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A THEOREM OF J. L. WALSH, REVISITED

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Dedicated to the memory of Ernst G. Straus

The well-known and beautiful result of J. L. Walsh, on the overconvergence of sequences of differences of polynomials interpolating a function $f(z)$ analytic in $|z| < \rho$ (but having a singularity on $|z| = \rho$), where $1 < \rho < \infty$, has been recently extended in a new direction by T. J. Rivlin. We give here three new extensions of Rivlin's result, which include Hermite and Birkhoff interpolation.

1. Introduction. Let A_ρ denote the collection of functions analytic in $|z| < \rho$ and having a singularity on the circle $|z| = \rho$ (where we assume throughout that $1 < \rho < \infty$). For each $f(z) = \sum_{k=0}^{\infty} a_k z^k$ in A_ρ and for each positive integer n, let

$$(1.1)\qquad s_n(z; f) := \sum_{k=0}^{n} a_k z^k$$

be the nth partial sum of $f(z)$, and let $L_n(z; f)$ similarly denote the unique Lagrange interpolation polynomial (of degree at most n) which interpolates $f(z)$ in the $(n + 1)$-st roots of unity, i.e., if ω is a primitive root of $\omega^{n+1} = 1$,

$$(1.2)\qquad L_n(\omega^k; f) = f(\omega^k), \quad \text{for all } k = 0, 1, 2, \ldots, n.$$

Then, a well-known and beautiful result of J. L. Walsh [**8**, p. 153] can be stated as

THEOREM A. ([**8**]). *For each $f \in A_\rho$, there holds*

$$(1.3)\qquad \lim_{n\to\infty} \{L_n(z; f) - s_n(z; f)\} = 0, \quad \textit{for all } |z| < \rho^2,$$

the convergence being uniform and geometric on any closed subset of $|z| < \rho^2$. More precisely, for any τ with $\rho \le \tau < \infty$, there holds

$$(1.4)\qquad \overline{\lim_{n\to\infty}} \left\{ \max_{|z|\le\tau} |L_n(z; f) - s_n(z; f)| \right\}^{1/n} \le \frac{\tau}{\rho^2}.$$

Further, the result of (1.3) *is best possible in the sense that there is some $\hat{f} \in A_\rho$ and some $\hat{z}$ with $|\hat{z}| = \rho^2$ for which the sequence*

$$\{L_n(\hat{z}; \hat{f}) - s_n(\hat{z}; \hat{f})\}_{n=1}^{\infty}$$

does not tend to zero as $n \to \infty$.

For general discussions of various extensions of Walsh's Theorem A, see for example [**2**] and [**7**]. Recently, Rivlin [**4**] has obtained some interesting new analogues of Walsh's Theorem. Here, we shall show that one of Rivlin's results [**4**, Theorem 1] can be further generalized. In order to describe these extensions, we introduce some needed notation.

First, let π_k as usual denote the collection of all complex polynomials of degree at most k. Next, consider all positive integers m of the form $m = qn + c$ where q and c are fixed positive integers, so that $m \geq n + 1$. With ω a primitive mth root of unity, and with r a fixed nonnegative integer, we propose to find, for each $f \in A_\rho$, the polynomial $P_{rm+n}(z; f)$ in π_{rm+n} which satisfies the Hermite interpolation conditions

$$(1.5)\quad P^{(\nu)}_{rm+n}(\omega^k; f) = f^{(\nu)}(\omega^k),\qquad \text{for all } k = 0,1,\ldots,m-1;\ \nu = 0,1,\ldots,r-1,\quad \text{if } r \geq 1,$$

and which also minimizes

$$(1.6)\qquad \sum_{k=0}^{m-1} \left|P^{(r)}_{rm+n}(\omega^k; f) - f^{(r)}(\omega^k)\right|^2,$$

over all polynomials in π_{rm+n} which satisfy the interpolation conditions of (1.5). (The existence and uniqueness of this polynomial $P_{rm+n}(z; f)$, while a basic consequence of approximation theory, will follow from the explicit representations of (2.4) and (2.8) in §2.)

In §2, we study the difference

$$P_{rm+n}(z; f) - s_{rm+n}(z; f)$$

in Theorem 1, and show that it tends to zero, as $n \to \infty$ in

$$|z| < \rho^{1+q/(1+rq)},$$

thereby extending Rivlin's result [**4**, Theorem 1]. In §3, we state extensions of Theorem 1 to Birkhoff interpolation, in which the Hermite interpolation condition of (1.5) is replaced by more general Birkhoff interpolation conditions (cf. (3.2)).

2. An Extension of Rivlin's Result.

We first establish

THEOREM 1. *For each $f \in A_\rho$ and for each nonnegative integer r, let the polynomials $P_{rm+n}(z; f)$ and $s_{rm+n}(z; f)$ be defined as in* (1.5)–(1.6) *and* (1.1). *With $m = nq + c$, where q and c are any fixed positive integers, there holds*

$$(2.1)\quad \lim_{n\to\infty} \{P_{rm+n}(z; f) - s_{rm+n}(z; f)\} = 0, \quad \textit{for all } |z| < \rho^{1+q/(1+rq)}$$

the convergence being uniform and geometric on any closed subset of $|z| < \rho^{1+q/(1+rq)}$. *More precisely, for any* τ *with* $\rho < \tau < \infty$, *there holds*

$$(2.2) \qquad \overline{\lim_{n\to\infty}} \left\{ \max_{|z|\le\tau} |P_{rm+n}(z;f) - s_{rm+n}(z;f)| \right\}^{1/n} \le \frac{\tau^{rq+1}}{\rho^{(r+1)q+1}}.$$

Further, the result of (2.1) *is best possible in the sense that there is some* $\hat{f} \in A_\rho$ *and some* $\hat{z}$ *with* $|\hat{z}| = \rho^{1+q/(1+rq)}$ *for which the sequence* $\{P_{rm+n}(\hat{z};\hat{f}) - s_{rm+n}(\hat{z};\hat{f})\}_{n=1}^{\infty}$ *does not tend to zero as* $n \to \infty$.

We remark that as the special case $r = 0$ of Theorem 1 reduces to Rivlin's result [**4**, Theorem 1], then the above result generalizes Rivlin's result.

To begin, for each $f \in A_\rho$, let $h_{rm-1}(z;f)$ be the unique Hermite interpolation polynomial of $f(z)$ in π_{rm-1} which satisfies (1.5), i.e.,

$$(2.3) \qquad h_{rm-1}^{(\nu)}(\omega^k;f) = f^{(\nu)}(\omega^k),$$

$$\text{for all } k = 0,1,\ldots,m-1,\ \nu = 0,1,\ldots,r-1,$$

if $r \ge 1$; otherwise, $h_{rm-1}(z;f) \equiv 0$ if $r = 0$. Then, any $P_{rm+n}(z;f)$ satisfying (1.5) necessarily has the form

$$(2.4) \qquad P_{rm+n}(z;f) = h_{rm-1}(z;f) - (z^m - 1)^r Q_n(z),$$

where $Q_n \in \pi_n$. Since

$$(2.5) \qquad \left.\frac{d^r}{dz^r}(z^m-1)^r\right|_{z=\omega^k} = \omega^{-kr}\sum_{\nu=0}^{r}(-1)^{r-\nu}\binom{r}{\nu}(m\nu)_r = \omega^{-kr}m^r r!,$$

(where $(x)_0 := 1$ and where $(x)_k := x(x-1)\cdots(x-k+1)$ when k is a positive integer), it easily follows from (2.4) that the problem of minimizing (1.6) is equivalent to finding the polynomial $Q_n(z)$ in π_n which solves

$$(2.6) \qquad \sum_{k=0}^{m-1}|g(\omega^k) - Q_n(\omega^k)|^2 = \min_{p_n\in\pi_n}\sum_{k=0}^{m-1}|g(\omega^k) - p_n(\omega^k)|^2,$$

where

$$(2.7) \qquad g(z) := z^r\{h_{rm-1}^{(r)}(z;f) - f^{(r)}(z)\} \,/\, [m^r \cdot r!].$$

We next establish

LEMMA 1. *The polynomial* $Q_n(z)$ *in* π_n *which solves* (2.6) *is explicitly given by*

$$(2.8) \qquad Q_n(z) = -\frac{1}{2\pi i}\int_\Gamma \frac{f(t)t^{m-n-1}(t^{n+1} - z^{n+1})\,dt}{(t-z)(t^m-1)^{r+1}},$$

where $\Gamma := \{z: |z| = R\}$ *and where* R *is any number satisfying* $1 < R < \rho$.

Proof. By Hermite's interpolation formula (cf. [2, p. 164]), we know that the polynomials $h_{rm-1}(z; f)$ of (2.3) and $s_{rm+n}(z; f)$ of (1.1) can be expressed as

$$(2.9)\quad \begin{cases} h_{rm-1}(z; f) = \dfrac{1}{2\pi i}\displaystyle\int_\Gamma \dfrac{f(t)\left[(t^m-1)^r-(z^m-1)^r\right]dt}{(t-z)(t^m-1)^r}, & \text{and} \\[2ex] s_{rm+n}(z; f) = \dfrac{1}{2\pi i}\displaystyle\int_\Gamma \dfrac{f(t)\left[t^{rm+n+1}-z^{rm+n+1}\right]dt}{(t-z)t^{rm+n+1}}. \end{cases}$$

Thus, from Cauchy's integral formula, we can write

$$(2.10)\qquad h_{rm-1}(z; f) - f(z) = -\frac{1}{2\pi i}\int_\Gamma \frac{f(t)K(t, z)\,dt}{(t^m-1)^r},$$

where

$$(2.11)\qquad K(t, z) := = \frac{(z^m-1)^r}{t-z}.$$

From (2.5), we see that

$$(2.12)\qquad z^r\frac{\partial^r}{\partial z^r}K(t, z)\bigg|_{z=\omega^k} = \frac{m^r \cdot r!}{t-\omega^k}, \qquad k = 0, 1, \ldots, m-1.$$

Thus, on differentiating r times with respect to z in (2.10) and using (2.12), it follows that the Lagrange polynomial interpolant $L_{m-1}(z; g)$ of (1.2) of $g(z)$ (defined in (2.7)) in the points ω^k, $k = 0, 1, \ldots, m-1$, is just

$$L_{m-1}(z; g) = -\frac{1}{2\pi i}\int_\Gamma \frac{f(t)(t^m-z^m)\,dt}{(t-z)(t^m-1)^{r+1}},$$

from which it follows (cf. (1.1)) that

$$(2.13)\quad s_n(z; L_{m-1}(z; g)) = -\frac{1}{2\pi i}\int_\Gamma \frac{f(t)t^{m-n-1}(t^{n+1}-z^{n+1})\,dt}{(t-z)(t^m-1)^{r+1}}.$$

But, Rivlin [4] has shown that the solution $Q_n(z)$ of (2.6) satisfies $Q_n(z) = s_n(z; L_{m-1}(z; g))$, so that (2.13) gives the desired integral representation for $Q_n(z)$ in (2.8). □

This brings us to the

Proof of Theorem 1. From (2.4), (2.8), and (2.9), we can write

$$(2.14)\quad P_{rm+n}(z; f) - s_{rm+n}(z; f) = \frac{1}{2\pi i}\int_\Gamma \frac{f(t)K_1(t, z)\,dt}{(t-z)},$$

where

$$(2.15)\qquad K_1(t,z) := \frac{z^{rm+n+1}}{t^{rm+n+1}} + \left(\frac{z^m-1}{t^m-1}\right)^r \left\{\frac{1-t^{m-n-1}z^{n+1}}{t^m-1}\right\}.$$

Next, set (cf. [**2**, p. 163])

$$(2.16)\qquad \beta_j(z^m) = \beta_j(z^m; r) := \sum_{k=0}^{r-1} \binom{r+j-1}{k} (z^m-1)^k,$$

for all $j = 1, 2, \ldots,$

so that $\beta_j(z^m)$ is in $\pi_{(r-1)m}$ for each $j \geq 1$. Moreover, the following identity holds (cf. [2]):

$$(2.17)\qquad \left(\frac{z^m-1}{t^m-1}\right)^r = \frac{z^{rm}}{t^{rm}} - \frac{(t^m-z^m)}{t^{(r+1)m}} \sum_{s=0}^{\infty} \frac{\beta_{s+1}(z^m)}{t^{sm}}.$$

We note from (2.16) that

$$(2.18)\qquad |\beta_j(z^m)| \leq 2^{r+j-1}(|z|^m+1)^{r-1} \quad \text{for all } j \geq 1,$$

so that the last sum in (2.17) converges absolutely for any t with $|t| > 1$, provided that m is sufficiently large. Inserting the identity of (2.17) in (2.15), it readily follows that $K_1(t, z)$ can be expressed as the sum

$$(2.19)\qquad K_1(t,z) = T_1(t,z) + T_2(t,z) + T_3(t,z),$$

where

$$(2.20)\qquad \begin{cases} T_1(t,z) := \dfrac{z^{rm}(t^{n+1}-z^{n+1})}{t^{(r+1)m+n+1}} \displaystyle\sum_{s=0}^{\infty} \frac{1}{t^{sm}}, \\ T_2(t,z) := \dfrac{z^{n+1}(t^m-z^m)}{t^{(r+1)m+n+1}} \displaystyle\sum_{s=0}^{\infty} \frac{\gamma_s(z^m)}{t^{sm}}, \\ T_3(t,z) := -\dfrac{(t^m-z^m)}{t^{(r+2)m}} \displaystyle\sum_{s=0}^{\infty} \frac{\gamma_s(z^m)}{t^{sm}}, \end{cases}$$

and where

$$(2.21)\qquad \gamma_s(z^m) := \sum_{j=0}^{s} \beta_{j+1}(z^m), \quad \text{for all } s = 0, 1, \ldots,$$

so that $\gamma_s(z^m)$ is in $\pi_{(r-1)m}$ for all $s \geq 0$.

If $\max_{|t|=R} |f(t)| =: M_R$, then for $|z| = \tau \geq \rho$ and for $|t| = R$ where $1 < R < \rho$, we have

$$\left|\frac{1}{2\pi i}\int_{\Gamma} \frac{f(t)T_2(t,z)\,dt}{t-z}\right| \leq \frac{M_R \tau^{n+1}(R^m+\tau^m)}{(\tau-R)R^{(r+1)m+n}} \left\{|\gamma_0(z^m)| + \frac{|\gamma_1(z^m)|}{R^m} + \cdots\right\}.$$

(3.4) *and* (1.1). *With* $m = nq + c$, *where* q *and* c *are any fixed positive integers, there holds*

$$\lim_{n\to\infty} \{P_N(z;f) - s_N(z;f)\} = 0, \quad \text{for all } |z| < \rho^{1+q/(1+rq)}, \tag{3.5}$$

the convergence being uniform and geometric on any closed subset of $|z| < \rho^{1+q/(1+rq)}$. *More precisely, for any* τ *with* $\rho \le \tau < \infty$, *there holds*

$$\overline{\lim_{n\to\infty}} \left\{ \max_{|z|\le\tau} |P_N(z;f) - s_N(z;f)| \right\}^{1/n} \le \frac{\tau^{rq+1}}{\rho^{(r+1)q+1}}.$$

Further, the result of (3.5) *is best possible in the sense that there is some* $\hat{f} \in A_\rho$ *and some* $\hat{z}$ *with* $|\hat{z}| = \rho^{1+q/(1+rq)}$ *for which the sequence*

$$\{P_N(\hat{z}:\hat{f}) - s_N(\hat{z}:\hat{f})\}_{n=1}^{\infty}$$

does not tend to zero as $n \to \infty$.

The proof of Theorem 3, while depending on the results of Riemenschneider and Sharma [**3**], and Saxena, Sharma, and Ziegler [**6**], follows along the lines of the proof of Theorem 1, and is omitted.

As further open questions, we finally ask if there are Theorem B-type and Theorem C-type extensions of Theorem 3.

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