Extended Numerical Computations on the "1/9" Conjecture In Rational Approximation Theory

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Abstract. The behavior of the constants $\lambda_{n,n}(e^{-x})$, denoting the errors of best uniform approximation to e^{-x} on the interval $[0,+\infty)$ by real rational functions having numerator and denominator polynomials of degree at most n, has generated much recent interest in the approximation theory literature. Based on high-precision calculations, we present here the table of constants $\{\lambda_{n,n}(e^{-x})\}_{n=0}^{30}$, rounded to forty significant digits, and we discuss their significance to related conjectures in this area.

1. Introduction

It is well-known (cf. [15, Chapter 8]) that rational approximations to e^{-x} permeate the numerical analysis and approximation theory literature, in that these approximations arise quite naturally in the numerical solution of heat-conduction problems and in the study of numerical methods for ordinary differential equations. Our interest here centers on the following specific problem. If $\pi_{m,n}$ denotes the class of rational functions $p_m(x)/q_n(x)$, where $p_m(x)$ is a real polynomial of degree at most m and $q_n(x)$ is a real polynomial of degree at most n, then what can be said about the quantities

$$\lambda_{m,n}(e^{-x}) := \min\left\{ \| e^{-x} - r_{m,n} \|_{L_\infty[0,+\infty)} : r_{m,n} \in \pi_{m,n} \right\},\ m \leq n \quad (1.1)$$

and

$$\Lambda_1 := \varliminf_{n\to\infty} \lambda_{n,n}^{1/n}(e^{-x})\ ;\ \Lambda_2 := \varlimsup_{n\to\infty} \lambda_{n,n}^{1/n}(e^{-x})\ \ ? \quad (1.2)$$

The history of that question has its beginnings in a 1969 paper of Cody, Meinardus, and Varga [4], where they showed that

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$$\lim_{n\to\infty} \left\{ \| \, e^{-x} - \frac{1}{s_n(x)} \, \|_{L_\infty[0,+\infty)} \right\}^{1/n} = \frac{1}{2} < 1 \,, \tag{1.3}$$

where $s_n(x) := \sum_{k=0}^{n} x^k/k!$ is the n-th partial sum of e^x. Thus (cf. (1.1)),

$$\overline{\lim_{n\to\infty}} \, \lambda_{0,n}^{1/n}(e^{-x}) \leq \frac{1}{2} \,. \tag{1.4}$$

In fact, a slightly better result than (1.3) was established in [4], namely

$$\overline{\lim_{n\to\infty}} \, \lambda_{0,n}^{1/n}(e^{-x}) \leq K \doteq \frac{1}{2.298} \,. \tag{1.5}$$

Since it is obvious from (1.1) that

$$\lambda_{0,n}(e^{-x}) \geq \lambda_{1,n}(e^{-x}) \geq \cdots \geq \lambda_{n,n}(e^{-x}) \,, \; n = 0, 1, \cdots \,, \tag{1.6}$$

it is evident that (1.5) implies, with (1.2) and (1.6), that

$$0 \leq \Lambda_1 \leq \Lambda_2 \leq \frac{1}{2.298} \,, \tag{1.7}$$

so that best rational approximation to e^{-x} on $[0,+\infty)$ exhibits *geometric convergence* to zero. It is this geometric convergence which has fascinated so many researchers.

In [4], a tabulation of the computed values $\{ \lambda_{0,n}(e^{-x}) \}_{n=0}^{9}$ and $\{ \lambda_{n,n}(e^{-x}) \}_{n=0}^{14}$ was also given. The numbers in these tabulations exhibited a striking regularity and added to the interest in the problem, despite the relatively low accuracy (about 4 significant digits) of these numbers computed. Two subsequent papers by Schönhage [12] in 1973 and Newman [7] in 1974 added significantly to this interest. First, Schönhage [12] obtained the very precise estimates

$$\frac{1}{6\sqrt{(4n+4)\log 3 + 2\log 2}} \leq 3^n \lambda_{0,n}(e^{-x}) \leq \sqrt{2} \,, \; n = 0, 1, \cdots \,, \tag{1.8}$$

so that in fact

$$\lim_{n\to\infty} \lambda_{0,n}^{1/n} = \frac{1}{3} \,, \tag{1.9}$$

whence (cf. (1.2) and (1.6))

$$\Lambda_2 \leq \frac{1}{3} \,. \tag{1.10}$$

Then, Newman [7] showed that the convergence of $\lambda_{n,n}(e^{-x})$ to zero was *at most* geometric, i.e., (cf. (1.2))

$$0 < \frac{1}{1280} \leq \Lambda_1 \ . \tag{1.11}$$

On the other hand, as the determination of $\lambda_{n,n}(e^{-x})$ depends on asymptotically twice as many coefficients (in $r_{n,n}(x)$ of (1.1)) as does the determination of $\lambda_{0,n}(e^{-x})$, one could wildly guess from (1.9) that

$$\lambda_{n,n}^{1/n} \sim \left(\frac{1}{3}\right)^2 = 1/9 \ . \tag{1.12}$$

But, as the computed values $\{\lambda_{n,n}(e^{-x})\}_{n=0}^{14}$ of [4] indeed seemed to *roughly* agree with (1.12), the following conjecture was born:

Conjecture 1 (cf.[11]). $\lim_{n\to\infty} \lambda_{n,n}^{1/n}(e^{-x}) = 1/9$, i.e., $\Lambda_1 = \Lambda_2 = 1/9$.

The race was then on to improve upon the bounds for Λ_1 of (1.11) and Λ_2 of (1.10). We list, in chronological order, the successive refinements for Λ_1 and Λ_2 :

1969	Cody/Meinardus/Varga [4]		$\Lambda_2 \leq \frac{1}{2.298}$
1973	Schönhage [12]		$\Lambda_2 \leq \frac{1}{3}$
1974	Newman [7]	$\frac{1}{1280} \leq \Lambda_1$	
1978	Rahman/Schmeisser [9]	$\frac{1}{380} \leq \Lambda_1$	
1978	Rahman/Schmeisser [10]		$\Lambda_2 < \frac{1}{4.091}$
1980	Blatt/Braess [1]	$\frac{1}{52} < \Lambda_1$	
1981	Németh [6]		$\Lambda_2 < \frac{1}{6.475}$
1982	Schönhage [13]	$\frac{1}{13.928} < \Lambda_1$	
1984	Opitz/Scherer [8]		$\Lambda_2 < \frac{1}{9.037}$

It is clear that this last result of Opitz and Scherer [8] rigorously *disproves* Conjecture 1; the geometrical convergence rate of $\lambda_{n,n}^{1/n}(e^{-x})$ is actually *better* than 1/9. What is curious is that Schönhage [13] and Trefethen and Gutknecht [14] simultaneously observed (by examining the ratios $\lambda_{n-1,n-1}(e^{-x}) / \lambda_{n,n}(e^{-x})$ of the numbers tabulated in [4]) that

Conjecture 1 was surely ***numerically*** false, even before this was rigorously established later by Opitz and Scherer [8]. In fact, Schönhage made the following conjecture:

Conjecture 2 ([13]): $\lim_{n \to \infty} \lambda_{n,n}^{1/n}(e^{-x}) = \frac{3}{2}\left(2-\sqrt{3}\right)^2 \doteq 1/9.285469$,

while Trefethen and Gutknecht [14] instead conjectured

Conjecture 3 ([14]): $\lim_{n \to \infty} \lambda_{n,n}^{1/n}\left(e^{-x}\right) \doteq 1/9.28903$.

Obviously, Conjectures 2 and 3 are very close, so close in fact that the tabulation given in [4] is simply not accurate enough to settle either of these conjectures.

2. Statement of Numerical Results

The numbers $\{\lambda_{n,n}(e^{-x})\}_{n=0}^{14}$ of [4] from 1969 no doubt contributed auxiliary interest to the theoretical problem of either settling Conjecture 1 or determining improved bounds for Λ_1 and Λ_2 of (1.2). But, in the intervening fifteen years since [4] appeared, we knew of no further *direct* computation of the numbers $\lambda_{n,n}(e^{-x})$ to either *improve* the accuracies of the calculations of [4], or to extend the *length* of the table $\{\lambda_{n,n}(e^{-x})\}_{n=0}^{14}$ of [4]. Thus, it was felt that a numerical *up-date* for such calculations was in order! (We note, however, that upper estimates of $\{\lambda_{n,n}(e^{-x})\}_{n=0}^{18}$, based on the Carathéodory-Fejér method, appear in [14].)

Based on calculations done on a VAX-11/780 in the Department of Mathematical Sciences at Kent State University, using Richard Brent's MP (multiple precision) package [2] with 230 decimal digits, the numbers $\{\lambda_{n,n}(e^{-x})\}_{n=0}^{30}$ were determined to an accuracy of about 200 decimal digits. These numbers are given below, rounded to forty significant digits.

n	$\lambda_{n,n}\ (e^{-x})$
0	5.0000000000000000000000000000000000000 (-01)
1	6.6831042161850463470611623827115147261 45 (-02)
2	7.3586701695805292800125541630806037567 45 (-03)
3	7.9938063633568782880811900971119616897 66 (-04)
4	8.6522406952888523482243458254146735250 07 (-05)
5	9.3457131530266464767536568207923979896 09 (-06)
6	1.0084543748996707079345287764100020604 07 (-06)
7	1.0874974913752479608665313072729334784 85 (-07)
8	1.1722652116334907177954323039388804735 11 (-08)
9	1.2632924833223141460949321009097283343 34 (-09)
10	1.3611205233454477498707881615368423764 73 (-10)
11	1.4663111949374871406681261995577526903 48 (-11)
12	1.5794568370512387714867567328183815746 85 (-12)
13	1.7011870763403529664164865499450815333 37 (-13)
14	1.8321743782540412751555017565131565305 59 (-14)
15	1.9731389966128034286256658020822992417 70 (-15)
16	2.1248537104952237487996344364187178090 45 (-16)
17	2.2881485632478919604052208612692419494 72 (-17)
18	2.4639157377651692748310829623232282977 74 (-18)
19	2.6531146580633127669264550346953305434 63 (-19)
20	2.8567773835490937066908938449300680288 30 (-20)
21	3.0760143495057905069144218639753086839 48 (-21)
22	3.3120205005513186907513737108226141460 29 (-22)
23	3.5660818606364245847698227997651372597 24 (-23)
24	3.8395825821681321269364868473011895629 43 (-24)
25	4.1340125172853630062707580554526301970 56 (-25)
26	4.4509753557304246897932636072797330395 12 (-26)
27	4.7921973758889041899314199978855209710 52 (-27)
28	5.1595368582571326546650112912554530364 11 (-28)
29	5.5549942137516226746420079038791349910 28 (-29)
30	5.9807228828496954372714270071247982846 42 (-30)

<u>Table 1:</u> $\{\ \lambda_{n,n}\ (e^{-x})\ \}_{n=0}^{30}$

A short description of how the actual computations for $\lambda_{n,n}\ (e^{-x})$ were performed, will be given in §3.

As the ratios $\lambda_{n-1,n-1}\ /\ \lambda_{n,n}\ (e^{-x})$ figured into the formulation of Conjectures 2 and 3 of §1, we next tabulate these ratios below, rounded to twenty digits.

n	ρ_n
1	7.4815532397221509829 (+00)
2	9.0819455991000169709 (+00)
3	9.2054646248528427538 (+00)
4	9.2390013695637342229 (+00)
5	9.2579780201008948071 (+00)
6	9.2673633886078728002 (+00)
7	9.2731650684028757880 (+00)
8	9.2768895688704833336 (+00)
9	9.2794442071765347804 (+00)
10	9.2812683495309755120 (+00)
11	9.2826170054814049413 (+00)
12	9.2836420758101343366 (+00)
13	9.2844394306651793616 (+00)
14	9.2850718606898552365 (+00)
15	9.2855819149043751996 (+00)
16	9.2859992519340952302 (+00)
17	9.2863450591648612312 (+00)
18	9.2866347991400778935 (+00)
19	9.2868799705918004397 (+00)
20	9.2870892682832631480 (+00)
21	9.2872693653333554026 (+00)
22	9.2874254522088778824 (+00)
23	9.2875616152014957848 (+00)
24	9.2876811067903443961 (+00)
25	9.2877865418013514399 (+00)
26	9.2878800417598657237 (+00)
27	9.2879633425048853224 (+00)
28	9.2880378753756707951 (+00)
29	9.2881048291364217039 (+00)
30	9.2881651976905816378 (+00)

Table 2: $\{ \rho_n := \lambda_{n-1,n-1}(e^{-x}) / \lambda_{n,n}(e^{-x}) \}_{n=1}^{30}$

Now, as the ratios $\rho_n := \lambda_{n-1,n-1}(e^{-x}) / \lambda_{n,n}(e^{-x})$ of Table 2, appear to be converging linearly in $1/n^2$ (an observation already used in [14]), i.e.,

$$\rho_n \doteq \rho + \frac{K_1}{n^2} + \cdots, \text{ as } n \to \infty, \tag{2.1}$$

where K_1 is independent of n, we have applied Richardson's extrapolation (cf. Brezinski [3, p.6]), with $x_n = 1/n^2$, to the last eleven entries in Table 2, to accelerate the convergence of the entries of Table 2. These are given in Table 3 - 10 below.

9.2890264097244993503 (+00)
9.2890262501648177328 (+00)
9.2890261238332083354 (+00)
9.2890260227584489782 (+00)
9.2890259411144138332 (+00)
9.2890258745847956720 (+00)
9.2890258199319277514 (+00)
9.2890257746993546055 (+00)
9.2890257370035920984 (+00)
9.2890257053863190015 (+00)

Table 3: 1st Richardson's extrapolation.

9.2890254903568100301 (+00)
9.2890254907395748783 (+00)
9.2890254910173236639 (+00)
9.2890254912217618362 (+00)
9.2890254913741950640 (+00)
9.2890254914892118820 (+00)
9.2890254915769523220 (+00)
9.2890254916445664944 (+00)
9.2890254916971628979 (+00)

Table 4: 2nd Richardson's extrapolation.

9.2890254919264426247 (+00)
9.2890254919246363634 (+00)
9.2890254919235212362 (+00)
9.2890254919227472919 (+00)
9.2890254919222163736 (+00)
9.2890254919218439885 (+00)
9.2890254919215797099 (+00)
9.2890254919213896706 (+00)

Table 5: 3rd Richardson's extrapolation.

9.2890254919205312241 (+00)
9.2890254919208485672 (+00)
9.2890254919207963074 (+00)
9.2890254919208120946 (+00)
9.2890254919208127682 (+00)
9.2890254919208150150 (+00)
9.2890254919208161591 (+00)

Table 6: 4th Richardson's extrapolation.

9.2890254919214127327 (+00)
9.2890254919206982369 (+00)
9.2890254919208432825 (+00)
9.2890254919208141655 (+00)
9.2890254919208198985 (+00)
9.2890254919208187594 (+00)

Table 7: 5th Richardson's extrapolation.

9.2890254919196627357 (+00)
9.2890254919210653837 (+00)
9.2890254919207671899 (+00)
9.2890254919208296190 (+00)
9.2890254919208167344 (+00)

Table 8: 6th Richardson's extrapolation.

9.2890254919227707308 (+00)
9.2890254919203837979 (+00)
9.2890254919209142567 (+00)
9.2890254919207983626 (+00)

Table 9: 7th Richardson's extrapolation.

9.2890254919178974094 (+00)
9.2890254919214990876 (+00)
9.2890254919206635242 (+00)

Table 10: 8th Richardson's extrapolation.

It is interesting to see from these Richardson extrapolations that the last 32 entries from Tables 5 - 10 *all* agree, when rounded to twelve digits, to the number $\rho = 9.28902549192$. It appears that the best extrapolated value of ρ of (2.1) comes from Table 6, which thus yields numerically, to fifteen decimals, that

$$1/\rho = \lim_{n\to\infty} \lambda_{n,n}^{1/n}(e^{-x}) \doteq \frac{1}{9.289\,025\,491\,920\,81} . \tag{2.2}$$

This would appear, numerically, to *refute* Conjecture 2 of §1. In addition, as the constant in (2.2) is distinctly different from $1/9.037$, it also appears that the claim of Opitz and Scherer [8], that their method might produce "optimal" results, is numerically surely false! Despite the general numerical agreement of the extrapolations of Tables 3 - 10 in estimating a common value (2.2) for Λ_1 and Λ_2, it must be emphasized, however, that we have presented here only numerical results.

As a second method for estimating the quantity ρ of (2.2), assume that, as $n \to +\infty$,

$$\lambda_{n,n}(e^{-x}) = \frac{1}{\rho^n}\left\{ \gamma_0 + \frac{\gamma_1}{n} + \frac{\gamma_2}{n^2} + \cdots \right\} + \text{lower order terms} , \tag{2.3}$$

so that

$$\left(\lambda_{n,n}(e^{-x}) \right)^{1/n} = \frac{1}{\rho} + \frac{c_1}{n} + \frac{c_2}{n^2} + \cdots + \text{lower order terms} , \tag{2.4}$$

where the constants c_j of (2.4) depend on ρ and the γ_j's. The form of (2.4) suggests that the convergence of $\left(\lambda_{n,n}\left(e^{-x}\right) \right)^{1/n}$ to $1/\rho$ can be similarly accelerated by Richardson's extrapolation, now with $x_n = 1/n$. The numerical results of this acceleration, applied to $\left\{ \left(\lambda_{n,n}\left(e^{-x}\right) \right)^{1/n} \right\}_{n=1}^{30}$, produced essentially the *same* value for ρ as that given in (2.2). (For brevity, we have not included here the analogs of Tables 3 - 10 for these extrapolations.)

With the assumption of (2.3), it follows that

$$\rho^n \lambda_{n,n}\left(e^{-x}\right) = \gamma_0 + \frac{\gamma_1}{n} + \frac{\gamma_2}{n^2} + \cdots + \text{lower order terms} , \qquad (2.5)$$

which further suggests that the constants γ_j in (2.5) can be successively determined from the known values of $\lambda_{n,n}\left(e^{-x}\right)$, the assumed values of ρ of (2.2), and Richardson's extrapolations, again with $x_n = 1/n$. The resulting estimates for the constants γ_j are given in Table II.

Table II:
Numerical estimates for the γ_j of (2.5)

j	γ_j
0	+0.656 213 133 75
1	-0.054 684 427 8
2	+0.029 620 072 8
3	-0.016 012 6
4	+0.008 627 4
5	-0.005 74

The approximation

$$\tilde{\lambda}_{n,n} := \frac{1}{\rho^n} \left\{ \gamma_0 + \frac{\gamma_1}{n} + \cdots + \frac{\gamma_5}{n^5} \right\}, \qquad (2.6)$$

based on the numbers of (2.2) and Table II, give excellent approximations of $\left\{ \lambda_{n,n}\left(e^{-x}\right) \right\}_{n=1}^{30}$, except for very small values of n,

To round out our discussion of the "1/9" Conjecture, we list, as in [4], the coefficients of the extremal polynomials $p_n(x)$ and $q_n(x)$ (with $q_n(0) = 1$) for which

$$\lambda_{n,n}\left(e^{-x}\right) = \left\| e^{-x} - \frac{p_n(x)}{q_n(x)} \right\|_{L_\infty[0,+\infty)} , \qquad (2.7)$$

for $n = 1,2, \cdots, 30$, these coefficients being rounded to twenty places to conserve space. These can be found in §4.

3. Description of the Numerical Computations.

Initially, our computations were done on $[0, +\infty)$ with an essentially standard Remez algorithm (cf. Meinardus [5]) using Brent's MP package [2] to handle the high-precision computations. The values $\{ \lambda_{n,n} (e^{-x}) \}_{n=1}^{13}$ were calculated in this fashion, using 43 decimal-digit arithmetic. However, $\lambda_{n+1,n+1} (e^{-x})$ had approximately 3 digits *less* accuracy than $\lambda_{n,n} (e^{-x})$, indicating that the method used in these initial computations was *highly* ill-conditioned.

To achieve a better conditioning, our original approximation problem (1.1) was restated in the form

$$\lambda_{n,n} (e^{-x}) = \min \left\{ \| e^{-c_n (1+t)/(1-t)} - \tilde{r}_{n,n}(t) \|_{L_\infty[-1,+1)} : \tilde{r}_{n,n} \in \pi_{n,n} \right\}, \quad (3.1)$$

resulting from the change of variables

$$x = c_n \left(\frac{1+t}{1-t} \right), c_n > 0 \text{ , where } x \in [0, +\infty) \text{ , } t \in [-1,1) \text{ .} \quad (3.2)$$

Ideally, the constant c_n should be chosen so as to distribute the set of $2n + 2$ alternant points (cf. [5]), associated with the interval $[-1, +1]$ of (3.1), as uniformly as possible in $[-1, +1]$.

The reformulated problem (3.1) was solved by the following implementation of the Remez algorithm:

1) Obtain an estimate for the alternants $\{ t_j \}_{j=0}^{2n+1}$ (where $-1 = t_0 < t_1 < \cdots < t_{2n+1} = 1$), and for a value for the constant c_n of (3.1), using previous data.
2) Find real polynomials $p_n(t)$ and $q_n(t)$ (with $q_n(0) := 1$), each of degree n, and a positive constant λ which satisfy

$$e^{-c_n (1+t_k)/(1-t_k)} - \frac{p_n(t_k)}{q_n(t_k)} - (-1)^k \lambda = 0 \text{ , } \left(k = 0,1, \cdots ,2n+1 \right) \text{ ,} \quad (3.3)$$

   on the current alternants $\{ t_k \}_{k=0}^{2n+1}$ in $[-1, +1]$. A Newton's method, involving $2n + 2$ parameters consisting of the $2n + 1$ coefficients of $p_n(t)$ and $q_n(t)$ and the constant λ, was used to solve the nonlinear problem of (3.3). To add stability to these calculations, the polynomials $p_n(t)$ and $q_n(t)$ were expressed in terms of the Chebyshev polynomial basis $\{ T_k (t) \}_{k=0}^{n}$.
3) A new estimate of the alternants was then found by finding a set of local extrema, with alternating signs, of the function

$$F(t) := e^{-c_n (1+t)/(1-t)} - p_n(t)/q_n(t) \text{ , defined on } [-1, +1] \text{ .}$$

If the new alternants were sufficiently close to the old alternants, the algorithm was terminated. Otherwise, step 2) above was repeated, etc.

With a sufficiently good estimate for the constants c_n, the new algorithm was *significantly* more stable than the standard Remez algorithm applied on $[0, +\infty)$: the converged value $\lambda_{n+1,n+1}(e^{-x})$ of this new algorithm had approximately *one* digit less accuracy than the previous converged value $\lambda_{n,n}(e^{-x})$. This is about as much as can be expected since $\lambda_{n+1,n+1}(e^{-x})$ is roughly 1/9.29 times $\lambda_{n,n}(e^{-x})$ with increasing n (cf. Table 2)!

The most time-consuming computer portion of our modified algorithm occurred in step 2) above. Now, each Newton step in 2) requires solving a $(2n+2) \times (2n+2)$ matrix equation, and this is clearly compounded by the extra computer time necessary in carrying out all calculations in very high precision. On starting the above algorithm with 20 digit accuracy in the associated parameters and on using 230 digit arithmetic from Brent's MP package, this algorithm only needed at most 8 Newton updates to achieve a final 200 digit accuracy in the associated parameters. But for $n = 30$, this required, for example, 15 cpu hours on our VAX-11/780 to determine $\lambda_{30,30}(e^{-x})$!

We remark that these costly computing times occurred, despite the fact that our initial estimates used in step 1) of the algorithm above were surprisingly good. If $\{t_j\}_{j=0}^{2n+1}$ denoted the alternants in the interval $[-1, +1]$ for our problem (3.1) and if $\{x_j\}_{j=0}^{2n+1}$ were the images, under the transformation in (3.2), of the alternants in the interval $[0, +\infty)$, then on choosing $c_n \cong \sqrt{x_n \cdot x_{n+1}}$, we found that the associated alternants $\{t_j\}_{j=0}^{2n+1}$ in $[-1, +1]$ became *unexpectedly* similar to the extrema $\{\tilde{t}_j := \cos[\pi(1 - \frac{j}{2n+1})]\}_{j=0}^{2n+1}$ of the Chebyshev polynomial $T_{2n+1}(t)$ on the interval $[-1, +1]$. More precisely, the ratios of these alternants,

$$u_j := \frac{t_j}{\tilde{t}_j} \quad (j = 0, 1, \cdots, 2n+1) \ , \tag{3.4}$$

formed a nearly symmetric inverted bell-shaped curve on $[0, 2n+1]$, i.e., these ratios were nearly one for j small or j near $2n+1$, and these ratios decreased slowly to about 0.76 as j approached the center point of the interval $[0, 2n+1]$. This observation led us to the following estimate

$$\hat{t}_j := \cos\left[\pi\left(-\frac{j}{2n+1}\right)\right]\left\{1 - 3.36\left(\frac{j}{2n+1}\right)^2\left(1 - \frac{j}{2n+1}\right)^2\right\}, \tag{3.5}$$
$$j = 0, 1, \cdots, 2n+1 \ ,$$

of the alternants $\{t_j\}_{j=0}^{2n+1}$ in $[-1, +1]$, which numerically achieved a relative deviation of at most 6% from the actual alternants $\{t_j\}_{j=0}^{2n+1}$, even when we used the estimates

$$\hat{c}_n := c_{n-1}^2 \ / \ c_{n-2} \tag{3.6}$$

Summarizing, using the estimates of (3.5) and (3.6), using the transformed problem of (3.1), and using the Chebyshev polynomial basis $\{T_k(x)\}_{k=0}^{n}$, resulted in a significantly better-conditioned computation for the values of $\lambda_{n,n}(e^{-x})$. (We stopped our computations with the case $n = 30$ from cpu-time considerations, rather than from accuracy considerations!)

4. Coefficients of $p_n(x)$ and $q_n(x)$.

Tables of coefficients for best approximants to e^{-x}

i	q	p
	n = 1	
0	1.0000000000000000000 (+00)	1.0668310421618504635 (+00)
1	1.7271172505820169235 (+00)	-1.1542504579210602494 (-01)
	n = 2	
0	1.0000000000000000000 (+00)	9.9264132983041947072 (-01)
1	6.6930154271087127186 (-01)	-1.8833350198927415815 (-01)
2	5.7224957904836489341 (-01)	4.2109959068982177855 (-03)
	n = 3	
0	1.0000000000000000000 (+00)	1.0007993806363356878 (+00)
1	7.9829357089752213329 (-01)	-2.2365742718351887787 (-01)
2	2.2040971161511489626 (-01)	1.2499601545398984435 (-02)
3	1.2485918642725863159 (-01)	-9.9810015898578281854 (-05)
	n = 4	
0	1.0000000000000000000 (+00)	9.9991347759304711148 (-01)
1	7.5668306888329708214 (-01)	-2.4025402432545953884 (-01)
2	2.9175397633746512345 (-01)	1.8400562307678039215 (-02)
3	4.5750548404322635677 (-02)	-4.4981502907081176448 (-04)
4	1.9376829538777680730 (-02)	1.6765299308108737248 (-06)
	n = 5	
0	1.0000000000000000000 (+00)	1.0000093457131530266 (+00)
1	7.5017443629508826484 (-01)	-2.5023100706418111745 (-01)
2	2.6991013134417674897 (-01)	2.2480613306965212876 (-02)
3	6.7668626041566587102 (-02)	-8.3363085734239059333 (-04)
4	6.9346135560032124409 (-03)	1.0779810679092561383 (-05)
5	2.3446790106210413736 (-03)	-2.1912697469186570498 (-08)
	n = 6	
0	1.0000000000000000000 (+00)	9.9999899154562510033 (-01)
1	7.4317310793725353126 (-01)	-2.5677508985594545088 (-01)
2	2.6898234032991615456 (-01)	2.5389670322537157467 (-02)
3	6.1593026160813895476 (-02)	-1.1769059339745022305 (-03)
4	1.1364907743793030262 (-02)	2.4820964613817624866 (-05)
5	8.2567981485296730024 (-04)	-1.9070014316258941354 (-07)
6	2.3230231175265367241 (-04)	2.3426628258627078732 (-10)

Tables of coefficients for best approximants to e^{-x}		
i	q	p
	n = 7	
0	1.0000000000000000000 (+00)	1.0000001087497491375 (+00)
1	7.3860755265403652073 (-01)	-2.6139890245157325374 (-01)
2	2.6609542167331571699 (-01)	2.7548737236512233353 (-02)
3	6.2210380540681505225 (-02)	-1.4675743675443654044 (-03)
4	1.0229633036518400366 (-02)	4.0604885270256007598 (-05)
5	1.4878817819751908909 (-03)	-5.3705891418335754085 (-07)
6	8.0883914233407339765 (-05)	2.6538677816935889717 (-09)
7	1.9484208914619273525 (-05)	-2.1189028316079702843 (-12)
	n = 8	
0	1.0000000000000000000 (+00)	9.9999998827734788367 (-01)
1	7.3516490200874711006 (-01)	-2.6483430847822992853 (-01)
2	2.6438044169339228008 (-01)	2.9207044273505478870 (-02)
3	6.1718734777325241574 (-02)	-1.7107715899069427972 (-03)
4	1.0520815163731983883 (-02)	5.6307821021754562350 (-05)
5	1.3283453064064323585 (-03)	-1.0147775758029899384 (-06)
6	1.5910254290684040602 (-04)	9.0013440975771190466 (-09)
7	6.7271456842176719503 (-06)	-3.0312488609110424332 (-11)
8	1.4167507178008938403 (-06)	1.6608075800347647028 (-14)
	n = 9	
0	1.0000000000000000000 (+00)	1.0000000012632924833 (+00)
1	7.3251419564078598353 (-01)	-2.6748589953942883104 (-01)
2	2.6303062535807977162 (-01)	3.0517580603500653821 (-02)
3	6.1530784916685721148 (-02)	-1.9147812452801917483 (-03)
4	1.0492608831391450658 (-02)	7.1103869037310644849 (-05)
5	1.3950029476562314910 (-03)	-1.5678080149581536439 (-06)
6	1.4116049483486023948 (-04)	1.9535758505225041515 (-08)
7	1.4351266620345007596 (-05)	-1.2209626369889029705 (-10)
8	4.8597568414012272418 (-07)	2.9287304416715685077 (-13)
9	9.0914553514881083438 (-08)	-1.1485167207995354803 (-16)
	n = 10	
0	1.0000000000000000000 (+00)	9.9999999986388794767 (-01)
1	7.3040628740509415862 (-01)	-2.6959370125162598429 (-01)
2	2.6198433793450755661 (-01)	3.1577898264638956869 (-02)

Tables of coefficients for best approximants to e^{-x}		
i	q	p
3	6.1359835762604515656 (-02)	-2.0872336141098597946 (-03)
4	1.0522214938304485349 (-02)	8.4694800522056656479 (-05)
5	1.4004400136031827693 (-03)	-2.1529632009223805112 (-06)
6	1.5162908445438365994 (-04)	3.3595495648065426068 (-08)
7	1.2670597578885658604 (-05)	-3.0243887922221152380 (-10)
8	1.1179771406674077683 (-06)	1.3835175675810288679 (-12)
9	3.1024563729347710543 (-08)	-2.4479596835781524301 (-15)
10	5.2207134645030587538 (-09)	7.1060202430410289851 (-19)
n = 11		
0	1.0000000000000000000 (+00)	1.0000000000146631119 (+00)
1	7.2869093514903338347 (-01)	-2.7130906619020039085 (-01)
2	2.6114358806002462135 (-01)	3.2452672671798064591 (-02)
3	6.1230547379183840216 (-02)	-2.2343529914673730280 (-03)
4	1.0537271208026105008 (-02)	9.7033279144419487693 (-05)
5	1.4150097974152344401 (-03)	-2.7417852073123794696 (-06)
6	1.5333091232704136771 (-04)	5.0236657702890298313 (-08)
7	1.3934767796316549369 (-05)	-5.7755502193955032327 (-10)
8	9.8315547981894275422 (-07)	3.8862322897518799981 (-12)
9	7.6575875007520006935 (-08)	-1.3413459199227922075 (-14)
10	1.7740700295278932878 (-09)	1.8010050564587997692 (-17)
11	2.7127219457634251348 (-10)	-3.9776945578255130913 (-21)
n = 12		
0	1.0000000000000000000 (+00)	9.9999999999842054316 (-01)
1	7.2726794442587404691 (-01)	-2.7273205541723861212 (-01)
2	2.6045423235897757189 (-01)	3.3186285409181941015 (-02)
3	6.1125881342176750711 (-02)	-2.3610296684603661423 (-03)
4	1.0551656317596859800 (-02)	1.0818210684852625702 (-04)
5	1.4248371121782907054 (-03)	-3.3170690870773473397 (-06)
6	1.5627772051583776268 (-04)	6.8564050007593925822 (-08)
7	1.4198481758017543395 (-05)	-9.4025623763914245637 (-10)
8	1.1076204551770729043 (-06)	8.2159261557706839823 (-12)
9	6.7122345531815556515 (-08)	-4.2460559325998987934 (-14)
10	4.6772292551801273223 (-09)	1.1335755380226379804 (-16)
11	9.1859907864599662598 (-11)	-1.1824226782945190429 (-19)
12	1.2870807304176959932 (-11)	2.0328884594951322377 (-23)

Tables of coefficients for best approximants to e^{-x}		
i	q	p
	n = 13	
0	1.0000000000000000000 (+00)	1.0000000000001701187 (+00)
1	7.2606856648187456846 (-01)	-2.7393143353638373887 (-01)
2	2.5987868966049413531 (-01)	3.3810123496675834027 (-02)
3	6.1040005208020200371 (-02)	-2.4710700386539540993 (-03)
4	1.0563659959887015315 (-02)	1.1824617529896343260 (-04)
5	1.4333960951724789950 (-03)	-3.8691771493804491498 (-06)
6	1.5836802154851644352 (-04)	8.7846766808947538873 (-08)
7	1.4609445035018602359 (-05)	-1.3766793766013695399 (-09)
8	1.1372692221789108095 (-06)	1.4546096489288275848 (-11)
9	7.7482794810543576337 (-08)	-9.9041130338816606894 (-14)
10	4.0886476353685167018 (-09)	4.0201482804353265611 (-16)
11	2.5766613699837803265 (-10)	-8.4753849754741511891 (-19)
12	4.3455473513268560364 (-12)	7.0050690177811160395 (-22)
13	5.6184959576874667073 (-13)	-9.5581127116884329757 (-26)
	n = 14	
0	1.0000000000000000000 (+00)	9.9999999999998167826 (-01)
1	7.2504400105794982801 (-01)	-2.7495599893993723129 (-01)
2	2.5939097352682523362 (-01)	3.4346972429252957162 (-02)
3	6.0968196803506752125 (-02)	-2.5674425707902795730 (-03)
4	1.0574051936471402715 (-02)	1.2734060684477834084 (-04)
5	1.4405532019175131023 (-03)	-4.3932761847806353276 (-06)
6	1.6019218139205361453 (-04)	1.0753187241762202746 (-07)
7	1.4908242237501135777 (-05)	-1.8710223403465319988 (-09)
8	1.1820298688854366697 (-06)	2.2849466635625509298 (-11)
9	8.0164052804759260531 (-08)	-1.9038591551025381666 (-13)
10	4.8361843519832766389 (-09)	1.0310120249635897805 (-15)
11	2.2472046467801857194 (-10)	-3.3490171642170337489 (-18)
12	1.2922822504467751142 (-11)	5.6743670690330211959 (-21)
13	1.8921825619816040160 (-13)	-3.7794532322670955969 (-24)
14	2.2710727625899900737 (-14)	4.1610013267680029528 (-28)
	n = 15	
0	1.0000000000000000000 (+00)	1.0000000000000019731 (+00)
1	7.2415866907620477911 (-01)	-2.7584133092403855215 (-01)

Tables of coefficients for best approximants to e^{-x}		
i	q	p
2	2.5897242355314705843 (-01)	3.4813754481829550920 (-02)
3	6.0907277733570110800 (-02)	-2.6524779865394277857 (-03)
4	1.0583087170044096379 (-02)	1.3557652521034370441 (-04)
5	1.4466872665940473451 (-03)	-4.8874813823443484833 (-06)
6	1.6173222871264890572 (-04)	1.2722046758585241530 (-07)
7	1.5171907890884190088 (-05)	-2.4081661405899631864 (-09)
8	1.2151064697266154881 (-06)	3.2966530346210070367 (-11)
9	8.4196487972429069905 (-08)	-3.2068470095050508776 (-13)
10	5.0409480029770692861 (-09)	2.1476966393883189321 (-15)
11	2.7233683230693980458 (-10)	-9.4109689260605595636 (-18)
12	1.1248069393139919485 (-11)	2.4856294581031545360 (-20)
13	5.9468225157795131984 (-13)	-3.4361705671901568590 (-23)
14	7.6317957265269706634 (-15)	1.8714400301425160987 (-26)
15	8.5472063683108394376 (-16)	-1.6864826197411413312 (-30)
n = 16		
0	1.0000000000000000000 (+00)	9.9999999999999978751 (-01)
1	7.2338602292194837605 (-01)	-2.7661397707802372040 (-01)
2	2.5860931332113989163 (-01)	3.5223290398593890975 (-02)
3	6.0854946127264699182 (-02)	-2.7280223945541603972 (-03)
4	1.0591015301175281646 (-02)	1.4305532540830578363 (-04)
5	1.4519917779283776247 (-03)	-5.3516653691319551221 (-06)
6	1.6306180422515751551 (-04)	1.4663511678787708208 (-07)
7	1.5396764745302603497 (-05)	-2.9748521541610146711 (-09)
8	1.2446986743617001063 (-06)	4.4667074036373546096 (-11)
9	8.7213007002183174707 (-08)	-4.9124750188063887407 (-13)
10	5.3517844906526438896 (-09)	3.8755282930914110147 (-15)
11	2.8593027098738583509 (-10)	-2.1199503237701174494 (-17)
12	1.3963572572460278486 (-11)	7.6277421016130829045 (-20)
13	5.1671893800279173586 (-13)	-1.6607979818792513091 (-22)
14	2.5276802007963606269 (-14)	1.8979708477544314608 (-25)
15	2.8667698160212141620 (-16)	-8.5606377701670908334 (-29)
16	3.0093104224231407541 (-17)	6.3943444171177598099 (-33)

Tables of coefficients for best approximants to e^{-x}		
i	q	p
	n = 17	
0	1.0000000000000000000 (+00)	1.0000000000000000229 (+00)
1	7.2270585462136953087 (-01)	-2.7729414537863365694 (-01)
2	2.5829131995630046565 (-01)	3.5585465335003458111 (-02)
3	6.0809507390072969545 (-02)	-2.7955519228564120076 (-03)
4	1.0598023489381217368 (-02)	1.4986697693060245133 (-04)
5	1.4566258090943959608 (-03)	-5.7867107664997448463 (-06)
6	1.6421910387421514909 (-04)	1.6558969231502529775 (-07)
7	1.5592669068294270482 (-05)	-3.5600331752523462572 (-09)
8	1.2701896118955897646 (-06)	5.7694610687453000688 (-11)
9	8.9952033533504590178 (-08)	-7.0111135292795387525 (-13)
10	5.5866285320057224635 (-09)	6.3018857979045448319 (-15)
11	3.0690410653109898959 (-10)	-4.0934546068922329038 (-17)
12	1.4763544085912781412 (-11)	1.8540997178336935538 (-19)
13	6.5692146982166977577 (-13)	-5.5478662451290864179 (-22)
14	2.1929711332561360156 (-14)	1.0077964568000473113 (-24)
15	9.9803140529347369232 (-16)	-9.6311299616434657123 (-28)
16	1.0076443559489818370 (-17)	3.6381008264169349244 (-31)
17	9.9533336444995010637 (-19)	-2.2774706078188437603 (-35)
	n = 18	
0	1.0000000000000000000 (+00)	9.9999999999999999754 (-01)
1	7.2210251162213760760 (-01)	-2.7789748837786202942 (-01)
2	2.5801053189666607896 (-01)	3.5908020274519729805 (-02)
3	6.0769684666293474499 (-02)	-2.8562580858878249220 (-03)
4	1.0604260828765657181 (-02)	1.5609017337127136364 (-04)
5	1.4607084457312575111 (-03)	-6.1940471597288549472 (-06)
6	1.6523598870223731736 (-04)	1.8396482187296904295 (-07)
7	1.5764526871377682191 (-05)	-4.1548085790428176917 (-09)
8	1.2926132501100778041 (-06)	7.1794537028167480115 (-11)
9	9.2334736432550722847 (-08)	-9.4773868843647724074 (-13)
10	5.8031481469953154253 (-09)	9.4758123475451790323 (-15)
11	3.2288400362818480622 (-10)	-7.0617684389507100553 (-17)
12	1.6023022340045240158 (-11)	3.8258499143063765432 (-19)
13	6.9925714572325997863 (-13)	-1.4520088830640599985 (-21)

Tables of coefficients for best approximants to e^{-x}		
i	q	p
14	2.8542945098191319824 (-14)	3.6534688613148991179 (-24)
15	8.6471483697462350423 (-16)	-5.5958102671944549554 (-27)
16	3.6787450097487522068 (-17)	4.5177277417421400742 (-30)
17	3.3278408185195843503 (-19)	-1.4434969618996796174 (-33)
18	3.1040251435758619580 (-20)	7.6480564016753553998 (-38)
	n = 19	
0	1.0000000000000000000 (+00)	1.0000000000000000003 (+00)
1	7.2156368259201663937 (-01)	-2.7843631740798340183 (-01)
2	2.5776078409791612555 (-01)	3.6197101505900533482 (-02)
3	6.0734497954567183445 (-02)	-2.9111115140177481016 (-03)
4	1.0609846263465004573 (-02)	1.6179325924143951781 (-04)
5	1.4643324939311864714 (-03)	-6.5753674125082918706 (-06)
6	1.6613650526094958487 (-04)	2.0168905169819757417 (-07)
7	1.5916587025359119477 (-05)	-4.7521805561279600593 (-09)
8	1.3124457578828000080 (-06)	8.6730006775887139868 (-11)
9	9.4451373237383608747 (-08)	-1.2276354018264591744 (-12)
10	5.9933092623793340401 (-09)	1.3411822822603597433 (-14)
11	3.3784614979453334011 (-10)	-1.1186004966950956669 (-16)
12	1.6989658352357050158 (-11)	6.9970429220569463847 (-19)
13	7.6742363437140323820 (-13)	-3.1972725111092105805 (-21)
14	3.0580043687699495419 (-14)	1.0273733186333559896 (-23)
15	1.1518577416976816152 (-15)	-2.1951396392431808895 (-26)
16	3.1835397068991986730 (-17)	2.8615408787772833807 (-29)
17	1.2713653178005351833 (-18)	-1.9694708209381382654 (-32)
18	1.0364344279541674044 (-20)	5.3703307695262059488 (-36)
19	9.1570582958910789228 (-22)	-2.4294725589568881359 (-40)
	n = 20	
0	1.0000000000000000000 (+00)	9.9999999999999999997 (-01)
1	7.2107955165146436895 (-01)	-2.7892044834853562639 (-01)
2	2.5753720158407837071 (-01)	3.6457649932613876971 (-02)
3	6.0703182600751914030 (-02)	-2.9609098242596261130 (-03)
4	1.0614875834295449959 (-02)	1.6703541699804974713 (-04)
5	1.4675710228275782849 (-03)	-6.9324562692610725077 (-06)
6	1.6693957290448895787 (-04)	2.1872483964129629603 (-07)
7	1.6052079646727331527 (-05)	-5.3467565092977333192 (-09)
8	1.3301225337360296890 (-06)	1.0228954715465658016 (-10)

Tables of coefficients for best approximants to e^{-x}

i	q	p
9	9.6338964603806016053 (-08)	-1.5368313511438283042 (-12)
10	6.1639056613443707964 (-09)	1.8096461988824864400 (-14)
11	3.5110926854287101220 (-10)	-1.6580745736734816354 (-16)
12	1.7908832457629530219 (-11)	1.1672198605029734819 (-18)
13	8.2008117841834923615 (-13)	-6.1936889537571481080 (-21)
14	3.3938864156618761186 (-14)	2.4105681922839642535 (-23)
15	1.2417688637946327705 (-15)	-6.6185196281694492006 (-26)
16	4.3385159620628243371 (-17)	1.2113724541256915285 (-28)
17	1.0990484330810003653 (-18)	-1.3552946153009542146 (-31)
18	4.1354019630164880823 (-20)	8.0169605314221382434 (-35)
19	3.0538918819268542585 (-22)	-1.8805470199361703461 (-38)
20	2.5628928552136086178 (-23)	7.3216143452337990707 (-43)
	n = 21	
0	1.0000000000000000000 (+00)	1.0000000000000000000 (+00)
1	7.2064219697569183005 (-01)	-2.7935780302430817048 (-01)
2	2.5733587895330020783 (-01)	3.6693681977608392584 (-02)
3	6.0675133266582940643 (-02)	-3.0063138655381823795 (-03)
4	1.0619427824397480596 (-02)	1.7186787185029200536 (-04)
5	1.4704823869739844457 (-03)	-7.2670890316381390456 (-06)
6	1.6766020704056002618 (-04)	2.3505832195838590627 (-07)
7	1.6173579847685692566 (-05)	-5.9344580351910059567 (-09)
8	1.3459762326494999013 (-06)	1.1828954830323715653 (-10)
9	9.8033984897602339980 (-08)	-1.8712232943943080645 (-12)
10	6.3173038544853450718 (-09)	2.3495449807922150045 (-14)
11	3.6312508159412913565 (-10)	-2.3313838537000271146 (-16)
12	1.8730981404230082919 (-11)	1.8118842180441427473 (-18)
13	8.7093081612820652481 (-13)	-1.0875663018513674485 (-20)
14	3.6548676757025251170 (-14)	4.9414360113574654171 (-23)
15	1.3936848498319246143 (-15)	-1.6523102799180557777 (-25)
16	4.7051616127896484209 (-17)	3.9082049594935053126 (-28)
17	1.5317669769113303944 (-18)	-6.1754498246162749622 (-31)
18	3.5714624755160836554 (-20)	5.9746919108510846444 (-34)
19	1.2703379067912114642 (-21)	-3.0598920596277695902 (-37)
20	8.5380603270749883481 (-24)	6.2192394025759794712 (-41)
21	6.8232509117901788034 (-25)	-2.0988417714945058814 (-45)

Tables of coefficients for best approximants to e^{-x}		
i	q	p
	n = 22	
0	1.0000000000000000000 (+00)	1.0000000000000000000 (+00)
1	7.2024515522426885778 (-01)	-2.7975484477573114216 (-01)
2	2.5715365085971926369 (-01)	3.6908495635450404166 (-02)
3	6.0649864512193881866 (-02)	-3.0478754020575993647 (-03)
4	1.0623566628151799912 (-02)	1.7633500843928321830 (-04)
5	1.4731137404802299197 (-03)	-7.5809738495069090674 (-06)
6	1.6831048917843324445 (-04)	2.5069191557238854811 (-07)
7	1.6283152234777024234 (-05)	-6.5122619757180812495 (-09)
8	1.3602761130470247007 (-06)	1.3457379364976452568 (-10)
9	9.9564343590539204149 (-08)	-2.2268026983155948583 (-12)
10	6.4560913902503564972 (-09)	2.9560245331729836665 (-14)
11	3.7401859742633337134 (-10)	-3.1410724352013750719 (-16)
12	1.9483106529080606242 (-11)	2.6553887718291344662 (-18)
13	9.1680999305772174791 (-13)	-1.7676644378244532847 (-20)
14	3.9107766743159333128 (-14)	9.1278680283845964971 (-23)
15	1.5123507567108144027 (-15)	-3.5805166566755439172 (-25)
16	5.3401899140075195528 (-17)	1.0365955917689346517 (-27)
17	1.6707425727208528640 (-18)	-2.1277784397335996581 (-30)
18	5.0886706159921823345 (-20)	2.9231399336791845047 (-33)
19	1.0961475446293379047 (-21)	-2.4623396257008074572 (-36)
20	3.6965545855001175030 (-23)	1.0991176527978871486 (-39)
21	2.2708806892898350258 (-25)	-1.9484008877459215375 (-43)
22	1.7320983079796653153 (-26)	5.7367451049989032790 (-48)
	n = 23	
0	1.0000000000000000000 (+00)	1.0000000000000000000 (+00)
1	7.1988310082459591732 (-01)	-2.8011689917540408269 (-01)
2	2.5698792471734873844 (-01)	3.7104823892752821325 (-02)
3	6.0626982524584947958 (-02)	-3.0860584471325009762 (-03)
4	1.0627345664071083966 (-02)	1.8047536072786826345 (-04)
5	1.4755035992707838480 (-03)	-7.8757210411818256606 (-06)
6	1.6890024843251790692 (-04)	2.6563900673673161074 (-07)
7	1.6382473993464912301 (-05)	-7.0779812242602691625 (-09)
8	1.3732404247572612432 (-06)	1.5101143424472028548 (-10)
9	1.0095302286038609121 (-07)	-2.5997957789837943232 (-12)

Tables of coefficients for best approximants to e^{-x}		
i	q	p
10	6.5822413182372985325 (-09)	3.6233532714776076309 (-14)
11	3.8394889629746827515 (-10)	-4.0861034248600722572 (-16)
12	2.0170542366772364610 (-11)	3.7137851529683374768 (-18)
13	9.5918906854811887872 (-13)	-2.7001985390889146513 (-20)
14	4.1436204320974462060 (-14)	1.5528939801402053295 (-22)
15	1.6304901228232053668 (-15)	-6.9530886094237998157 (-25)
16	5.8386383707137038815 (-17)	2.3720625335812422216 (-27)
17	1.9175029935153725110 (-18)	-5.9872927305549306726 (-30)
18	5.5808542257176158674 (-20)	1.0736346677944520471 (-32)
19	1.5960495536735968784 (-21)	-1.2905749729715581657 (-35)
20	3.1871496109206449820 (-23)	9.5241140961329381142 (-39)
21	1.0217421583117155147 (-24)	-3.7278671871533691988 (-42)
22	5.7595314217244562516 (-27)	5.7982136100082272330 (-46)
23	4.2015926344320482739 (-28)	-1.4983223279431735600 (-50)
n = 24		
0	1.0000000000000000000 (+00)	1.0000000000000000000 (+00)
1	7.1955160628060490012 (-01)	-2.8044839371939509988 (-01)
2	2.5683655680548874312 (-01)	3.7284950524883842970 (-02)
3	6.0606164481744409458 (-02)	-3.1212558501085499598 (-03)
4	1.0630809594431807553 (-02)	1.8432246866428426565 (-04)
5	1.4776837304194045942 (-03)	-8.1528290297762335410 (-06)
6	1.6943756353916543523 (-04)	2.7992015745235962935 (-07)
7	1.6472921423080865550 (-05)	-7.6300845409641694341 (-09)
8	1.3850483386300453833 (-06)	1.6749426980810398258 (-10)
9	1.0221886364132138740 (-07)	-2.9867405776081327288 (-12)
10	6.6974106945354655796 (-09)	4.3453522989129354348 (-14)
11	3.9303598949948647010 (-10)	-5.1625886299634749395 (-16)
12	2.0801904728762698738 (-11)	4.9974970771678653853 (-18)
13	9.9823495547390858172 (-13)	-3.9209132343147921877 (-20)
14	4.3607684640656254187 (-14)	2.4722281164775052693 (-22)
15	1.7388589608575720948 (-15)	-1.2376529579577131695 (-24)
16	6.3423901723117428381 (-17)	4.8389987731625314556 (-27)
17	2.1120571959754931088 (-18)	-1.4452601356069529738 (-29)
18	6.4766664595094796687 (-20)	3.2004921170130934113 (-32)
19	1.7596348598140616683 (-21)	-5.0438750863387113193 (-35)

i	q	p
	Tables of coefficients for best approximants to e^{-x}	
20	4.7406321676460227043 (-23)	5.3360424355118835887 (-38)
21	8.8030256384366476434 (-25)	-3.4694587438114214128 (-41)
22	2.6892325986479079123 (-26)	1.1974216241920071748 (-44)
23	1.3959597685006867882 (-28)	-1.6430816452663319539 (-48)
24	9.7582905119403701973 (-30)	3.7467762281382790571 (-53)
	n = 25	
0	1.0000000000000000000 (+00)	1.0000000000000000000 (+00)
1	7.1924696060030549152 (-01)	-2.8075303939969450848 (-01)
2	2.5669775918887669273 (-01)	3.7450798588571201206 (-02)
3	6.0587143251270532876 (-02)	-3.1538023041200807920 (-03)
4	1.0633996033260288963 (-02)	1.8790560971052078695 (-04)
5	1.4796805670782497197 (-03)	-8.4136803468272165599 (-06)
6	1.6992913597947736280 (-04)	2.9356040915848232932 (-07)
7	1.6555634671491244526 (-05)	-8.1675511125608142608 (-09)
8	1.3958482728461237761 (-06)	1.8393384178735986993 (-10)
9	1.0337750836440238611 (-07)	-3.3845209609800166958 (-12)
10	6.8029738899716716771 (-09)	5.1157161639534070020 (-14)
11	4.0138303634825116937 (-10)	-6.3644923388319778193 (-16)
12	2.1383556797380942966 (-11)	6.5117237108438679018 (-18)
13	1.0343638215272956605 (-12)	-5.4595347139888024431 (-20)
14	4.5624243518361710181 (-14)	3.7273813754702834958 (-22)
15	1.8408841519514984572 (-15)	-2.0530724455614180911 (-24)
16	6.8081267555534785131 (-17)	9.0077109606214317012 (-27)
17	2.3116269500240241604 (-18)	-3.0948366861501914604 (-29)
18	7.1858776322636732020 (-20)	8.1407882328111744921 (-32)
19	2.0647764426693640793 (-21)	-1.5906727254440811555 (-34)
20	5.2528760787303468994 (-23)	2.2153039233678690245 (-37)
21	1.3370915947538380277 (-24)	-2.0736074544841591910 (-40)
22	2.3154233915482175675 (-26)	1.1940545565030128827 (-43)
23	6.7551774268161192942 (-28)	-3.6523509679158950883 (-47)
24	3.2397074059310773266 (-30)	4.4437519314004684085 (-51)
25	2.1738952552926457167 (-31)	-8.9869101966470571763 (-56)

Tables of coefficients for best approximants to e^{-x}

i	q	p
	n = 26	
0	1.0000000000000000000 (+00)	1.0000000000000000000 (+00)
1	7.1896603004635407628 (-01)	-2.8103396995364592372 (-01)
2	2.5657002887003749352 (-01)	3.7603998823683417241 (-02)
3	6.0569695881738960595 (-02)	-3.1839846317881614457 (-03)
4	1.0636936873433653069 (-02)	1.9125041898775982050 (-04)
5	1.4815162820564590937 (-03)	-8.6595435825426075478 (-06)
6	1.7038057158576408119 (-04)	3.0658737713813768797 (-07)
7	1.6631566345893463369 (-05)	-8.6897545335970724597 (-09)
8	1.4057642432742099133 (-06)	2.0025861713950916330 (-10)
9	1.0444204490518077866 (-07)	-3.7903724436935358376 (-12)
10	6.9000857608018811274 (-09)	5.9282411666625107531 (-14)
11	4.0907662030930223269 (-10)	-7.6842585097056414965 (-16)
12	2.1921125846523355118 (-11)	8.2570666493171204585 (-18)
13	1.0678718214314360999 (-12)	-7.3392007703057431530 (-20)
14	4.7503981805777539255 (-14)	5.3712708293521833434 (-22)
15	1.9363701772525106571 (-15)	-3.2137958886540163074 (-24)
16	7.2507250514592475300 (-17)	1.5565082769644566430 (-26)
17	2.4975442158093370662 (-18)	-6.0214447480232786212 (-29)
18	7.9241460923961745271 (-20)	1.8284175028861242452 (-31)
19	2.3072848547429883203 (-21)	-4.2589594095915223042 (-34)
20	6.2318770493623148075 (-23)	7.3811490698898474272 (-37)
21	1.4887308305062634160 (-24)	-9.1298836604138566248 (-40)
22	3.5900145501058512309 (-26)	7.5983415064438853460 (-43)
23	5.8126365856240802587 (-28)	-3.8935547750075112746 (-46)
24	1.6227799580300107200 (-29)	1.0604684624677904373 (-49)
25	7.2122228957098244519 (-32)	-1.1493650186177443495 (-53)
26	4.6529920015189527423 (-33)	2.0710352729171641461 (-58)
	n = 27	
0	1.0000000000000000000 (+00)	1.0000000000000000000 (+00)
1	7.1870615013494964452 (-01)	-2.8129384986505035548 (-01)
2	2.5645209321774275677 (-01)	3.7745943082793112249 (-02)
3	6.0553634833924406861 (-02)	-3.2120499830101943170 (-03)
4	1.0639659327332069240 (-02)	1.9437941312076668187 (-04)
5	1.4832096123430519539 (-03)	-8.8915786943714812065 (-06)

Tables of coefficients for best approximants to e^{-x}		
i	q	p
6	1.7079659574127093742 (-04)	3.1902991377590222703 (-07)
7	1.6701518602605391310 (-05)	-9.1963709811680089503 (-09)
8	1.4149006878178168423 (-06)	2.1641140277115697685 (-10)
9	1.0542351021349550139 (-07)	-4.2018705007566813448 (-12)
10	6.9897228455967626327 (-09)	6.7769798099866907653 (-14)
11	4.1619044516026066335 (-10)	-9.1133406626101867879 (-16)
12	2.2419392381250479060 (-11)	1.0230254833623119193 (-17)
13	1.0990305718118410865 (-12)	-9.5764061479699012137 (-20)
14	4.9259005778171551594 (-14)	7.4512740901471078481 (-22)
15	2.0260314782637897578 (-15)	-4.7931707838444243896 (-24)
16	7.6681358771735639629 (-17)	2.5293126309095483331 (-26)
17	2.6758680762905260877 (-18)	-1.0834661026569340573 (-28)
18	8.6169847303890952011 (-20)	3.7161403305502155654 (-31)
19	2.5634235918361219624 (-21)	-1.0024877187066737147 (-33)
20	7.0127193297565789436 (-23)	2.0780918351231215531 (-36)
21	1.7855742180920933932 (-24)	-3.2097160980790598559 (-39)
22	4.0156816304849403139 (-26)	3.5424654128735486313 (-42)
23	9.1964363393424183436 (-28)	-2.6331524240973874715 (-45)
24	1.3955671627874349543 (-29)	1.2060058776911720646 (-48)
25	3.7352134986647930254 (-31)	-2.9375861296063709736 (-52)
26	1.5427084484811426672 (-33)	2.8484004094090161063 (-56)
27	9.5834724343683731142 (-35)	-4.5925891451883766222 (-61)
	n = 28	
0	1.0000000000000000000 (+00)	1.0000000000000000000 (+00)
1	7.1846504099977450858 (-01)	-2.8153495900022549142 (-01)
2	2.5634286747132473766 (-01)	3.7877826471550229077 (-02)
3	6.0538801220168057895 (-02)	-3.2382124179360921395 (-03)
4	1.0642186749988000749 (-02)	1.9731243218656025214 (-04)
5	1.4847764994109842628 (-03)	-9.1108440565059548173 (-06)
6	1.7118121976301399562 (-04)	3.3091718065250326410 (-07)
7	1.6766171748975058231 (-05)	-9.6873069289091329257 (-09)
8	1.4233461897756964635 (-06)	2.3234704839756243162 (-10)
9	1.0633127738462594514 (-07)	-4.6169087559105242168 (-12)
10	7.0727167212993401677 (-09)	7.6563385687590018588 (-14)
11	4.2278753905749988235 (-10)	-1.0642631862580357359 (-15)
12	2.2882484070110947837 (-11)	1.2424881652802987399 (-17)

Tables of coefficients for best approximants to e^{-x}

i	q	p
13	1.1280736630891817950 (-12)	-1.2181315591198636789 (-19)
14	5.0900982136752396390 (-14)	1.0008004746340252594 (-21)
15	2.1102989402066160773 (-15)	-6.8628244877211120581 (-24)
16	8.0629352903914881994 (-17)	3.9041630685993974915 (-26)
17	2.8453127951750166544 (-18)	-1.8273116597117645173 (-28)
18	9.2876348488845996777 (-20)	6.9600084535166773350 (-31)
19	2.8055162407493957644 (-21)	-2.1270994500225483862 (-33)
20	7.8493728999571373376 (-23)	5.1221768334261579751 (-36)
21	2.0231282677112118888 (-24)	-9.4924644132126223194 (-39)
22	4.8688630300634044605 (-26)	1.3124217192547875882 (-41)
23	1.0332584543810187690 (-27)	-1.2979704782618576593 (-44)
24	2.2522930190976109514 (-29)	8.6529029938315506542 (-48)
25	3.2105491540280196138 (-31)	-3.5567748438917412531 (-51)
26	8.2519544489273895218 (-33)	7.7792379820555470653 (-55)
27	3.1755253582770512366 (-35)	-6.7753365272078992201 (-59)
28	1.9020809159366749124 (-36)	9.8138565931627609198 (-64)
	n = 29	
0	1.0000000000000000000 (+00)	1.0000000000000000000 (+00)
1	7.1824074043079310993 (-01)	-2.8175925956920689007 (-01)
2	2.5624142131497740435 (-01)	3.8000680884184294416 (-02)
3	6.0525059535231118368 (-02)	-3.2626582310163976817 (-03)
4	1.0644539295547595500 (-02)	2.0006701267299431847 (-04)
5	1.4862305915618925334 (-03)	-9.3183042499982610991 (-06)
6	1.7153787091780660524 (-04)	3.4227801419691581593 (-07)
7	1.6826106556886635018 (-05)	-1.0162642462709301594 (-08)
8	1.4311763809427150035 (-06)	2.4803044888644507226 (-10)
9	1.0717335838264462378 (-07)	-5.0336720080283779322 (-12)
10	7.1497799310138340033 (-09)	8.5611335339544449737 (-14)
11	4.2892213438580353073 (-10)	-1.2262802013924015446 (-15)
12	2.3313973576581319597 (-11)	1.4832100877490882467 (-17)
13	1.1552060390438050984 (-12)	-1.5158313990467185641 (-19)
14	5.2440069076448366312 (-14)	1.3074657548508829771 (-21)
15	2.1896191080064318492 (-15)	-9.4902885995221209857 (-24)
16	8.4364220579657309921 (-17)	5.7698403519637535860 (-26)
17	3.0067321387130736625 (-18)	-2.9188924120576619426 (-28)
18	9.9296151586572481448 (-20)	1.2179775468499820477 (-30)

Tables of coefficients for best approximants to e^{-x}		
i	q	p
19	3.0419849906975770270 (-21)	-4.1449405346182376688 (-33)
20	8.6456065781235580902 (-23)	1.1339292479243459054 (-35)
21	2.2812963979843922853 (-24)	-2.4481138035452179148 (-38)
22	5.5537857813866255467 (-26)	4.0730588289709086731 (-41)
23	1.2663351963339644273 (-27)	-5.0614377161525805088 (-44)
24	2.5413421113019371806 (-29)	4.5033487174180149912 (-47)
25	5.2836220838809921666 (-31)	-2.7029531453596257962 (-50)
26	7.0894030065454535655 (-33)	1.0000296137524799591 (-53)
27	1.7525818103023344647 (-34)	-1.9731092335931916942 (-57)
28	6.2991336260625001585 (-37)	1.5493207620467243315 (-61)
29	3.6427133054769394938 (-38)	-2.0235251334280446010 (-66)
	n = 30	
0	1.0000000000000000000 (+00)	1.0000000000000000000 (+00)
1	7.1803155042772127612 (-01)	-2.8196844957227872388 (-01)
2	2.5614695234169101695 (-01)	3.8115401913969740830 (-02)
3	6.0512293508462864581 (-02)	-3.2855502860341809776 (-03)
4	1.0646734444855619090 (-02)	2.0265870261805029904 (-04)
5	1.4875836417649791494 (-03)	-9.5148379859906376412 (-06)
6	1.7186949502113754284 (-04)	3.5314050182780261684 (-07)
7	1.6881821841067465703 (-05)	-1.0622586963885646562 (-08)
8	1.4384562284711296335 (-06)	2.6343483216755972319 (-10)
9	1.0795664283018752186 (-07)	-5.4506073294691798165 (-12)
10	7.2215265693946799983 (-09)	9.4866156508756075292 (-14)
11	4.3464114384271304094 (-10)	-1.3964554205647248956 (-15)
12	2.3716969565318967420 (-11)	1.7441252036441660423 (-17)
13	1.1806074393066802850 (-12)	-1.8506688650091489877 (-19)
14	5.3885346845345913909 (-14)	1.6676820706603970154 (-21)
15	2.2643856170276607118 (-15)	-1.2737164571814720528 (-23)
16	8.7901123390735385090 (-17)	8.2159993047662503208 (-26)
17	3.1604309408297432480 (-18)	-4.4525490040349927562 (-28)
18	1.0545543403758406495 (-19)	2.0130724366066200126 (-30)
19	3.2699924326595110152 (-21)	-7.5239822024090788005 (-33)
20	9.4303280463436687932 (-23)	2.2978810217829931996 (-35)
21	2.5286339977569998853 (-24)	-5.6508242227847947169 (-38)
22	6.3086412701711342899 (-26)	1.0982155012464389275 (-40)

Tables of coefficients for best approximants to e^{-x}		
i	q	p
23	1.4540002113970188434 (-27)	-1.6467595005222406689 (-43)
24	3.1480227761306722217 (-29)	1.8462045089123337587 (-46)
25	5.9861194489977205340 (-31)	-1.4832272593982169255 (-49)
26	1.1893082101510933116 (-32)	8.0441303745414552340 (-53)
27	1.5049933964043840759 (-34)	-2.6930919888571651980 (-56)
28	3.5836255739795333870 (-36)	4.8015689571472672268 (-60)
29	1.2057358269368541653 (-38)	-3.4109451463472630706 (-64)
30	6.7397798848782102402 (-40)	4.0308755782861197983 (-69)

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