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Comparisons of Regular Splittings of Matrices

Dedicated to Fritz Bauer on the occasion of his 60th birthday

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Summary. In this article, new comparison theorems for regular splittings of matrices are derived. In so doing, the initial results of Varga in 1960 on regular splittings of matrices, and the subsequent unpublished results of Woźnicki in 1973 on regular splittings of matrices, will be seen to be special cases of these new comparison theorems.

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§ 1. Introduction

The theory of regular splittings of matrices has been a useful tool in the analysis of iterative methods for solving large systems of linear equations (cf. Berman and Plemmons [1, p. 130], Ortega and Rheinboldt [5, p. 57], Varga [7, p. 88], and Young [9, p. 123]). Our basic purpose here is to derive new comparison theorems (Theorems 2 and 4) for regular splittings of matrices, which generalize the original results in 1960 of Varga [6] and the subsequent unpublished thesis results in 1973 of Woźnicki [8]. A secondary objective of this work is to popularize here the useful but little known results of Woźnicki [8].

In the remainder of this section, we give the older results and background for the regular splitting theory of matrices. In § 2, our new results are stated along with supplementary discussions and examples, while in § 3, the proofs of our new results are given.

For our theoretical background, let A, M, and N all be complex $n \times n$ matrices. Then, $A = M - N$ is said to be a *regular splitting of* A (cf. [6, 7]) if M is nonsingular and if M^{-1} and N have all their entries nonnegative (written

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$M^{-1} \geqq 0$ and $N \geqq 0$). This concept arises most naturally in the iterative solution of large linear systems of equations. Specifically, suppose we are given the following system of n linear equations in n unknowns:

$$A\mathbf{x} = \mathbf{k}, \tag{1.1}$$

where A is a given $n \times n$ matrix, and where $\mathbf{x}$ and $\mathbf{k}$ are column vectors with n components, with $\mathbf{k}$ being given. If $A = M - N$ is a regular splitting of A, then (1.1) can be expressed as

$$\mathbf{x} = M^{-1}N\mathbf{x} + M^{-1}\mathbf{k}, \tag{1.2}$$

which suggests the following iterative method:

$$\mathbf{x}^{(m+1)} = M^{-1}N\mathbf{x}^{(m)} + M^{-1}\mathbf{k}, \qquad (m = 0, 1, \dots), \tag{1.3}$$

where $\mathbf{x}^{(0)}$ is an arbitrarily chosen initial column vector. That this iterative method is, when $A^{-1} \geqq 0$, necessarily *convergent* to the unique solution of (1.1), independent of the initial choice of $\mathbf{x}^{(0)}$, is contained in Theorem A below. For additional notation, we write $C > 0$ if all entries of an $n \times n$ matrix C are positive, and $A \geqq B$ and $A > B$ if $A - B \geqq 0$ and if $A - B > 0$, respectively. Also, $\rho(C)$ will denote the *spectral radius* of C. With these definitions and notations, the following results of Theorems A and B are well-known.

Theorem A ([6, 7]). *Let $A = M - N$ be a regular splitting of A. If $A^{-1} \geqq 0$, then*

$$\rho(M^{-1}N) = \frac{\rho(A^{-1}N)}{1 + \rho(A^{-1}N)} < 1. \tag{1.4}$$

Conversely, if $\rho(M^{-1}N) < 1$, then $A^{-1} \geqq 0$.

Theorem B ([6, 7]). *Let $A = M_1 - N_1 = M_2 - N_2$ be two regular splittings of A, where $A^{-1} \geqq 0$. If $N_2 \geqq N_1$, then*

$$\rho(M_2^{-1}N_2) \geqq \rho(M_1^{-1}N_1). \tag{1.5}$$

In particular, if $N_2 \geqq N_1$ with $N_2 \neq N_1$, and if $A^{-1} > 0$, then

$$\rho(M_2^{-1}N_2) > \rho(M_1^{-1}N_1). \tag{1.6}$$

Less well-known, but nonetheless useful in applications, is the following thesis result of Woźnicki [8].

Theorem C ([8]). *Let $A = M_1 - N_1 = M_2 - N_2$ be two regular splitting of A, where $A^{-1} \geqq 0$. If $M_1^{-1} \geqq M_2^{-1}$, then*

$$\rho(M_2^{-1}N_2) \geqq \rho(M_1^{-1}N_1). \tag{1.7}$$

In particular, if $M_1^{-1} > M_2^{-1}$, and if $A^{-1} > 0$, then

$$\rho(M_2^{-1}N_2) > \rho(M_1^{-1}N_1). \tag{1.8}$$

§ 2. Statements of New Results

To motivate our generalization of Theorems B and C on the comparison of regular splittings of a matrix, we state

Proposition 1. *Let $A=M_1-N_1=M_2-N_2$ be two regular splittings of A, where $A^{-1}\geqq 0$. Then,*

i) *$N_2\geqq N_1$ implies that $M_1^{-1}\geqq M_2^{-1}$;*
ii) *$M_1^{-1}\geqq M_2^{-1}$ implies that $A^{-1}N_2A^{-1}\geqq A^{-1}N_1A^{-1}$;*
iii) *$A^{-1}N_2A^{-1}\geqq A^{-1}N_1A^{-1}$ implies that $(A^{-1}N_2)^jA^{-1}\geqq(A^{-1}N_1)^jA^{-1}$, for each positive integers $j>1$.*

Moreover, the reverse implication in i), ii), *or* iii) *is not in general valid.*

We remark that assertions i) and ii) of Proposition 1 can be found in Woźnicki [8, pp. 46, 47], as well as an example showing that the reverse implication of i) can fail (cf. [8, p. 54]). For completeness however, Proposition 1 will be established in its entirety in § 3.

It is now evident from Proposition 1 that the key hypotheses of Theorems B and C (namely, that $N_2\geqq N_1$, and that $M_1^{-1}\geqq M_2^{-1}$) are progressively *weaker* hypotheses. As our generalization of Theorems B and C, we impose now a still weaker hypothesis (cf. (2.1)).

Theorem 2. *Let $A=M_1-N_1=M_2-N_2$ be two regular splittings of A, where $A^{-1}\geqq 0$. Assume that there exists a positive integer j for which*

$$(A^{-1}N_2)^jA^{-1}\geqq(A^{-1}N_1)^jA^{-1}. \tag{2.1}$$

Then,

$$1>\rho(M_2^{-1}N_2)\geqq\rho(M_1^{-1}N_1). \tag{2.2}$$

That Theorem 2 (to be proved in § 3) generalizes the first parts of Theorems B and C is an immediate consequence of Proposition 1.

Continuing, let $A=M_1-N_1=M_2-N_2$ be two regular splittings of A, where it is also assumed that $A^{-1}\geqq 0$. Then, we define the set S (which depends on the matrices A, M_1, N_1, M_2, and N_2) as

$$S:=\{\text{positive integers } j\text{: } (A^{-1}N_2)^jA^{-1}\geqq(A^{-1}N_1)^jA^{-1}\}. \tag{2.3}$$

With this definition, we further have

Proposition 3. *The set S is closed under addition.*

As a consequence of Proposition 3, we of course have that if S is not empty, then S has infinitely many elements. In particular, if $1\in S$, i.e., $A^{-1}N_2A^{-1}\geqq A^{-1}N_1A^{-1}$, then S consists of all positive integers from Proposition 3, which thus established iii) of Proposition 1. We further remark that if k and j in S are relatively prime, then S must contain all sufficiently large positive integers.

It is natural to ask if there are regular splittings $A=M_1-N_1=M_2-N_2$ of a matrix A satisfying $A^{-1}\geqq 0$, for which

$$(A^{-1}N_2)^jA^{-1}\geqq(A^{-1}N_1)^jA^{-1}$$

for some positive integer $j>1$, but for which $A^{-1}N_2A^{-1}\ngeqq A^{-1}N_1A^{-1}$. To this end, set

$$\lambda(S):=\min\{j: j\in S\} \quad (\text{where } \lambda(S):=+\infty \text{ if } S=\emptyset). \tag{2.4}$$

In terms of this notation, the question above simply asks if there are examples where $\lambda(S)>1$. To affirmatively answer this question, consider the following (fixed) matrices

$$A:=\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad \text{so that} \quad A^{-1}=\tfrac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \tag{2.5}$$

$$M_1:=\begin{bmatrix} 2 & -1/2 \\ -1 & 2 \end{bmatrix}, \quad \text{and} \quad N_1:=\begin{bmatrix} 0 & 1/2 \\ 0 & 0 \end{bmatrix}, \tag{2.6}$$

and the variable matrices

$$M_1(\alpha):=\begin{bmatrix} 2 & -1 \\ -1+\alpha & 2 \end{bmatrix}, \quad \text{and} \quad N_2(\alpha):=\begin{bmatrix} 0 & 0 \\ \alpha & 0 \end{bmatrix}, \quad \text{where } 0\leqq\alpha\leqq 1. \tag{2.7}$$

For each choice of α with $0\leqq\alpha\leqq 1$, $A=M_1-N_1=M_1(\alpha)-N_2(\alpha)$ are regular splittings of A. It can be verified that k is the *least* positive integer for which $(A^{-1}N_2(\alpha))^kA^{-1}\geqq(A^{-1}N_1)^kA^{-1}$ if and only if

$$(2\alpha)^{k-1}<4, \quad \text{and} \quad (2\alpha)^k\geqq 4. \tag{2.8}$$

Considering the case of equality in the second part of (2.8), we set

$$\alpha(k):=4^{1/k}/2 \qquad \text{for each positive integer } k\geqq 2, \tag{2.9}$$

so that $0<\alpha(k)\leqq 1$. On further setting

$$M_1^{(k)}:=\begin{bmatrix} 2 & -1 \\ -1+\alpha(k) & 2 \end{bmatrix}, \quad \text{and} \quad N_1^{(k)}:=\begin{bmatrix} 0 & 0 \\ \alpha(k) & 0 \end{bmatrix}, \tag{2.10}$$

then, for the regular splittings $A=M_1-N_1=M_1^{(k)}-N_1^{(k)}$, the associated set S of (2.3) satisfies (cf. (2.4))

$$\lambda(S)=k, \qquad \text{for } \textit{each} \text{ positive integer } k\geqq 2, \tag{2.11}$$

which shows that reverse implication of iii) in Proposition 1 can fail.

For results which provide partial converses to Theorem 2, as suggested by the second parts of Theorems B and C, we next state

Theorem 4. *Let $A=M_1-N_1=M_2-N_2$ be two regular splittings of A, where it is assumed that $A^{-1}>0$. If*

$$\rho(M_2^{-1}N_2)>\rho(M_1^{-1}N_1), \tag{2.12}$$

there exists a positive integer j_0 for which

$$(A^{-1}N_2)^jA^{-1}>(A^{-1}N_1)^jA^{-1}, \qquad \textit{for all } j\geqq j_0. \tag{2.13}$$

Consequently, if $\rho(M_2^{-1}N_2)>\rho(M_1^{-1}N_1)$, then the set S of (2.3) *is not empty, and, in fact, S contains all sufficiently large positive integers. Conversely, if there is a* *positive integer j for which*

$$(A^{-1}N_2)^j A^{-1} > (A^{-1}N_1)^j A^{-1}, \tag{2.14}$$

then (2.12) *is valid.*

As a useful but immediate corollary of Theorem 4, corresponding to the case $j=1$ of (2.14), we have

Corollary 5. *Let* $A=M_1-N_1=M_2-N_2$ *be two regular splittings of* A, *where it is assumed that* $A^{-1}>0$. *If*

$$A^{-1}N_2A^{-1} > A^{-1}N_1A^{-1}, \tag{2.15}$$

then $\rho(M_2^{-1}N_2)>\rho(M_1^{-1}N_1)$.

The reason *why* this Corollary 5 is interesting to us is that it generalizes the second parts of *both* Theorems B and C. To see this, assume (as in Theorem B) that $A=M_1-N_1=M_2-N_2$ are two regular splittings of A where $A^{-1}>0$, and where $N_2 \geqq N_1$ with $N_2 \neq N_1$ so that

$$N_2=N_1+E, \quad \text{where } E \geqq 0 \text{ with } E \neq 0. \tag{2.16}$$

From the assumption that $A^{-1}>0$, it readily follows that $A^{-1}EA^{-1}>0$, so that from (2.16), we have $A^{-1}N_2A^{-1}>A^{-1}N_1A^{-1}$, the hypothesis of (2.15), whence $\rho(M_2^{-1}N_2)>\rho(M_1^{-1}N_1)$ from Corollary 5. Similarly, assume (as in Theorem C) that $A=M_1-N_1=M_2-N_2$ are two regular splittings of A where $A^{-1}>0$, and where $M_1^{-1}>M_2^{-1}$. A straightforward calculation (to be given in § 3) shows again that $A^{-1}N_2A^{-1}>A^{-1}N_1A^{-1}$, so that $\rho(M_2^{-1}N_2)>\rho(M_1^{-1}N_1)$ follows once more from Corollary 5.

Returning to Theorem 4, it is natural to try to weaken (2.12) of Theorem 4, to say

$$\rho(A^{-1}N_2)=\rho(A^{-1}N_1), \tag{2.12'}$$

and to try to establish the weaker form of (2.13), namely

$$(A^{-1}N_2)^j A^{-1} \geqq (A^{-1}N_1)^j A^{-1}, \tag{2.13'}$$

for all sufficiently large positive integers j.

To show that this fails, consider the following 3×3 matrix

$$A:=\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad \text{so that} \quad A^{-1}=\tfrac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}>0, \tag{2.17}$$

and set

$$M_1:=\begin{bmatrix} 2 & -1/2 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad N_1:=\begin{bmatrix} 0 & 1/2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \tag{2.18}$$

$$M_2 := \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 7/3 \end{bmatrix}, \quad N_2 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}. \tag{2.19}$$

As is easily verified, $M_1^{-1} > 0$ and $M_2^{-1} > 0$, so that $A = M_1 - N_1 = M_2 - N_2$ are two regular splittings of A. Furthermore,

$$\rho(A^{-1}N_1) = \tfrac{1}{4} = \rho(A^{-1}N_2),$$

so that (2.12′) above is satisfied. On the other hand, as can be directly computed, we have

$$(A^{-1}N_1)^j A^{-1} = \frac{1}{4^{j+1}} \begin{bmatrix} 3 & 6 & 3 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix},$$

$$(A^{-1}N_2)^j A^{-1} = \frac{1}{4^{j+1}} \begin{bmatrix} 1/3 & 2/3 & 3/3 \\ 2/3 & 4/3 & 6/3 \\ 1 & 2 & 3 \end{bmatrix} \tag{2.20}$$

for each $j \geqq 1$. But, the comparison of the matrices in (2.20) shows that (2.13′) above fails and that (2.13′) also fails with N_1 and N_2 interchanged, for *each* positive integer j. In this sense, the result of Theorem 4 may be viewed as being *best possible*. It is, however, an open question if, under the hypotheses of Theorem 4, the associated set S of (2.3) can have some initial *gaps*.

To conclude this section, we remark that the results of Theorems B and C are very useful in comparing the spectral radii of associated iteration matrices, primarily because the conditions $N_2 \geqq N_1$ and/or $M_1^{-1} \geqq M_2^{-1}$ are relatively easy to check. We remark that our new condition, namely $(A^{-1}N_2)^j A^{-1} \geqq (A^{-1}N_1)^j A^{-1}$, while weaker than the conditions $N_2 \geqq N_1$ or $M_1^{-1} \geqq M_2^{-1}$, *may* be more cumbersome to apply in actual practice.

§ 3. Proofs

In this section, we give the proofs of the new results stated in § 2. We begin with the

Proof of Proposition 3. Suppose that j and k are arbitrary (not necessarily distinct) elements of the set S of (2.3), so that

$$(A^{-1}N_2)^j A^{-1} \geqq (A^{-1}N_1)^j A^{-1}, \tag{3.1}$$

and

$$(A^{-1}N_2)^k A^{-1} \geqq (A^{-1}N_1)^k A^{-1}. \tag{3.2}$$

Postmultiplying throughout in (3.1) by $N_2(A^{-1}N_2)^{k-1}A^{-1}$ gives

$$(A^{-1}N_2)^{j+k} A^{-1} \geqq (A^{-1}N_1)^j (A^{-1}N_2)^k A^{-1}, \tag{3.3}$$

while premultiplying throughout in (3.2) by $(A^{-1}N_1)^j$ gives

$$(A^{-1}N_1)^j(A^{-1}N_2)^k A^{-1} \geqq (A^{-1}N_1)^{j+k}A^{-1}. \tag{3.4}$$

Thus, combining (3.3) and (3.4) yields

$$(A^{-1}N_2)^{j+k}A^{-1} \geqq (A^{-1}N_1)^{j+k}A^{-1}, \tag{3.5}$$

so that $(j+k)\in S$. □

Proof of Proposition 1. i) *Assume that* $N_2 \geqq N_1$ (where $A=M_1-N_1=M_2-N_2$ are two regular splittings of A with $A^{-1}\geqq 0$), so that $N_2-N_1\geqq 0$. Since $N_i=M_i-A$, then $N_2-N_1=M_2-M_1\geqq 0$, which can be factored as

$$M_2-M_1=M_1(M_1^{-1}-M_2^{-1})M_2\geqq 0. \tag{3.6}$$

Since, by hypothesis, M_1^{-1} and M_2^{-1} are nonnegative matrices, then premultiplying by M_1^{-1} and postmultiplying by M_2^{-1} in (3.6) gives

$$M_1^{-1}-M_2^{-1}\geqq 0, \quad \text{i.e.,} \quad M_1^{-1}\geqq M_2^{-1}, \tag{3.7}$$

the desired conclusions of i).

ii) *Assume that* $M_1^{-1}\geqq M_2^{-1}$. Since $M_i=A+N_i$, then $M_1^{-1}\geqq M_2^{-1}$ implies that $(A+N_1)^{-1}\geqq(A+N_2)^{-1}$, which can be represented as

$$(I+A^{-1}N_1)^{-1}A^{-1}\geqq A^{-1}(I+N_2A^{-1})^{-1}. \tag{3.8}$$

Since the matrices $I+A^{-1}N_i$ are nonnegative, then premultiplying by $(I+A^{-1}N_1)$ and postmultiplying by $(I+N_2A^{-1})$ in (3.8) gives

$$A^{-1}+A^{-1}N_2A^{-1}\geqq A^{-1}+A^{-1}N_1A^{-1},$$

which reduces to

$$A^{-1}N_2A^{-1}\geqq A^{-1}N_1A^{-1}, \tag{3.9}$$

the desired conclusion of ii).

iii) *Assume* $A^{-1}N_2A^{-1}\geqq A^{-1}N_1A^{-1}$, so that $1\in S$ (cf. (2.3)). From Proposition 3, it follows that S contains all positive integers j, whence

$$(A^{-1}N_2)^jA^{-1}\geqq(A^{-1}N_1)^jA^{-1}, \quad \text{for each positive integers } j1, \tag{3.10}$$

the desired conclusion of iii).

To complete the proof of Proposition 1, we give examples where the reverse implication of i) or ii) of Proposition 1, fails. Set

$$A:=\tfrac{1}{2}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad \text{so that} \quad A^{-1}=\tfrac{2}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}>0,$$

and set

$$M_1:=\tfrac{1}{4}\begin{bmatrix} 4 & -2 \\ -2 & 5 \end{bmatrix}, \quad N_1:=\tfrac{1}{4}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{so that} \quad M_1^{-1}=\tfrac{1}{4}\begin{bmatrix} 5 & 2 \\ 2 & 4 \end{bmatrix}, \tag{3.12}$$

$$M_2 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad N_2 := \tfrac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \text{so that} \quad M_2^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{3.13}$$

and

$$M_3 := \tfrac{1}{4}\begin{bmatrix} 5 & 0 \\ -1 & 5 \end{bmatrix}, \qquad N_3 := \tfrac{1}{4}\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \quad \text{so that} \quad M_3^{-1} = \tfrac{4}{25}\begin{bmatrix} 5 & 0 \\ 1 & 5 \end{bmatrix}. \tag{3.14}$$

With these definitions, it is evident that $A = M_i - N_i$ $(i = 1, 2, 3)$ are all regular splittings of A. Further, from the above equations, we see that

$$M_1^{-1} \geqq M_2^{-1}, \quad \text{but } N_2 \ngeqq N_1 \text{ and } N_1 \ngeqq N_2, \tag{3.15}$$

and

$$A^{-1}N_3A^{-1} = \tfrac{1}{9}\begin{bmatrix} 11 & 13 \\ 10 & 11 \end{bmatrix} \geqq A^{-1}N_2A^{-1} = \tfrac{1}{9}\begin{bmatrix} 8 & 10 \\ 10 & 8 \end{bmatrix}, \text{ but}$$

$$M_2^{-1} \ngeqq M_3^{-1}, \quad \text{and} \quad M_3^{-1} \ngeqq M_2^{-1}, \tag{3.16}$$

showing that the reverse implications of i) and ii) of Proposition 1 can fail. □

Proof of Theorem 2. From (1.1) of Theorem A, it suffices to show that $\rho(M_2^{-1}N_2) \geqq \rho(M_1^{-1}N_1)$. From the hypothesis,

$$(A^{-1}N_2)^j A^{-1} \geqq (A^{-1}N_1)^j A^{-1} \geqq 0, \tag{3.17}$$

for some integer j. Postmultiplying throughout in (3.17) by the nonnegative matrix $N_2(A^{-1}N_2)^{j-1}$ gives

$$(A^{-1}N_2)^{2j} \geqq (A^{-1}N_1)^j (A^{-1}N_2)^j \geqq 0, \tag{3.18}$$

while postmultiplying throughout in (3.17) by the nonnegative matrix $N_1(A^{-1}N_1)^{j-1}$ gives

$$(A^{-1}N_2)^j (A^{-1}N_1)^j \geqq (A^{-1}N_1)^{2j} \geqq 0. \tag{3.19}$$

Clearly, from the Perron-Frobenius Theorem on nonnegative matrices, (3.18) and (3.19) respectively imply that

$$\rho^{2j}(A^{-1}N_2) \geqq \rho^j((A^{-1}N_1)(A^{-1}N_2)) \tag{3.20}$$

and

$$\rho^j((A^{-1}N_2)(A^{-1}N_1)) \geqq \rho^{2j}(A^{-1}N_1). \tag{3.21}$$

But, as it is well-known that $\rho(EF) = \rho(FE)$ for any two complex $n \times n$ matrices, then (3.20) and (3.21) give that

$$\rho^{2j}(A^{-1}N_2) \geqq \rho^{2j}(A^{-1}N_1), \tag{3.22}$$

whence

$$\rho(A^{-1}N_2) \geqq \rho(A^{-1}N_1). \tag{3.23}$$

Since from (1.1) we see that $\rho(M^{-1}N)$ is a strictly monotone increasing function of $\rho(A^{-1}N)$, then (3.23) implies that

$$\rho(M_2^{-1}N_2) \geqq \rho(M_1^{-1}N_1), \tag{3.24}$$

the desired conclusion. □

In order to prove Theorem 4, we need the following result of

Lemma 6. *Assume that the* $n \times n$ *matrices* A *and* N *satisfy* $A^{-1} > 0$ *and* $N \geqq 0$. *Then, the following are mutually exclusive and collectively exhaustive:*

i) $N \equiv 0$, *in which case* $A^{-1}N \equiv 0$;

ii) $A^{-1}N \not\equiv 0$ *and reducible, in which case there exists an* $n \times n$ *permutation matrix* P *such that*

$$P(A^{-1}N)P^T = \left[\begin{array}{c|c} 0 & R_{1,2} \\ \hline 0 & R_{2,2} \end{array}\right], \quad \textit{where } R_{1,2} > 0 \textit{ and where } R_{2,2} > 0; \tag{3.25}$$

iii) $A^{-1}N$ *is irreducible, in which case* $A^{-1}N > 0$.

Proof. Set $A^{-1}N =: B = [b_{i,j}]$. On writing $A^{-1} := [\alpha_{i,j}]$, suppose that some $b_{i,j} = 0$, so that if $N := [n_{i,j}]$, then

$$b_{i,j} = \sum_{k=1}^{n} \alpha_{i,k} n_{k,j} = 0. \tag{3.26}$$

As $A^{-1} > 0$ and as $N \geqq 0$ by hypothesis, then evidently $n_{k,j} = 0$ for all $1 \leqq k \leqq n$. Hence, the j-th column of N must vanish, which implies the j-th column of $A^{-1}N$ must also vanish. Thus, both N and $A^{-1}N$ are reducible. Assume that $A^{-1}N$ is reducible. Then (cf. [7, p. 46]), there is an $n \times n$ permutation matrix P such that

$$P(A^{-1}N)P^T = \begin{bmatrix} \tilde{R}_{1,1} & & /// \\ & \ddots & \\ 0 & & \tilde{R}_{l,l} \end{bmatrix}, \tag{3.27}$$

the so-called *canonical reduced form* for $A^{-1}N$, where each $\tilde{R}_{j,j}$ is square and irreducible, or a 1×1 null matrix. But, from the argument above, any column of the matrix on the right in (3.27) having a zero entry must vanish identically. Thus, the display in (3.27) reduces to

$$P(A^{-1}N)P^T = \left[\begin{array}{c|c} 0 & R_{1,2} \\ \hline 0 & R_{2,2} \end{array}\right], \tag{3.28}$$

where $R_{2,2}$ is irreducible or a 1×1 null matrix. If $R_{2,2}$ is a 1×1 null matrix, then, by the above argument, $A^{-1}N \equiv 0$, so that $N \equiv 0$. If $R_{2,2}$ is irreducible, the same argument shows that $R_{1,2} > 0$ and $R_{2,2} > 0$. Finally, if $A^{-1}N$ is irreducible, then we similarly conclude that $A^{-1}N > 0$. □

Proof of Theorem 4. Assume (2.12), which from Theorem A gives $\rho(A^{-1}N_2) > \rho(A^{-1}N_1)$. Since $\rho(A^{-1}N_2) > 0$, only cases ii) and iii) of Lemma 6 are relevant. Assuming $A^{-1}N_2$ is reducible, we have (up to a permutation transformation) that (cf. (3.25) of Lemma 6)

$$A^{-1}N_2=\left[\begin{array}{c|c}0 & R_{1,2}\\ \hline 0 & R_{2,2}\end{array}\right], \quad \text{where } R_{1,2}>0 \text{ and where } R_{2,2}>0. \tag{3.29}$$

On setting $\rho:=\rho(A^{-1}N_2)$, it follows that

$$(A^{-1}N_2/\rho)^j=\left[\begin{array}{c|c}0 & R_{1,2}R_{2,2}^{j-1}/\rho^j\\ \hline 0 & R_{2,2}^j/\rho^j\end{array}\right], \tag{3.30}$$

for all positive integers j. But as $\rho=\rho(A^{-1}N_2)=\rho(R_{2,2})$ and as $R_{2,2}>0$ from (3.29), it is known (cf. [1, p. 45]) that

$$\lim_{j\to\infty}\left(\frac{R_{2,2}}{\rho}\right)^j=:\hat{R}_{2,2}>0, \tag{3.31}$$

so that from (3.30),

$$\lim_{j\to\infty}(A^{-1}N_2/\rho)^j=\left[\begin{array}{c|c}0 & R_{1,2}\hat{R}_{2,2}/\rho\\ \hline 0 & \hat{R}_{2,2}\end{array}\right]. \tag{3.32}$$

Further, partitioning A^{-1} conformally with respect to the partitioning in (3.29), we can write

$$A^{-1}=\left[\begin{array}{c|c}D_{1,1} & D_{1,2}\\ \hline D_{2,1} & D_{2,2}\end{array}\right], \quad \text{where } D_{i,j}>0 \text{ for all } 1\leqq i,j\leqq 2, \tag{3.33}$$

and, similar to (3.32), we also deduce that

$$\lim_{j\to\infty}(A^{-1}N_2/\rho)^jA^{-1}=:E=\left[\begin{array}{c|c}R_{1,2}\hat{R}_{2,2}D_{2,1}/\rho & R_{1,2}\hat{R}_{2,2}D_{2,2}/\rho\\ \hline \hat{R}_{2,2}D_{2,1} & \hat{R}_{2,2}D_{2,2}\end{array}\right] \tag{3.34}$$

so that all entries in the matrix E are *positive*. On the other hand, $\rho(A^{-1}N_1)/\rho(A^{-1}N_2)<1$ implies (cf. [7, p. 13]) that $\lim_{j\to\infty}(A^{-1}N_1/\rho)^j=0$, so that

$$\lim_{j\to\infty}(A^{-1}N_1/\rho)^jA^{-1}=0. \tag{3.35}$$

Hence, there exists a positive integer j_0 such that

$$(A^{-1}N_2/\rho)^jA^{-1}>(A^{-1}N_1/\rho)^jA^{-1}, \quad \text{for all } j\geqq j_0,$$

whence

$$(A^{-1}N_2)^jA^{-1}>(A^{-1}N_1)^jA^{-1}, \quad \text{for all } j\geqq j_0, \tag{3.36}$$

the desired result of (2.13) of Theorem 4.

Continuing, assume (cf. (2.14)) there is a positive integer j such that

$$(A^{-1}N_2)^jA^{-1}>(A^{-1}N_1)^jA^{-1},$$

which we express as

$$(A^{-1}N_2)^jA^{-1}-(A^{-1}N_1)^jA^{-1}=:F_j, \quad \text{where} \quad F_j>0. \tag{3.37}$$

Postmultiplying (3.37) by $N_2(A^{-1}N_2)^{j-1}$ gives

$$(A^{-1}N_2)^{2j}=(A^{-1}N_1)^j(A^{-1}N_2)^j+F_jN_2(A^{-1}N_2)^{j-1}, \tag{3.38}$$

while postmultiplying (3.37) by $N_1(A^{-1}N_1)^{j-1}$ similarly gives

$$(A^{-1}N_2)^j(A^{-1}N_1)^j=(A^{-1}N_1)^{2j}+F_jN_1(A^{-1}N_1)^{j-1}. \tag{3.39}$$

First, it is evident from (3.37) that $N_2\not\equiv 0$. Thus, in applying Lemma 6 to A^{-1} and N_i, $i=1,2$, a number of different cases must be considered. If $N_1\equiv 0$, then either case ii) or iii) of Lemma 6, applied to A^{-1} and N_2, gives that $\rho(A^{-1}N_2)>0$, whence

$$\rho(A^{-1}N_2)>0=\rho(A^{-1}N_1).$$

Thus, from (1.4) of Theorem A, $\rho(M_2^{-1}N_2)>\rho(M_1^{-1}N_1)=0$, the desired result of (2.12).

Next, assume that $A^{-1}N_1$ is irreducible, so that, from Lemma 6, $A^{-1}N_1>0$ implies that no column of N_1 can vanish. Hence, as $F_j>0$, it follows that

$$F_jN_1(A^{-1}N_1)^{j-1}>0. \tag{3.40}$$

Thus, from (3.39), $(A^{-1}N_2)^j(A^{-1}N_1)^j>(A^{-1}N_1)^{2j}$, and as $(A^{-1}N_1)^{2j}$ is a positive matrix and hence irreducible, we deduce from (3.39) that (cf. [5, p. 57])

$$\rho^j((A^{-1}N_2)(A^{-1}N_1))>\rho^{2j}(A^{-1}N_1). \tag{3.41}$$

On the other hand, it is obvious from (3.38) that

$$\rho^{2j}(A^{-1}N_2)\geqq\rho^j((A^{-1}N_1)(A^{-1}N_2)), \tag{3.42}$$

so that (3.41) and (3.42) yield

$$\rho(A^{-1}N_2)>\rho(A^{-1}N_1), \tag{3.43}$$

which, from (1.1) of Theorem A, gives $\rho(M_2^{-1}N_2)>\rho(M_1^{-1}N_1)$, the desired result of (2.12).

In the final case that $A^{-1}N_1\not\equiv 0$ and reducible, we use the representation (3.25) for $P(A^{-1}N_1)P^T$, where $R_{1,2}>0$ and where $R_{2,2}>0$. In addition, it can be verified that

$$PN_1P^T=\left[\begin{array}{c|c}0 & S_{1,2}\\ \hline 0 & S_{2,2}\end{array}\right], \tag{3.44}$$

where the partitioning in (3.44) is conformal with respect to the partitioning in (3.25). Moreover, $S_{1,2}\geqq 0$, $S_{2,2}\geqq 0$, and no column of $(S_{1,2},S_{2,2})^T$ can vanish. Dropping the (fixed) permutation matrix P now for convenience it easily follows (cf. 3.30)) that

$$(A^{-1}N_1)^{2j}=\left[\begin{array}{c|c}0 & R_{1,2}R_{2,2}^{2j-1}\\ \hline 0 & R_{2,2}^{2j}\end{array}\right], \quad \text{and} \quad N_1(A^{-1}N_1)^{j-1}=\left[\begin{array}{c|c}0 & S_{1,2}R_{2,2}^{j-1}\\ \hline 0 & S_{2,2}R_{2,2}^{j-1}\end{array}\right]. \tag{3.45}$$

Thus, if F_j of (3.37) is conformally partitioned relative to (3.44), i.e.,

$$F_j := \left[\begin{array}{c|c} T_{1,1} & T_{1,2} \\ \hline T_{2,1} & T_{2,2} \end{array}\right], \quad \text{where } T_{i,k} > 0 \text{ for all } 1 \leqq i,\ k \leqq 2, \tag{3.46}$$

then

$$F_j N_1 (A^{-1}N_1)^{j-1} = \left[\begin{array}{c|c} 0 & T_{1,1}S_{1,2}R_{2,2}^{j-1} + T_{1,2}S_{2,2}R_{2,2}^{j-1} \\ \hline 0 & T_{2,1}S_{1,2}R_{2,2}^{j-1} + T_{2,2}S_{2,2}R_{2,2}^{j-1} \end{array}\right]. \tag{3.47}$$

Because no column of $(S_{1,2}, S_{2,2})^T$ can vanish and because the $T_{i,k}$ and $R_{i,2}^{j-1}$ are all positive, the final diagonal block in (3.47) is evidently positive. This implies, from (3.39) and (3.45), that

$$\rho^j((A^{-1}N_2)(A^{-1}N_1)) > \rho^{2j}(A^{-1}N_1), \tag{3.48}$$

while from (3.38) we have the obvious inequality

$$\rho^{2j}(A^{-1}N_2) \geqq \rho^j((A^{-1}N_1)(A^{-1}N_2)). \tag{3.49}$$

Thus, (3.48) and (3.49) together yield

$$\rho(A^{-1}N_2) > \rho(A^{-1}N_1), \tag{3.50}$$

which, from (1.1) of Theorem A, gives $\rho(M_2^{-1}N_2) > \rho(M_1^{-1}N_1)$, the desired result of (2.12). □

We remark that it is clear from our proof of Theorem 4 that, with the initial hypotheses of Theorem 4, if $\rho(M_2^{-1}N_2) > \rho(M_1^{-1}N_1)$, then for any matrix $B > 0$, there exists a positive integer j_0 for which

$$(A^{-1}N_2)^j B > (A^{-1}N_1)^j B, \quad \text{for all } j \geqq j_0. \tag{3.51}$$

Finally, we show that Corollary 5 generalizes the second part of Woźnicki's Theorem C. Let $A = M_1 - N_1 = M_2 - N_2$ be two regular splittings of A, where $A^{-1} > 0$, and assume $M_1^{-1} > M_2^{-1}$. Consider the matrix $A^{-1}(N_2 - N_1)A^{-1}$, which is at least nonnegative from ii) of Proposition 1. Then, following Woźnicki [8, p. 48],

$$\begin{aligned} A^{-1}(N_2 - N_1)A^{-1} &= A^{-1}(M_2 - M_1)A^{-1} = A^{-1}M_1(M_1^{-1} - M_2^{-1})M_2A^{-1} \\ &= A^{-1}(A + N_1)(M_1^{-1} - M_2^{-1})(A + N_2)A^{-1} \\ &= (I + A^{-1}N_1)(M_1^{-1} - M_2^{-1})(I + N_2A^{-1}), \end{aligned}$$

which can be expressed as

$$\begin{aligned} A^{-1}(N_2 - N_1)A^{-1} &= (M_1^{-1} - M_2^{-1}) + A^{-1}N_1(M_1^{-1} - M_2^{-1}) \\ &\quad + (M_1^{-1} - M_2^{-1})N_2A^{-1} + A^{-1}N_1(M_1^{-1} - M_2^{-1})N_2A^{-1}. \end{aligned} \tag{3.52}$$

Clearly, the first term on the right is a positive matrix by hypothesis and the remaining terms on the right are all nonnegative matrices, so that

$$A^{-1}(N_2-N_1)A^{-1}>0, \quad \text{or equivalently,} \quad A^{-1}N_2A^{-1}>A^{-1}N_1A^{-1}, \quad (3.53)$$

which is hypothesis (2.15) of Corollary 5, whence $\rho(M_2^{-1}N_2)>\rho(M_1^{-1}N_1)$.

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