatrix}. \tag{1.15}$$

On choosing $\alpha=\{2\}$, so that $A_4[\alpha]=[0]$ is singular and irreducible, we see from the directed graph $G_4(A_4)$ that there is a path from vertex v_4 to vertex v_2, as well as a path from vertex v_2 to vertex v_3. Consequently, A does not admit an LU factorization into M-matrices with either nonsingular L or with nonsingular U.

The example of the matrix A_4 in (1.15) leaves open the question of whether A_4 admits an LU factorization into M-matrices, without regard to the singularity or nonsingularity of either L or U. As we shall see, the answer to this question is *no*, and we are thus lead to the more general problem of

Problem 2. Characterize those M-matrices which Admit an LU Factorization into M-matrices

Obviously, as in the case of Problem 1, it remains only to determine which singular and reducible M-matrices admit an LU factorization into M-matrices. As an easy extension of Theorem 1, a solution to Problem 2 is stated below in Theorem 5.

Theorem 5. *Let A be an $n\times n$ M-matrix. Then, the following are equivalent:*

i) A admits an LU factorization into M-matrices;

ii) for every proper subset $\alpha=\{\alpha_1,\alpha_2,\dots,\alpha_k\}$ of $\langle n\rangle$ for which $A[\alpha]$ is singular and irreducible, there do not simultaneously exist paths in the directed graph $G_n(A)$ of A from vertex v_t to vertex v_{α_j} (for some $t>\alpha_k$ and some $1\leqq j\leqq k$), and from vertex v_{α_i} to vertex v_s (for some $1\leqq i\leqq k$ and some $s>\alpha_k$).

From the discussion following the definition of the matrix A_4 in (1.15), it is evident that A_4 does not satisfy *ii)* of Theorem 5, so that A_4 does not admit an LU factorization into M-matrices. To further illustrate Theorem 5, consider the matrix

$$A_5=\begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & -2 & -2 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & -3 & 0 \end{bmatrix}, \tag{1.16}$$

which is a singular reducible M-matrix. Now $\alpha_i:=\{i\}$, $i=1,2,3,4$ are the only proper subsets of $\langle 4\rangle$ for which $A_5[\alpha_i]$ is a singular irreducible M-matrix. As *ii)* of Theorem 5 is valid for each α_i, $i=1,2,3,4$, then A_5 admits an LU factorization into M-matrices. Such an LU factorization is explicitly given in

$$A_5=\begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & -3 & 1 \end{bmatrix}\cdot\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{1.17}$$

As a final remark, we can, for any $n\times n$ M-matrix A, analogously set (cf. (1.9)–(1.10))

$$\mathscr{P}_n^G(A):=\{P\in\mathscr{P}_n\colon PAP^T \text{ admits an } LU \text{ factorization into } M\text{-matrices}\}, \tag{1.18}$$

and

$$\mathcal{P}_n^B(A) := \mathcal{P}_n \setminus \mathcal{P}_n^G(A). \tag{1.19}$$

By definition, we have in general that $\mathcal{P}_n^G(A) \supseteq \mathcal{P}_n^g(A)$. We remark, for the special structure of $\tilde{A}$ in (1.12) in Theorem 4, that $\mathcal{P}_n^G(\tilde{A}) = \mathcal{P}_n$, and thus $\mathcal{P}_n^G(\tilde{A}) \supsetneqq \mathcal{P}_n^g(\tilde{A})$.

2. Preliminaries

In this section, we establish a number of needed lemmas before proceeding to the proofs of Theorems 1, 3, 4 and 5 in § 3.

Lemma 1. *Assume that the three* $n \times n$ *matrices* $A = [a_{i,j}]$, L, *and* U *satisfy* $A = LU$, *where* L *is lower triangular and where* U *is upper triangular. If* L *is nonsingular, then* $a_{1,1} = 0$ *implies* $a_{j,1} = 0$ *for all* $1 \leqq j \leqq n$, *while if* U *is nonsingular, then* $a_{1,1} = 0$ *implies* $a_{1,j} = 0$ *for all* $1 \leqq j \leqq n$.

Proof. Immediate! ∎

Our first interest is in Problem 1, and we now examine carefully the application of Gaussian elimination, by successive columns, to an $n \times n$ M-matrix $A = [a_{i,j}]$ to see if A admits an LU factorization into M-matrices *with nonsingular* L. First, suppose $a_{1,1} = 0$. If some $a_{j,1} < 0$ for $1 < j \leqq n$, this factorization of A *fails* from Lemma 1. Otherwise, all entries in the first column of A are zero so that A has the form

$$A_1 := \left[\begin{array}{c|ccc} a_{1,1}^{(0)} & a_{1,2}^{(0)} & \cdots & a_{1,n}^{(0)} \\ \hline 0 & a_{2,2}^{(1)} & \cdots & a_{2,n}^{(1)} \\ \vdots & \vdots & & \vdots \\ 0 & a_{n,2}^{(1)} & \cdots & a_{n,n}^{(1)} \end{array}\right] = \left[\begin{array}{c|c} a_{1,1}^{(0)} & \mathbf{a}^{(0)T} \\ \hline \mathbf{0} & \tilde{A}_1 \end{array}\right], \tag{2.1}$$

where $A = A_0 := [a_{i,j}^{(0)}]$, and where

$$\tilde{A}_1 := [a_{j,\ell}^{(1)}], \quad \text{where} \quad 1 < j, \ell \leqq n, \quad \text{with} \quad a_{j,\ell}^{(1)} := a_{j,\ell}^{(0)} := a_{j,\ell}. \tag{2.2}$$

Thus, if $a_{1,1}^{(0)} = 0$ and if the factorization does not fail, then A has the form (2.1), so that, from the hypothesis that $A = A_1$ is an M-matrix, we see that $\tilde{A}_1$ is also an M-matrix.

Continuing, suppose $a_{1,1} \neq 0$, so that $a_{1,1} > 0$ as A is an M-matrix. We can add $(-a_{j,1}/a_{1,1})[a_{1,1}, a_{1,2}, \ldots, a_{1,n}]^T$ to the j-th row of A, thereby forming $[0, a_{j,2}^{(1)}, \ldots, a_{j,n}^{(1)}]^T$, where in general

$$a_{j,\ell}^{(1)} := \frac{-a_{j,1}^{(0)} a_{1,\ell}^{(0)}}{a_{1,1}^{(0)}} + a_{j,\ell}^{(0)}, \quad 2 \leqq j, \ell \leqq n. \tag{2.3}$$

Thus, if $a_{1,1} \neq 0$, we can perform Gaussian elimination on the first column of A, and we obtain the matrix A_1 of (2.1), where the entries of the $(n-1) \times (n-1)$ matrix $\tilde{A}_1$ in (2.2) are given now by (2.3). Note that we can express A_1 as $L_1^{-1} A_0$, where

$$L_1 := \begin{bmatrix} 1 & & & & \\ a_{2,1}^{(0)}/a_{1,1}^{(0)} & 1 & & 0 & \\ a_{3,1}^{(0)}/a_{1,1}^{(0)} & 0 & \ddots & & \\ \vdots & \vdots & & & \\ a_{n,1}^{(0)}/a_{1,1}^{(0)} & 0 & \dots & 0 & 1 \end{bmatrix} \text{ if } a_{1,1}^{(0)} \neq 0, \text{ and } L_1 := I \text{ otherwise.} \tag{2.4}$$

In either case, it is evident that L_1 is a unit lower triangular M-matrix.

By a result of Ky Fan [3, Lemma 1], whom we are honoring, the $(n-1)\times(n-1)$ matrix $\tilde{A}_1$ of (2.1) is itself an M-matrix, so that the process can be continued. Specifically, if a $a_{2,2}^{(1)}=0$, Lemma 1 gives us that either $a_{j,2}^{(1)}=0$ for all $2 \leqq j \leqq n$, or the factorization *fails* at this step. If $a_{2,2}^{(1)} \neq 0$ (so that $a_{2,2}^{(1)}>0$ since $\tilde{A}_1$ is an M-matrix), Gaussian elimination can be performed on the second column of A_1. It is then clear that, if the M-matrix A admits an LU factorization into M-matrices *with nonsingular* L, the matrix A is, after k steps $(1 \leqq k < n)$ of Gaussian elimination, given by

$$A_k = \left[\begin{array}{cc|c} a_{1,1}^{(0)} & /// & //// \\ 0 & a_{k,k}^{(k-1)} & \\ \hline \multicolumn{2}{c|}{0} & \tilde{A}_k \end{array}\right] \tag{2.5}$$

where if $\tilde{A}_k := [a_{j,\ell}^{(k)}]$ where $k<j,\ \ell \leqq n$, then recursively,

$$\begin{aligned} a_{j,\ell}^{(k)} &= a_{j,\ell}^{(k-1)} \quad \text{for all } k \leqq j,\ \ell \leqq n \quad \text{if } a_{k,k}^{(k-1)}=0; \\ a_{j,\ell}^{(k)} &= -\frac{a_{j,k}^{(k-1)}\, a_{k,\ell}^{(k-1)}}{a_{k,k}^{(k-1)}} + a_{j,\ell}^{(k-1)}, \quad \text{for all } k<j,\ \ell \leqq n \quad \text{if } a_{k,k}^{(k-1)} \neq 0. \end{aligned} \tag{2.6}$$

In addition, we have that

$$A_k = L_k^{-1} A_{k-1}, \qquad k=1,2,\dots,n-1, \tag{2.7}$$

where for $k>1$,

$$L_k := \left[\begin{array}{c|ccccc} I_{k-1} & & & 0 & & \\ \hline & 1 & & & & \\ & a_{k+1,k}^{(k-1)}/a_{k,k}^{(k-1)} & & & & \\ 0 & \vdots & 1 & & 0 & \\ & & 0 & \ddots & & \\ & & \vdots & & & \\ & a_{n,k}^{(k-1)}/a_{k,k}^{(k-1)} & 0 & \dots & 0 & 1 \end{array}\right] \begin{array}{l} \text{if } a_{k,k}^{(k-1)} \neq 0, \text{ and} \\ L_k = I_n \text{ otherwise;} \end{array} \tag{2.8}$$

here, I_{k-1} denotes the $(k-1)\times(k-1)$ identity matrix. If the procedure does not fail at any step, we see from (2.7) that

$$A = (L_1 \cdot L_2 \dots L_{n-1}) \cdot A_{n-1} = L \cdot U, \tag{2.9}$$

where $L := L_1 \cdot L_2 \dots L_{n-1}$, by construction, is a unit lower triangular M-matrix, and $U := A_{n-1}$ is, from the sign properties of (2.6), an upper triangular M-

matrix. This gives us the desired LU factorization of A into M-matrices with nonsingular L.

We now look at the graph-theoretic implications of the above procedure.

Lemma 2. *Let $A=[a_{i,j}]$ be an $n \times n$ M-matrix which admits an LU factorization into M-matrices with nonsingular L. Then, i) there is a path* (cf. (1.5)) *from vertex v_r to vertex v_s in the directed graph $G_n(A)$ for A for which $r \neq s$ and min $\{r; s\} > k$, iff ii) there is an associated path from vertex v_r to vertex v_s in the directed graph $G_{n-k}(\tilde{A}_k)$ of the matrix $\tilde{A}_k$, arising in the k-th step of Gaussian elimination applied to A* (cf. (2.5), *where the vertices for $G_{n-k}(\tilde{A}_k)$ are defined to be $v_{k+1}, v_{k+2}, \ldots, v_n$.*

Proof. It suffices to consider only the case $k=1$, as the other cases follow recursively.

i) ⇒ ii). Let the given path be determined from the sequence $\{a^{(0)}_{k_r,k_{r+1}}\}_{r=1}^{\ell}$ with $a_{k_r,k_{r+1}} \neq 0$, where $k_1=r$ and $k_{\ell+1}=s$. Suppose that there are two successive terms, say $a^{(0)}_{t,1}$ and $a^{(0)}_{1,q}$ in this sequence, with $t \neq q$. By definition, $a^{(0)}_{t,1} \neq 0$, so that $a^{(0)}_{1,1} > 0$ from Lemma 1. Consider the display in (2.3). Since we are dealing with M-matrices, $a^{(0)}_{j,\ell} \leqq 0$ for any $j \neq \ell$, so that for any pair (j, ℓ) with $1 < j, \ell \leqq n$ with $j \neq \ell$, both terms on the right of the display in (2.3) are of the *same* sign, i.e., they are both nonpositive. In particular, we also see from (2.3) that

$$a^{(1)}_{t,q} = \frac{-a^{(0)}_{t,1}\, a^{(0)}_{1,q}}{a^{(0)}_{1,1}} + a^{(0)}_{t,q} < 0, \tag{2.10}$$

since the first term on the right is negative. Similarly, $a^{(0)}_{k_r,k_{r+1}} < 0$ with $k_r \neq 1$ and $k_{r+1} \neq 1$ implies from (2.3) that $a^{(1)}_{k_r,k_{r+1}} < 0$. Hence, the sequence $\{a^{(0)}_{k_r,k_{r+1}}\}_{r=1}^{\ell}$ generates a new sequence $\{a^{(1)}_{k'_r,k'_{r+1}}\}_{r=1}^{\ell'}$ with $k'_1=r$ and $k'_{\ell'}=s$, where $\ell' < \ell$ if some $k_r=1$, and $\ell'=\ell$ otherwise. This new sequence evidently can be interpreted as a path from vertex v_r to v_s in the directed graph $G_{n-1}(\tilde{A}_1)$ for $\tilde{A}_1$ on the vertices $v_2, v_3, \ldots, v_n$, which gives ii) for the case $k=1$.

ii) = i). Suppose $a^{(1)}_{t,q} \neq 0$ with min $(t; q) \geqq 2$, so that there is an arc joining v_t to v_q in $G_{n-1}(\tilde{A}_1)$. Assuming first that $a^{(0)}_{1,1} > 0$, it follows from (2.10) that there is an arc or path joining v_t to v_q in $G_n(A)$. If $a^{(0)}_{1,1}=0$, then A has the form (2.1), and there is an arc joining v_t to v_q in $G_n(A)$. ∎

As an immediate consequence of Lemma 2, based on the well-known (cf. [9, p. 20]) equivalence between irreducible matrices and strongly connected directed graphs, we have the

Corollary. *Let $A=[a_{i,j}]$ be an $n \times n$ M-matrix which admits an LU factorization into M-matrices with nonsingular L. If A is irreducible, then so is each submatrix $\tilde{A}_k$* (cf. (2.5)), *for $k=1, 2, \ldots, n-1$.*

With the notation of (1.6), we next establish

Lemma 3. *Let $A=[a_{i,j}]$ be an $n \times n$ M-matrix. Then, a proper subset $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ of $\langle n \rangle$ exists such that $A[\alpha]$ is singular iff A is a singular reducible M-matrix. Moreover, if α is a proper subset of $\langle n \rangle$ such that $A[\alpha]$ is singular and irreducible, then $A[\alpha]$, after a suitable permutation of indices, is one of the singular irreducible diagonal matrices $\tilde{A}_{j,j}$ in the normal reduced form* (cf. (2.15)) *for A.*

Proof. Since A is an M-matrix by hypothesis, we can write, as usual, that $A = sI - B$, where $B \geqq \mathcal{O}$ and where

$$s \geqq \rho(B), \text{ with equality only if } A \text{ is singular.} \tag{2.11}$$

Assuming first that there exists a proper subset $\alpha = \{\alpha_1, \ldots, \alpha_k\}$ of $\langle n \rangle$ such that $A[\alpha]$ is singular, then we can also write $A[\alpha] = sI - B[\alpha]$, where $B[\alpha]$ is the associated principal submatrix of B. A well-known consequence of the Perron-Frobenius Theorem (cf. [9, p. 46]) gives that

$$\rho(B) \geqq \rho(B[\alpha]). \tag{2.12}$$

But, as $A[\alpha]$ is a singular M-matrix by hypothesis, then from the statement in (2.11),

$$\rho(B[\alpha]) = s. \tag{2.13}$$

Combining the inequalities of (2.11)–(2.13), we see that

$$s = \rho(B) = \rho(B[\alpha]) = s, \tag{2.14}$$

which establishes that A is a singular M-matrix. If A (and hence B) were irreducible, we would conclude, since α is a proper subset of $\langle n \rangle$, that $\rho(B[\alpha]) < \rho(B)$ (cf. [9, p. 30]), which contradicts (2.14). Thus, A is a singular reducible M-matrix.

Conversely, suppose that A is a singular and reducible M-matrix. Since A is a singular M-matrix, we can write $A = \rho(B)I - B$ where $B \geqq \mathcal{O}$, and as A is reducible, so is B. Putting B into normal reduced form (cf. [9, p. 46]) equivalently implies that there is a $P \in \mathscr{P}_n$ such that

$$P A P^T = \begin{bmatrix} \tilde{A}_{1,1} & \tilde{A}_{1,2} & \cdots & \tilde{A}_{1,s} \\ & \tilde{A}_{2,2} & \cdots & \tilde{A}_{2,s} \\ & & \ddots & \vdots \\ 0 & & & \tilde{A}_{s,s} \end{bmatrix}, \tag{2.15}$$

which we call the *normal reduced form of* A, where each $\tilde{A}_{j,j}$ is an irreducible M-matrix, and there is a j, with $1 \leqq j \leqq s$, such that $\tilde{A}_{j,j}$ is singular. On permuting back indices, it is evident that $\tilde{A}_{j,j}$ defines a proper subset $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_k\}$ of $\langle n \rangle$ such that $A[\alpha]$ is singular (and irreducible), which completes the first part of this lemma. Moreover, it is also clear from the arguments given above that any proper subset α of $\langle n \rangle$ for which $A[\alpha]$ is singular and irreducible is such that $A[\alpha]$, after a suitable permutation of indices, is precisely one of the singular diagonal matrix $\tilde{A}_{j,j}$ in (2.15). ∎

Lemma 4. *Let $A = [a_{i,j}]$ be an $n \times n$ M-matrix which admits an LU factorization into M-matrices with nonsingular L, and let $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ be the largest subset of $\langle \alpha_k \rangle$ such that $A[\alpha]$ is an irreducible M-matrix. Then, at the α_k-th column Gaussian elimination step applied to A, the α_k-th diagonal entry of A_{α_k}* (cf. (2.5)) *is zero iff $A[\alpha]$ is singular.*

Proof. Suppose first that $C = [c_{i,j}]$ is any $k \times k$ irreducible M-matrix. From Kuo's result [7], C admits an LU factorization into M-matrices with nonsingular L. If $k > 1$, then $\alpha^{(j)} := \{1, 2, \ldots, j\}$ defines the proper leading submatrix $C[\alpha^{(j)}]$ of C for each j with $1 \leq j < k$. Because C is irreducible, $C[\alpha^{(j)}]$, from the proof of Lemma 3, is necessarily a nonsingular M-matrix for each $1 \leq j < k$, whence (cf. [1, p. 134, (A 1)]) $\det C[\alpha^{(j)}] > 0$ for $1 \leq j < k$. But, as is well-known, the diagonal entries $c_{j,j}^{(j-1)}$ of the upper triangular matrix (cf. (2.5)), derived from applying Gaussian elimination to C, satisfy

$$c_{j,j}^{(j-1)} = \frac{\det C[\alpha^{(j)}]}{\det C[\alpha^{(j-1)}]} > 0, \quad \text{for } 1 \leq j < k, \quad \text{where } \det C[\alpha^{(0)}] := 1,$$

while the final diagonal entry $c_{k,k}^{(k-1)}$ is zero iff C is singular, i.e.,

$$c_{k,k}^{(k-1)} = 0.$$

Thus, only at the final step of the application of Gaussian elimination to C can one encounter a zero diagonal entry. This will be useful below.

Continuing, let $A = [a_{i,j}]$ an M-matrix which admits an LU factorization into M-matrices with nonsingular L. If $\alpha = \{\alpha_1, \ldots, \alpha_k\}$ is a largest subset of $\langle \alpha_k \rangle$ such that $A[\alpha]$ is irreducible, we know from Lemma 3 that $A[\alpha]$, after a suitable permutation of indices, is one of the irreducible diagonal matrices $\tilde{A}_{j,j}$ in the normal reduced form (cf. (2.15)) for $A[\langle \alpha_k \rangle]$. This implies, in the terminology of Rothblum [8], that α, one of the equivalence classes of the communication relation induced by the directed graph of $A[\langle \alpha_k \rangle]$, communicates with no vertex v_t, $1 \leq t \leq \alpha_k$, not in α. This further implies, as is easily seen, that the particular diagonal entries $a_{\alpha_j, \alpha_j}^{(\alpha_j - 1)}$, $1 \leq j \leq k$, arising in the elimination process applied to A, are just the successive diagonal entries obtained by applying Gaussian elimination directly to the submatrix $A[\alpha]$. But, from the preceding discussion, $a_{\alpha_k, \alpha_k}^{(\alpha_k - 1)} = 0$ iff $A[\alpha]$ is singular. ∎

Having considered Problem 1, we next wish to determine (Problem 2) if an $n \times n$ M-matrix $A = [a_{i,j}]$ admits an LU factorization into M-matrices, without regard to L or U being singular or nonsingular. First, suppose that $a_{1,1} = 0$. If some $a_{j,1} \neq 0$ *and* some $a_{1,t} \neq 0$ for $1 < j$, $t \leq n$, such a factorization of A *fails* from Lemma 1. Otherwise, either all entries in the first column of A are zero, or all entries in the first row of A are zero, i.e., we can express A as

$$A = \left[\begin{array}{c|c} 0 & a_{1,2} \ldots a_{1,n} \\ \hline \mathbf{0} & \tilde{A}_1 \end{array}\right], \quad \text{or as } A = \left[\begin{array}{c|c} 0 & \mathbf{0}^T \\ \hline a_{2,1} & \\ \vdots & \tilde{A}_1 \\ a_{n,1} & \end{array}\right], \tag{2.16}$$

where $\tilde{A}_1$ is evidently an $(n-1) \times (n-1)$ M-matrix. Similarly, if $a_{1,1} > 0$, we can apply, as before, Gaussian elimination to the first column of A, and we obtain (cf. (2.1))

$$A_1 = \left[\begin{array}{c|c} a_{1,1} & a_{1,2} \ldots a_{1,n} \\ \hline \underline{0} & \tilde{A}_1 \end{array}\right], \tag{2.17}$$

where $A_1 = L_1^{-1} A$, with L_1 a unit lower triangular M-matrix (cf. (2.4)). In either case, the problem is reduced to determining if $\tilde{A}_1$ admits an LU factorization into M-matrices. Indeed, if $\tilde{A}_1 = \tilde{L} \cdot \tilde{U}$ is such a factorization of $\tilde{A}_1$, we see from (2.16) (when $a_{1,1} = 0$) that

$$A=\left[\begin{array}{c|c}1 & \mathbf{0}^T \\ \hline \mathbf{0} & \tilde{L}\end{array}\right]\cdot\left[\begin{array}{c|c}0 & a_{1,2}\dots a_{1,n} \\ \hline \mathbf{0} & \tilde{U}\end{array}\right],$$

or (2.18)

$$A=\left[\begin{array}{c|c}0 & \mathbf{0}^T \\ \hline a_{2,1} & \\ \vdots & \tilde{L} \\ a_{n,1} & \end{array}\right]\cdot\left[\begin{array}{c|c}1 & \mathbf{0}^T \\ \hline \mathbf{0} & \tilde{U}\end{array}\right]$$

gives an LU factorization of A into M-matrices, while from (2.17) (when $a_{1,1}>0$),

$$A=\left[\begin{array}{c|c}1 & \mathbf{0}^T \\ \hline a_{2,1}/a_{1,1} & \\ \vdots & \tilde{L} \\ a_{n,1}/a_{1,1} & \end{array}\right]\cdot\left[\begin{array}{c|c}a_{1,1} & a_{1,2}\dots a_{1,n} \\ \hline \mathbf{0} & \tilde{U}\end{array}\right], \tag{2.19}$$

similarly gives an LU factorization of A into M-matrices. Thus, this decision procedure can be successively applied to the lower order M-matrices $\tilde{A}_k$ (as in (2.5)), to determine if A admits an LU factorization in M-matrices.

3. Proofs of Main Results

With the results of § 2, we now give the

Proof of Theorem 1. i)⇒ii). Assuming A admits an LU factorization into M-matrices with nonsingular L, suppose $\alpha=\{\alpha_1,\alpha_2,\dots,\alpha_k\}$ is any proper subset of $\langle n\rangle$ for which $A[\alpha]$ is singular and irreducible. Then from Lemma 3, A is a singular reducible M-matrix, and from Lemma 4, at the α_k-th Gaussian elimination step applied to A, $a^{(\alpha_k-1)}_{\alpha_k,\alpha_k}=0$. Hence, from Lemma 1, it is necessary that $a^{(\alpha_k-1)}_{t,\alpha_k}=0$ for all $\alpha_k<t\leqq n$. However, from Lemma 2, this implies that there is no path in the directed graph $G_n(A)$ for A from any vertex v_t to the vertex v_{α_k} for all $\alpha_k<t\leqq n$. Because $A[\alpha]$ is by hypothesis irreducible, this further implies that there is no path from vertex v_t to vertex v_{α_j} for any $t>\alpha_k$ and any $1\leqq j\leqq k$. Thus, *i)⇒ii).*

not i)=not ii). Assuming that the $n\times n$ M-matrix A does not admit an LU factorization into M-matrices with nonsingular L, there exists a positive integer k with $1\leqq k<n$ such that the factorization procedure of § 2, applied to A, *fails* at the $(k-1)$st step, i.e. (cf. (2.5)), $a^{(k-1)}_{k,k}=0$ and $a^{(k-1)}_{r,k}\neq 0$ for some r with $k<r\leqq n$. This means that the factorization procedure *does* apply to $A[\langle k\rangle]$, but as $a^{(k-1)}_{k,k}=0$, then (cf. (2.9)) $A[\langle k\rangle]$ is a singular M-matrix. Next, let $\alpha=\{\alpha_1,\alpha_2,\dots,\alpha_j\}$ with $1\leqq\alpha_1,<\alpha_2<\dots<\alpha_j=k$ be the largest subset of $\langle k\rangle$ for which $A[\alpha]$ is irreducible. From Lemma 4, $A[\alpha]$ is both irreducible and singular. Because $a^{(k-1)}_{r,k}\neq 0$, it follows from Lemma 2 that there is a path in $G_n(A)$ from v_r to v_k. Thus, *ii)* in Theorem 1 cannot hold. ∎

We now establish Theorem 3 as a consequence of Theorem 1.

Proof of Theorem 3. i)⇒ii). This has already been established in §1. (cf. [6]).

ii)⇒iii). Assume that PAP^T admits an LU factorization into M-matrices with nonsingular L for *all* $P \in \mathscr{P}_n$, and assume that $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ is any proper subset of $\langle n \rangle$ for which $A[\alpha]$ is singular and irreducible. From Lemma 3, A is necessarily singular and reducible, and moreover, $A[\alpha]$ is, after a suitable permutation of indices, one of the singular irreducible matrices $\tilde{A}_{j,j}$ in the normal reduced form (2.15) for A. Next, from Theorem 1, there is no path in the directed graph $G_n(A)$ for A from vertex v_t to vertex v_{α_j} for any $t > \alpha_k$ and any $1 \leqq j \leqq k$. But as this must hold for *any* permutation matrix P in $\mathscr{P}_n$, it follows that there is no path in the directed graph $G_n(A)$ for A from vertex v_t to vertex v_p for any $t \notin \alpha$, and any $p \in \alpha$, whence $a_{t,p} = 0$ for all $t \notin \alpha$ and all $p \in \alpha$. Thus, *ii)* implies *iii)*.

iii)⇒i). Assuming *iii)*, this means that $\tilde{A}_{\ell,j} = \mathcal{O}$ for any $\ell \neq j$ in the normal reduced form (2.15) for A. On taking transposes and using a result from Berman, Varga, and Ward [2, Theorem 1 *(ii)*], this implies that there is an $\mathbf{x} > \mathbf{0}$ for which $A^T \mathbf{x} \geqq \mathbf{0}$, whence $\mathbf{x}^T A \geqq \mathbf{0}^T$. Thus, A satisfies condition (1.3), and *iii)* implies *i)*. ∎

Proof of Theorem 4. With the hypotheses of Theorem 4, set $S_1 := \{1, 2, \ldots, m_1\}$, and set $S_2 := \langle n \rangle \setminus S_1$, so that $|S_1| = m_1$ and $|S_2| = m_2$. For the matrix $\tilde{A}$ of (1.12), we remark that the only proper subset α of $\langle n \rangle$ for which $\tilde{A}[\alpha]$ is singular and irreducible is $\alpha = S_2$.

First, consider any permutation of the elements of $\langle n \rangle$ for which the final element of this permutation is from the set S_2. As is readily verified, the number of distinct ways in which this can be done is $m_2 \cdot (n-1)!$. For any such permutation, let Q denote the associated permutation matrix in $\mathscr{P}_n$. Then, we claim that *ii)* of Theorem 1 vacuously holds for $Q \tilde{A} Q^T$. Indeed, if $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ is a proper subset of $\langle n \rangle$ for which $Q \tilde{A} Q^T[\alpha]$ is singular and irreducible, then α is a renumbering of S_2 with $\alpha_k = n$. Hence, from Theorem 1 and the definition of (1.9), it follows that $Q \in \mathscr{P}_n^g(\tilde{A})$.

Next, any remaining permutation of $\langle n \rangle$ is such that the final element is from the set S_1. If R denotes the associated permutation matrix in $\mathscr{P}_n$, and if $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ is any proper subset of $\langle n \rangle$ for which $R \tilde{A} R^T[\alpha]$ is singular and irreducible, then $\alpha_k < n$. From the irreducibility of $\tilde{A}_{1,1}$ and $\tilde{A}_{2,2}$ and from $\tilde{A}_{1,2} \neq \mathcal{O}$ in (1.12), it is easy to see that there is a path in $G_n(R \tilde{A} R^T)$ from a vertex v_t to vertex v_{α_k} for some $t > \alpha_k$. Hence, from Theorem 1 and from (1.10), it follows that $R \in \mathscr{P}_n^b(\tilde{A})$. Thus, $|\mathscr{P}_n^b(\tilde{A})| = n! - |\mathscr{P}_n^g(\tilde{A})| = m_1 \cdot (n-1)!$ ∎

Proof of Theorem 5. Based on the discussion in §2, this proof follows easily along the lines of the proof of Theorem 1 above. ∎

Acknowledgments. The authors wish to thank Dr. R.E. Funderlic and Professors C.R. Johnson and R.J. Plemmons for stimulating discussions on these problems. In particular, Professor Plemmons raised the question of finding a necessary and sufficient condition that an M-matrix A be such that either A or A^T satisfies (1.3), which can be deduced from Theorem 1.

References

1. Berman, A., Plemmons, R.J.: Nonnegative matrices in the mathematical sciences. New York: Academic Press, 1979

2. Berman, A., Varga, R.S., Ward, R.C. ALPS: Matrices with nonpositive off-diagonal entries. Linear Algebra and Appl. **21**, 233–244 (1978)
3. Fan, Ky: Note on M-matrices. Quart. J. Math. Oxford Ser. (2), **11**, 43–49 (1960)
4. Fiedler, M., Pták, V.: On matrices with nonpositive off-diagonal elements and positive principal minors. Czech. Math. J. **12**, 382–400 (1962)
5. Funderlic, R.E., Mankin, J.B.: Solution of homogeneous systems of linear equations arising from compartmental models. SIAM J. Sci. Statist. Comput. (in press, 1981)
6. Funderlic, R.E., Plemmons, R.J.: *LU* decompositions of M-matrices by elimination without pivoting. Linear Algebra and Appl. (in press, 1981)
7. Kuo, I-wen: A note on factorization of singular M-matrices. Linear Algebra and Appl. **16**, 217–220
8. Rothblum, U.G.: A rank characterization of the number of final classes of a nonnegative matrix. Linear Algebra and Appl. **23**, 65–68 (1979)
9. Varga, R.S.: Matrix iterative analysis. Englewood Cliffs, New Jersey: Prentice-Hall 1962

Received May 6, 1981/Revised September 17, 1981