ON THE MINIMUM MODULI OF NORMALIZED POLYNOMIALS

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Abstract. Consider any complex polynomial $p_n(z) = 1 + \sum_{j=1}^{n} a_j z^j$ which satisfies $\sum_{j=1}^{n} |a_j| = 1$, and let Γ_n denote the supremum of the minimum moduli on $|z| = 1$ of all such polynomials $p_n(z)$. We show that

$$1 - \frac{1}{n} \leq \Gamma_n \leq \sqrt{1 - \frac{1}{n}}, \quad \text{for all } n \geq 1 .$$

If the coefficients of $p_n(z)$ are further restricted to be positive numbers and if $\tilde{\Gamma}_n$ denotes the analogous supremum of the minimum moduli on $|z| = 1$ of such polynomials, we similarly show that

$$1 - \frac{1}{n} \leq \tilde{\Gamma}_n \leq \sqrt{1 - \frac{3}{(2n+1)}}, \quad \text{for all } n \geq 1 .$$

We also include some recent numerical experiments on the behavior of Γ_n, as well as some related conjectures.

1. Introduction

Consider any non-constant complex polynomial $p_n(z) = \sum_{j=0}^{n} a_j z^j$ with $p_n(0) \neq 0$, and normalize $p_n(z)$ so that

$$(1.1) \qquad p_n(z) = 1 + \sum_{j=1}^{n} a_j z^j, \quad \text{where} \quad \sum_{j=1}^{n} |a_j| \neq 0 .$$

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By a well-known result of Cauchy (cf. Marden [4, p. 126]), if R (the Cauchy radius of $p_n(z)$) is the unique positive zero of

(1.2) $$1 - |a_1|\cdot R - \dots - |a_n|\cdot R^n \quad ,$$

then each zero $\hat{z}$ of $p_n(z)$ of (1.1) satisfies $|\hat{z}| \geq R$. On further normalizing the Cauchy radius R of $p_n(z)$ to be unity, i.e., on assuming

(1.3) $$\sum_{j=1}^{n} |a_j| = 1 \quad ,$$

then any polynomial $p_n(z)$ in (1.1) which satisfies (1.3) evidently has no zeros in $|z| < 1$. (It may have zeros on $|z| = 1$, as the examples $1 + z^n$ show.)

Our interest is in the following problem, which is related to the recent study of global descent functions for determining zeros of polynomials (cf. Henrici [2,3] and Ruscheweyh [6]). Consider the set of normalized polynomials

(1.4) $$S_n := \{p_n(z) = 1 + \sum_{j=1}^{n} a_j z^j : \sum_{j=1}^{n} |a_j| = 1\} \text{ , for each } n \geq 1 \quad ,$$

and put

(1.5) $$m(p_n) := \min\{|p_n(e^{i\theta})| : \theta \text{ real}\} \text{ , for any } p_n \in S_n \quad .$$

Our main question is: How large can $m(p_n)$ be on the set S_n ? Thus, on setting

(1.6) $$\Gamma_n := \sup\{m(p_n) : p_n \in S_n\} \quad , \qquad (n \geq 1) \quad ,$$

our goal here is to establish rigorous upper and lower bounds for Γ_n, as a function of n. We also report on some numerical experiments, which in turn have inspired some related mathematical conjectures.

2. Upper and Lower Bounds for Γ_n

The following inequality, based on conformal mappings, was derived in Ruscheweyh [6]:

$$(2.1)\qquad m^2(p_n) + \sum_{j=1}^{n} |a_j|^2 \le 1 \quad , \quad \text{for any} \quad p_n \in S_n \quad .$$

As the Cauchy-Schwarz inequality applied to (1.3) gives that $\sum_{j=1}^{n} |a_j|^2 \ge \frac{1}{n}$, it follows from (2.1) that

$$(2.2)\qquad m(p_n) \le \sqrt{1 - \frac{1}{n}} \quad , \quad \text{for any} \quad p_n \in S_n \quad ,$$

which yields (cf. (1.6)) the upper bound

$$(2.3)\qquad \Gamma_n \le \sqrt{1 - \frac{1}{n}} \quad , \quad \text{for all} \quad n \ge 1 \quad .$$

To similarly derive a lower bound for Γ_n , consider the specific polynomial

$$(2.4)\qquad Q_n(z) := 1 + \frac{2}{n(n+1)} \sum_{k=1}^{n} (n+1-k) z^k \quad ,$$

which is an element of S_n for each $n \ge 1$. Now, $Q_n(z)$ can also be expressed as

$$Q_n(z) = \frac{n(n+1) - 2n^2 z + (n-2)(n+1)z^2 + 2z^{n+2}}{n(n+1)(1-z)^2} \quad ,$$

and evaluating the above expression for $z = -1$ gives

$$(2.5)\qquad Q_n(-1) = \begin{cases} 1 - \frac{1}{n+1} \quad , & \text{for } n \text{ an even positive integer;} \\ 1 - \frac{1}{n} \quad , & \text{for } n \text{ an odd positive integer .} \end{cases}$$

Next, on defining

$$(2.6)\qquad g_n(z) := \frac{1}{(n+1)} \sum_{k=0}^{n} (n+1-k) z^k \quad , \qquad (n = 1,2,\ldots) \quad ,$$

then $Q_n(z)$ and $g_n(z)$ are related through

$$(2.7)\qquad Q_n(z) = \frac{n-2}{n} + \frac{2}{n} g_n(z) \quad .$$

Writing $g_n(z)$ of (2.6) as

(2.8) $$g_n(z) = \sum_{k=0}^{\infty} \alpha_k(n) z^k \ , \text{ where } \alpha_k(n) := \begin{cases} \frac{n+1-k}{n+1} \ , & k = 0,1,\ldots,n \ ; \\ 0 \ , & k \geq n+1 \ , \end{cases}$$

then the coefficients $\alpha_k(n)$ of $g_n(z)$ can be seen to be doubly monotonic, i.e.,

(2.9) $$\alpha_k(n) \geq 0 \ , \quad \text{and} \quad \alpha_k(n) - \alpha_{k+1}(n) \geq \alpha_{k+1}(n) - \alpha_{k+2}(n) \ ,$$

for all $k \geq 0$ and all $n \geq 1$. From a well-known result of Fejér [1], it follows that

(2.10) $$\text{Re } g_n(z) \geq \frac{1}{2} \ , \quad \text{for all } |z| \leq 1 \ .$$

Consequently (cf. (2.7)),

(2.11) $$\text{Re } Q_n(z) = \frac{n-2}{n} + \frac{2}{n} \text{Re } g_n(z) \geq 1 - \frac{1}{n} \ , \quad \text{for all } |z| \leq 1 \ ,$$

which implies that

(2.12) $$m(Q_n) \geq 1 - \frac{1}{n} \ , \quad \text{for all } n \geq 1 \ .$$

(Note that because of (2.5), equality evidently holds in (2.12) for every odd positive integer n.) But, as $Q_n(z)$ is an element of S_n, (2.12) implies that

(2.13) $$1 - \frac{1}{n} \leq \Gamma_n \ , \quad \text{for all } n \geq 1 \ .$$

Combining (2.13) and (2.3) then yields our first result of

Proposition 1. For each positive integer n,

(2.14) $$1 - \frac{1}{n} \leq \Gamma_n \leq \sqrt{1 - \frac{1}{n}} \ .$$

Obviously, the bounds of (2.14) are tight for $n = 1$ and give $\Gamma_1 = 0$. For $n > 1$, there is however a gap between the upper and lower bounds in (2.14).

Because of the bounds of (2.14), it is reasonable to express Γ_n as

$$(2.15)\qquad \Gamma_n := 1 - \frac{\gamma_n}{n}\quad,\quad \text{for all}\quad n \geq 1\quad,$$

so that from (2.14),

$$(2.16)\qquad n\left(1 - \sqrt{1 - \frac{1}{n}}\right) \leq \gamma_n \leq 1\quad,\quad \text{for all}\quad n \geq 1\quad.$$

Now, the lower bound in (2.16) is strictly decreasing as a function of n and has the limit $1/2$, so that

$$(2.17)\qquad \frac{1}{2} < \gamma_n \leq 1\quad,\quad \text{for all}\quad n \geq 1\quad.$$

Next, consider the following subset, $\tilde{S}_n$, of S_n :

$$(2.18)\qquad \tilde{S}_n := \{p_n(z) = 1 + \sum_{j=1}^{n} a_j z^j : p_n \in S_n \text{ and } a_j > 0 \text{ for all } 1 \leq j \leq n\}.$$

For $\tilde{S}_n$, we can associate the analogous quantity $\tilde{\Gamma}_n$:

$$(2.19)\qquad \tilde{\Gamma}_n := \sup\{m(p_n) : p_n \in \tilde{S}_n\}\quad,\qquad (n \geq 1)\quad.$$

Obviously, as $\tilde{S}_n \subset S_n$, then

$$(2.20)\qquad \tilde{\Gamma}_n \leq \Gamma_n\quad,\quad \text{for all}\quad n \geq 1\quad.$$

But, noting that $Q_n(z)$ of (2.4) is also an element of $\tilde{\Gamma}_n$, then from (2.12), we also deduce

$$(2.21)\qquad 1 - \frac{1}{n} \leq \tilde{\Gamma}_n\quad,\quad \text{for all}\quad n \geq 1\quad.$$

From (2.20) and (2.21), the upper and lower bounds of (2.14) apply equally well to $\tilde{\Gamma}_n$. But, for an improved upper bound for $\tilde{\Gamma}_n$, we establish

<u>Proposition</u> 2. <u>For each positive integer</u> n,

$$(2.22)\qquad 1 - \frac{1}{n} \leq \tilde{\Gamma}_n \leq \sqrt{1 - \frac{3}{(2n+1)}}\quad.$$

<u>Proof</u>. To obtain the upper bound in (2.22), set

$$(2.23)\qquad M(p_n) := \max\{|p_n(e^{i\theta})| : \theta \text{ real}\}\quad,\quad \text{for any}\quad p_n \in S_n\quad.$$

Note that $M(p_n) = 2$ for any $p_n \in \tilde{S}_n$, for each $n \geq 1$. Next, for each $p_n(z) = \sum_{j=0}^{n} a_j z^j$ (with $a_0 := 1$) in $\tilde{S}_n$, consider the real trigonometric polynomial

$$(2.24) \qquad T_n(\theta; p_n) := |p_n(e^{i\theta})|^2 - \sum_{j=0}^{n} a_j^2 ,$$

which has the explicit form (without constant term)

$$(2.25) \qquad T_n(\theta; p_n) = 2 \sum_{k=1}^{n} \cos(k\theta) \sum_{j=0}^{n-k} a_j a_{j+k} .$$

Setting $\lambda_0 := \sum_{j=0}^{n} a_j^2$, it is evident from (2.24) that

$$(2.26) \quad \begin{cases} \max\limits_{\theta \text{ real}} T_n(\theta; p_n) = M^2(p_n) - \lambda_0 = 4 - \lambda_0 , \quad \text{and} \\ -\min\limits_{\theta \text{ real}} T_n(\theta; p_n) = \lambda_0 - m^2(p_n) . \end{cases}$$

From Pólya-Szegö [5, p. 84, Exercise 58], it is known that

$$\max_{\theta} T_n(\theta; p_n) \leq -n \cdot \min_{\theta} T_n(\theta; p_n) ,$$

or equivalently, using (2.26),

$$(2.27) \qquad 4 + n \cdot m^2(p_n) \leq (n+1)\lambda_0 .$$

But since $\sum_{j=1}^{n} a_j^2 \leq 1 - m^2(p_n)$ from (2.1), then

$$\lambda_0 := 1 + \sum_{j=1}^{n} a_j^2 \leq 2 - m^2(p_n) .$$

Substituting the above in (2.27) then yields

$$(2.28) \qquad m^2(p_n) \leq \frac{2n-2}{2n+1} = 1 - \frac{3}{2n+1} , \quad \text{for all } p_n \in \tilde{S}_n ,$$

which gives the desired upper bound of (2.22). □

In analogy with (2.15), we similarly define

$$(2.29)\qquad \tilde{\Gamma}_n := 1 - \frac{\tilde{\gamma}_n}{n}\ , \quad \text{for all} \quad n \geq 1\ .$$

Thus, from (2.22),

$$(2.30)\qquad n\left\{1 - \sqrt{1 - \frac{3}{2n+1}}\right\} \leq \gamma_n \leq 1\ , \quad \text{for all} \quad n \geq 1\ ,$$

which further yields

$$(2.31)\qquad 0.732213\ \ldots = 3\left(1 - \sqrt{\frac{4}{7}}\right) \leq \tilde{\gamma}_n \leq 1\ , \quad \text{for all} \quad n \geq 1\ ,$$

as well as

$$(2.32)\qquad \frac{3}{4} \leq \underline{\lim}_{n \to \infty} \tilde{\gamma}_n \leq \overline{\lim}_{n \to \infty} \tilde{\gamma}_n \leq 1\ .$$

3. Computational Results

Intrigued by these numbers Γ_n, $\tilde{\Gamma}_n$, γ_n, and $\tilde{\gamma}_n$, we embarked on some numerical calculations to give further insight into their behavior as $n \to \infty$. First, we conjecture that $\Gamma_n = \tilde{\Gamma}_n$ for all $n \geq 1$, and we hope to establish this in a later work. Thus, our calculations (to be described below) were aimed at determining sharp lower bounds for $\tilde{\Gamma}_n$.

The idea in our calculations was to find an "extremal" $\tilde{p}_n(z)$ in $\tilde{S}_n$ such that

$$(3.1)\qquad \tilde{p}_n(e^{i\theta}) = m(\tilde{p}_n) \quad \text{in precisely } n \text{ distinct points} \quad \{\theta_j\}_{j=1}^{n}\ .$$

For convenience, all calculations were performed for n even, say $n = 2k$, $k \geq 1$. Then, for $p_{2k}(z) = 1 + \sum_{j=1}^{2k} a_j z^j$ in $\tilde{S}_{2k}$ (so that $a_j > 0$ for $j = 1,\ldots,2k$, and $a_1 + a_2 + \ldots + a_{2k} = 1$), write

$$(3.2)\qquad \left|1 + \sum_{j=1}^{2k} a_j e^{ij\theta}\right|^2 = m^2(p_{2k}) + 2^{2k}\, a_{2k} \prod_{j=1}^{k} (\varepsilon_j + \cos\theta)^2\ ,$$

with the objective of maximizing $m^2(p_{2k})$. Writing

$$(3.3)\qquad \left|1+\sum_{j=1}^{2k} a_j e^{ij\theta}\right|^2 = \sum_{j=0}^{2k} A_j \cos j\theta \text{ , and } \prod_{j=1}^{k} (\varepsilon_j + \cos\theta)^2 = \sum_{j=0}^{2k} B_j \cos j\theta ,$$

where $A_j = A_j(a_1,\ldots,a_{2k})$, and where $B_j = B_j(\varepsilon_1,\ldots,\varepsilon_k)$, then (3.2) and (3.3) give us that

$$(3.4)\qquad \begin{cases} m^2(p_{2k}) = A_0 - 2^{2k} a_{2k} B_0 , \\ A_j - 2^{2k} a_{2k} B_j = 0 , \quad j = 1,\ldots,2k-1 , \\ a_1 + a_2 + \ldots + a_{2k} = 1 . \end{cases}$$

We then formulate this as the following <u>Lagrange multiplier problem</u>. Consider

$$(3.5)\qquad F(a_1,\ldots,a_{2k},\varepsilon_1,\ldots,\varepsilon_k,\lambda_1,\ldots\lambda_{2k})$$
$$:= (A_0 - 2^{2k} a_{2k} B_0) + \sum_{j=1}^{2k-1} \lambda_j (A_j - 2^{2k} a_{2k} B_j) + \lambda_{2k}(a_1 + \ldots + a_{2k} - 1),$$

subject to the conditions

$$(3.6)\qquad \frac{\partial F}{\partial a_j} = 0 , \ j = 1,\ldots,2k ; \ \frac{\partial F}{\partial \varepsilon_j} = 0, \ j = 1,2,\ldots k ;$$
$$\frac{\partial F}{\partial \lambda_j} = 0, \ j = 1,2,\ldots,2k .$$

As a start-vector, we used the polynomials $Q_{2k}(z)$ of (2.4), to give the initial estimates for the coefficients $\{a_j\}_{j=1}^{2k}$. Similarly, the negative cosines of the points of local minima of $Q_{2k}(e^{i\theta})$ were used as initial estimates for $\{\varepsilon_j\}_{j=1}^{k}$, from which the associated constants $\{A_j\}_0^{2k}$ and $\{B_j\}_0^{2k}$ were determined. Then, solving the linear system $\{\frac{\partial F}{\partial a_j} = 0\}_{j=1}^{2k}$ (in the parameters $\lambda_1,\ldots,\lambda_{2k}$) determined our initial estimates for $\{\lambda_j\}_{j=1}^{2k}$. From this point, a standard nonlinear Newton procedure was used and this converged quadratically in all cases treated, thanks, no doubt, to the good start polynomials $Q_{2k}(z)$. These calculations were carried out by Timothy S. Norfolk using Richard Brent's MP (multiple precision) package on the VAX-11/780 in the Department of Mathematical Sciences at Kent State University.

Below, we give the converged values $\{\hat{\Gamma}_{2k}\}_{k=1}^{11}$ from our numerical experiments (rounded to twelve decimals), along with the associated numbers $\hat{\gamma}_{2k} := 2k(1-\hat{\Gamma}_{2k})$. (These constants $\hat{\Gamma}_{2k}$ are surely lower bounds for $\tilde{\Gamma}_{2k}$ and Γ_{2k}, but we conjecture that $\hat{\Gamma}_{2k} = \Gamma_{2k}$ in all cases below.)

k	$\hat{\Gamma}_{2k}$	$\hat{\gamma}_{2k}$
1	0.544 331 053 952	0.911 337 892 096
2	0.778 192 979 320	0.887 228 082 721
3	0.853 294 443 051	0.880 233 341 695
4	0.890 391 158 846	0.876 870 729 236
5	0.912 511 021 366	0.874 889 786 340
6	0.927 201 419 083	0.873 582 971 010
7	0.937 667 439 454	0.872 655 847 650
8	0.945 502 263 242	0.871 963 788 123
9	0.951 587 367 216	0.871 427 390 110
10	0.956 450 029 314	0.870 999 413 712
11	0.960 425 000 586	0.870 649 987 104

Table 1

Finally, it seems reasonable to suppose that constants μ_0, $\mu_1, \ldots$ exist, independent of $2k$, such that

$$(3.7) \qquad 2k(1-\hat{\Gamma}_{2k}) = \mu_0 + \frac{\mu_1}{2k} + \frac{\mu_2}{(2k)^2} + \cdots, \quad \text{for } k \to \infty .$$

Assuming (3.7), the Richardson extrapolation method (with $x_k = \frac{1}{2k}$) was then numerically applied to the column of numbers $\{\hat{\gamma}_{2k}\}_{k=1}^{11}$ of Table 1, to accelerate the convergence of these numbers. Numerically, this Richardson extrapolation converged very rapidly, so much so that we were led to our final two conjectures. We conjecture that

$\lim_{k \to \infty} 2k(1 - \hat{\Gamma}_{2k})$ exists, i.e.,

(3.8) $$\lim_{k \to \infty} 2k(1 - \hat{\Gamma}_{2k}) = \mu_0 ,$$

and we further conjecture that

(3.9) $$\mu_0 \doteq 0.867\ 189\ 051 .$$

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