ON THE RATE OF OVERCONVERGENCE OF THE GENERALIZED ENESTRÖM-KAKEYA FUNCTIONAL FOR POLYNOMIALS*

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Abstract

The classical Eneström-Kakeya Theorem, which provides an upper bound for the moduli of the zeros of any polynomial with positive coefficients, has been recently extended by Anderson, Saff and Varga to the case of any complex polynomial having no zeros on the ray $[0, +\infty)$. Their extension is sharp in the sense that, given such a complex polynomial $p_n(z)$ of degree $n \geqslant 1$, a sequence of multiplier polynomials $\{Q_{m_i}(z)\}_{i=1}^{\infty}$ can be found for which the Eneström-Kakeya upper bound, applied to the products $Q_{m_i}(z) \cdot p_n(z)$, converges, in the limit as i tends to ∞, to the maximum of the moduli of the zeros of $p_n(z)$. Here, the rate of convergence of these upper bounds (to the maximum of the moduli of the zeros of $p_n(z)$) is studied. It is shown that the obtained rate of convergence is best possible.

§ 1. Introduction

With π_n denoting the set of all complex polynomials of degree exactly n, and with

$$\pi_n^+ := \left\{ p_n(z) = \sum_{j=0}^{n} a_j z^j : \ a_j > 0 \text{ for all } j = 0, 1, \cdots, n \right\}, \tag{1.1}$$

a useful form of the classical Eneström-Kakeya Theorem[3,5], due in fact to Eneström[3], is the following

Theorem A. *For any* $p_n(z) = \sum_{j=0}^{n} a_j z^j$ *in* π_n^+ *with* $n \geqslant 1$, *define*

$$\alpha[p_n] := \min_{0 \leqslant i < n} \left\{ \frac{a_i}{a_{i+1}} \right\} \quad \text{and} \quad \beta[p_n] := \max_{0 \leqslant i < n} \left\{ \frac{a_i}{a_{i+1}} \right\}. \tag{1.2}$$

Then, all the zeros of $p_n(z)$ *lie in the annulus*

$$\alpha[p_n] \leqslant |z| \leqslant \beta[p_n]. \tag{1.3}$$

Evidently, if

$$\rho(p_n) := \max\{|z_j| : p_n(z_j) = 0\}$$

denotes the spectral radius of any complex polynomial $p_n(z)$ in π_n with $n \geqslant 1$, then the Eneström-Kakeya Theorem asserts that

$$\beta[p_n] \geqslant \rho(p_n), \text{ for all } p_n \in \pi_n^+, \text{ for all } n \geqslant 1. \tag{1.4}$$

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for all m sufficiently large. This implies that the rate of convergence of $\tau_m(p_n)$ to $\rho(p_n)$ is no worse than linear in $1/m$, while the example $\tilde{p}_2(z):=(1+z)^2$ in (1.14) shows that this rate can be exactly linear in $1/m$.

§ 2. Proof of (1.3) of Theorem 1

Fix any $p_n(z)$ in $\hat{\pi}_n$, where $n\geqslant 1$. Without loss of generality (cf. [2]), we may assume that $p_n(z)$ is monic, real, and is normalized so that $\rho(p_n)=1$. Thus, we can write

$$p_n(z)=\prod_{j=1}^{n}(z-\xi_j), \tag{2.1}$$

and we can set

$$\xi_j:=r_je^{2\pi i\theta_j}, \text{ where } 0<r_j\leqslant 1,\ 0<\theta_j<1\ (1\leqslant j\leqslant n). \tag{2.2}$$

With Z^+ denoting the set of all positive integers, we next define the sets

$$S_j:=\{t\in Z^+: \xi_j^t\notin[0,+\infty)\},\quad 1\leqslant j\leqslant n, \tag{2.3}$$

so that (cf. [2]) $S_j=Z^+\backslash\{sd_j\}_{s=1}^{\infty}$ if $\theta_j=n_j/d_j$ it rational (where n_j and d_j are positive integers, in lowest terms), as well as $S_j=Z^+$ if θ_j is irrational. Thus,

$$T:=\bigcap_{j=1}^{n}S_j=Z^+\backslash\bigcup_{\theta_j \text{ rational}}\{sd_j\}_{s=1}^{\infty}. \tag{2.4}$$

Since $1\in S_j$ for all $1\leqslant j\leqslant n$, we can write

$$T:=\{t_k\}_{k=1}^{\infty}, \text{ where } 1=t_1<t_2<\cdots, \text{ with } \lim_{k\to\infty}t_k=\infty. \tag{2.5}$$

For some elementary number-theoretic properties of the set T, define

$$D:=\text{l.c.m. }\{d_j\}_{j=1}^{n}\ (\text{where } D:=1 \text{ if all } \theta_j\text{'s are irrational}). \tag{2.6}$$

Then, as a consequence of the Chinese Remainder Theorem (cf. [6, p. 249]), it is easily verified that T is periodic with period D, i.e., if, for any positive integer a, $t_{j+1}, t_{j+2}, \cdots, t_{j+l}$ are the consecutive elements of T in the interval $[a, a+D)$, then the consecutive elements of T in $[a+D, a+2D)$ are exactly given by $t_{j+l+k}=t_{j+k}+D$ for all $1\leqslant k\leqslant l$. This implies that T is an infinite set (cf. (2.5)), and that the maximum gap, $\tilde{g}$, between consecutive terms of T is finite:

$$\tilde{g}:=\max_{k\geqslant 1}[t_{k+1}-t_k]<\infty. \tag{2.7}$$

Next, since $p_n(z)$ is also real, we can write (2.1) in the form

$$p_n(z)=\prod_{i=1}^{\alpha_1}(z+\delta_i)\prod_{j=1}^{\alpha_2}[z^2-2r_j\cos(2\pi\theta_j)z+r_j^2], \tag{2.8}$$

where $0<\delta_i\leqslant 1$ (if the first product is not vacuous), and where $0<\theta_j<1/2$ (if the second product is not vacuous). Since the quadratic factors in the second product of (2.8) are derived only from non-real zeros of $p_n(z)$ in the open upper-half plane, we set

$$I:=\{\theta_k: \theta_k \text{ is irrational, with } 0<\theta_k<1/2\}. \tag{2.9}$$

Clearly, I is either empty or a finite non-empty set.

For additional notation, let $\langle x\rangle$ denote x minus its nearest integer, for x any real number, with the convention that $\langle x\rangle:=1/2$ when $x=n+1/2$, n an integer.

Our objective now is to construct a suitable subset U of the set T, defined in (2.4). If $I=\emptyset$, set $U:=T$. If $I\neq\emptyset$, let r be the largest positive integer for which numbers $\theta_1, \theta_2, \cdots, \theta_r$ can be found in I such that $\{\theta_j\}_{j=1}^r$ and $\{1\}$ are linearly independent over the rational numbers. Recalling the periodic nature of the set $T=\{t_j\}_{j=1}^\infty$ and the fact that $t_1=1$, then $\{1+mD\}_{m=1}^\infty$ is an infinite subset of T, where D is defined in (2.6) and where m is any positive integer.

Fixing one such set $\{\theta_j\}_{j=1}^r$ such that $\{\theta_j\}_{j=1}^r$ and $\{1\}$ are linearly independent over the rational numbers, consider next the r-dimensional real vector

$$[\langle(1+mD)\theta_1\rangle, \langle(1+mD)\theta_2\rangle, \cdots, \langle(1+mD)\theta_r\rangle] \quad \text{for each } m\geqslant 1, \tag{2.10}$$

which, by definition, lies in the r-dimensional "unit" hypercube $|x_i|\leqslant 1/2$ $(1\leqslant i\leqslant r)$. Now, the linear independence of the $\{\theta_j\}_{j=1}^r$ and $\{1\}$ insures (cf. [7]) that the points of (2.10) are asymptotically uniformly distributed in this unit hypercube. Next, let $\{\alpha_j\}_{j=1}^r$ and δ be any $r+1$ positive constants satisfying

$$0<\alpha_j-\delta<\alpha_j+\delta<1/2, \text{ for all } 1\leqslant j\leqslant r. \tag{2.11}$$

(Later, a determination of the α_j's and δ will be made.)

This brings us to

Lemma 1. *For any choice of positive numbers $\{\alpha_j\}_{j=1}^r$ and δ satisfying (2.11), there exists an infinite sequence $\{m_i\}_{i=1}^\infty$ of positive integers with $m_1<m_2<\cdots$, such that*

$$\alpha_j-\delta\leqslant\langle(1+m_iD)\theta_j\rangle\leqslant\alpha_j+\delta, \text{ for all } i\geqslant 1, \text{ all } 1\leqslant j\leqslant r. \tag{2.12}$$

Thus, if $U:=\{u_i:=1+m_iD\}_{i=1}^\infty$, then U is an infinite subset of T. Moreover, the maximum gap, g, between successive elements of U is bounded:

$$g:=\max_{l\geqslant 1}\{u_{l+1}-u_l\}<+\infty. \tag{2.13}$$

Proof. If $I=\emptyset$, then $U=T$ by definition, and U is independent of the α_j's and δ. That U is an infinite set possessing a finite maximum gap between successive elements, is a consequence of (2.7). Thus, assume that $I\neq\emptyset$, and let $\{\theta_j\}_{j=1}^r$ be a largest subset of I such that $\{\theta_j\}_{j=1}^r$ and $\{1\}$ are linearly independent over the rational numbers, and let $\{\hat{\theta}_j\}_{j=r+1}^s$ denote the elements of the (possibly empty) set $I\setminus\{\theta_j\}_{j=1}^r$.

A result of Slater[7, p. 182] gives that, for any irrational numbers $\{\psi_j\}_{j=1}^r$ which, with $\{1\}$, are linearly independent over the rational numbers, and for any closed convex subset V (with nonempty interior) of the unit hypercube $(x_1, x_2, \cdots, x_r)$, where $|x_i|\leqslant 1/2$ for $1\leqslant i\leqslant r$, there exist infinitely many positive integers $m_1<m_2, \cdots$ such that

$$[\langle m_i\psi_1\rangle, \langle m_i\psi_2\rangle, \cdots\langle m_i\psi_r\rangle]\in V \text{ for all } i\geqslant 1, \tag{2.14}$$

and such that

$$\max_{i\geqslant 1}[m_{i+1}-m_i]<+\infty. \tag{2.15}$$

Now, the irrational numbers $\{D\theta_i\}_{i=1}^r$ are, with $\{1\}$, evidently linearly independent over the rational numbers, and (cf. (2.11))

$$\tilde{V}:=\{(x_1, x_2, \cdots, x_r):\alpha_j-\delta\leqslant x_j\leqslant\alpha_j+\delta \text{ for all } 1\leqslant j\leqslant r\} \tag{2.16}$$

is a particular closed convex subset of our unit hypercube. Thus, an easy consequence of Slater's result is that there exist infinitely many positive integers $m_1 < m_2 < \cdots$ such that

$$[\langle(1+m_iD)\theta_1\rangle, \langle(1+m_iD)\theta_2\rangle, \cdots, \langle(1+m_iD)\theta_r\rangle] \in \tilde{V}, \quad \text{for all } i \geqslant 1, \tag{2.17}$$

and such that

$$\max_{i\geqslant 1}[m_{i+1}-m_i] < +\infty. \tag{2.18}$$

Clearly, (2.17) and (2.18) imply the desired results of (2.12) and (2.13). ▮

This brings us to the selection of the positive numbers $\{\alpha_j\}_{j=1}^r$ and δ, which satisfy (2.11). If $I \neq \emptyset$, assume first that the irrational numbers $\{\theta_j\}_{j=1}^r$ (which, with $\{1\}$, are linearly independent over the rational numbers) exhaust the set I of (2.9). In this case, any choice of positive numbers $\{\alpha_j\}_{j=1}^r$ and δ, satisfying (2.11), is acceptable. If $\{\theta_j\}_{j=1}^r$ does not exhaust I, let $\{\hat{\theta}_j\}_{j=r+1}^s$ denote the remaining elements of I. As $\{\theta_j\}_{j=1}^r$ is a largest subset of I for which $\{\theta_j\}_{j=1}^r$ and $\{1\}$ are linearly independent over the rational numbers, then each $\hat{\theta}_l$ (for $r+1 \leqslant l \leqslant s$) is a linear combination (with rational coefficients) of $\{\theta_i\}_{i=1}^r$ and $\{1\}$, i.e., with $\theta_0 := 1$ for convenience, we can write

$$\hat{\theta}_l = \sum_{k=0}^r c_{l,k}\,\theta_k, \text{ for all } r+1 \leqslant l \leqslant s, \tag{2.19}$$

where $c_{l,k}$ are rational numbers. On multiplying (2.19) by the least common multiple of the denominators of the non-zero $|c_{l,k}|$, (2.19) becomes

$$\hat{\theta}_l = \frac{1}{C_l}\sum_{k=0}^r C_{l,k}\,\theta_k, \quad \text{for all } r+1 \leqslant l \leqslant s, \tag{2.20}$$

where C_l and $C_{l,k}$ are integers. It is evident that $C_l \neq 0$ $(r+1 \leqslant l \leqslant s)$, and, as $\hat{\theta}_l$ is irrational, that $\sum_{k=1}^r |C_{l,k}| > 0$ $(r+1 \leqslant l \leqslant s)$.

With D defined as in (2.6) and for any positive integer m, (2.20) yields

$$(1+mD)\hat{\theta}_l = \frac{1}{C_l}\sum_{k=0}^r C_{l,k}(1+mD)\theta_k, \quad r+1 \leqslant l \leqslant s. \tag{2.21}$$

From the definition of $\langle x\rangle$, it is evident from (2.21) that

$$\langle(1+mD)\hat{\theta}_l\rangle = \frac{\mu_l(m)}{C_l} + \sum_{k=1}^r c_{l,k}\langle(1+mD)\theta_k\rangle,$$

where $\mu_l(m)$ is some integer, so that

$$|\langle(1+mD)\hat{\theta}_l\rangle| = \left|\frac{\mu_l(m)}{C_l} + \sum_{k=1}^r c_{l,k}\langle(1+mD)\theta_k\rangle\right|. \tag{2.22}$$

Now, for each l with $r+1 \leqslant l \leqslant s$,

$$\sum_{k=1}^r c_{l,k}x_k = 0 \tag{2.23}$$

is a plane through the origin in r-dimensional space. As the number of such planes is finite, there is a cone (with nonempty interior) in the positive-hyperoctant of this real r-dimensional space, having as boundaries the coordinate planes and/or the planes of (2.23). Thus, we can extract a closed hypercube H from the interior of this cone, whose coordinate intervals are $\{\alpha_k-\delta, \alpha_k+\delta\}$ for $1 \leqslant k \leqslant r$. In fact, we can choose the positive numbers $\{\alpha_j\}_{j=1}^r$ and δ so that

$$0<\alpha_k-\delta<\alpha_k+\delta<1/2, \quad \text{for all } 1\leqslant k\leqslant r, \tag{2.24}$$

and so that the planes

$$\frac{\mu_1(m)}{C_l}+\sum_{k=1}^r c_{l,k}x_k=0, \quad l+1\leqslant r\leqslant s, \tag{2.25}$$

which are translates of the planes of (2.23), do not intersect our hypercube H, for all choices of $\mu_l(m)$ and for all $r+1\leqslant l\leqslant s$.

Fixing our hypercube H in this way, we have, from Lemma 1, the existence of the sequence $\{m_i\}_{i=1}^\infty$ of positive integers such that (2.12) and (2.13) are valid. In particular, (2.12) implies that the vector

$$[\langle(1+m_iD)\theta_1\rangle, \langle(1+m_iD)\theta_2\rangle, \cdots, \langle(1+m_iD)\theta_r\rangle]$$

is in our hypercube H for all m_i. Moreover, with this sequence $\{m_i\}_{i=1}^\infty$, the fact that the planes of (2.25) do not intersect H, gives us, from (2.22), that

$$\min\{|\langle(1+mD)\hat{\theta}_l\rangle|:m\in\{m_i\}_{i=1}^\infty \text{ and } r+1\leqslant l\leqslant s\}=:\tau>0. \tag{2.26}$$

This will be used below.

Next, with $u_l:=1+m_lD$ (cf. Lemma 1), we set (cf. (2.1))

$$P_l(w):=\prod_{i=1}^n(w-\xi_i^{u_l})<\text{for each } l\geqslant 1. \tag{2.27}$$

Lemma 2. *For each $l\geqslant 1$, there exists a polynomial $G_l(w)$ such that*

i) *$G_l(w)$ is monic with all its coefficients positive;*

ii) *$P_l(w)$ divides $G_l(w)$;*

iii) *$\beta[G_l(w)]\leqslant n$ (where β is defined in (1.2));* (2.28)

iv) *there is a constant M (independent of l) such that* deg $(G_l(w))\leqslant M$.

Proof. Consider the representation of $p_n(z)$ in (2.8). If the first product in (2.8) is not vacuous, then $-\delta_j=|\delta_j|e^{2\pi i(1/2)}$ are the real negative zeros of $p_n(z)$. In this case, the set T of (2.4) contains only odd positive integers, so that the same is true for the set $U:=\{u_l:=1+m_lD\}_{l=1}^\infty$, since D is necessarily even in this case. In general, it follows from (2.27) and (2.8) that

$$P_l(w)=\prod_{i=1}^{\alpha_1}(w+\delta_i^{u_l})\prod_{j=1}^{\alpha_2}[w^2-2r_j^{u_l}\cos(2\pi u_l\theta_j)w+r_j^{2u_l}], \tag{2.29}$$

for $l\geqslant 1$.

Consider any term of the first product of (2.29). As $\delta_i>0$, then $\beta[w+\delta_i^{u_l}]=\delta_i^{u_l}\leqslant 1$. Consider next any term of the second product in (2.29), say

$$r_j^{2u_l}-2r_j^{u_l}\cos(2\pi u_l\theta_j)\cdot w+w^2. \tag{2.30}$$

For any rational θ_j, it follows from (2.4) and (2.6) that

$$\cos(2\pi u_l\theta_j)\leqslant\cos(2\pi/D)<1, \text{ for all } l\geqslant 1. \tag{2.31}$$

Thus, on applying the angle-doubling procedure of the proof of [2, Prop. 1] a finite number of times (depending only on D), there exists a monic real polynomial $Q_m(w)$, depending on l and θ_j but whose degree (which depends only on D) is bounded, such that $Q_m(w)$ times the polynomial in (2.30) has positive coefficients, and such that (cf. (1.2))

$$\beta\{Q_m(w)\cdot[r_j^{2u_l}-2r_j^{u_l}\cos(2\pi u_l\theta_j)\cdot w+w^2]\}\leqslant 2, \tag{2.32}$$

for all $l\geqslant 1$, all θ_j rational.

Next, we consider the quadratic in (2.30) when θ_j is irrational. First, suppose that θ_j is one of the $\{\theta_l\}_{l=1}^r$ which, with $\{1\}$, are linearly independent over the rational numbers. Referring to (2.12) of Lemma 1 and the definition $u_l:=1+m_lD$, then

$$0<\alpha_j-\delta\leqslant\langle u_l\theta_j\rangle\leqslant\alpha_j+\delta<1/2, \quad \text{for all } l\geqslant 1, \tag{2.33}$$

where $\{\alpha_j\}_{j=1}^r$ and δ satisfy (2.11). Thus, as in (2.31), (2.33) insures that

$$\cos(2\pi u_l\theta_j)\leqslant\cos[2\pi(\alpha_j-\delta)]<1, \quad \text{for all } l\geqslant 1.$$

As in the previous case of θ_j rational, the angle-doubling procedure of [2, Prop. 1] can be applied a finite number of times (depending only on $\alpha_j-\delta$), and there is again a monic real polynomial $Q_m(w)$ whose degree is bounded, such that (2.32) is valid for all $l\geqslant 1$, when θ_j is an irrational number from the particular set $\{\theta_l\}_{l=1}^r$.

Finally, suppose that θ_j is in $\{\hat{\theta}_l\}_{l=r+1}^s:=I\setminus\{\theta_l\}_{l=1}^r$. Recalling (2.26), we see that

$$\min|\langle u_l\theta_j\rangle|\geqslant\tau>0, \quad \text{for all } l\geqslant 1,$$

so that

$$\cos(2\pi u_l\theta_j)\leqslant\cos\tau<1, \quad \text{for all } l\geqslant 1.$$

As in the previous cases of θ_j rational and $\theta_j\in\{\theta_l\}_{l=1}^r$, the angle-doubling procedure applied a finite number of times (depending only on τ) produces a real monic polynomial $Q_m(w)$, depending on l and θ_j, but whose degree is bounded, such that (2.32) is valid for all $l\geqslant 1$.

Now, on multiplying all the terms of (2.29) together, along with their associated multipliers $Q_m(w)$, we obtain a monic real polynomial $G_l(w)$, where $P_l(w)$ divides $G_l(w)$ and where deg $[G_l(w)]\leqslant M$ for all $l\leqslant 1$. Moreover, from [2, Lemma 1], (2.28iii) is further satisfied, i.e., $\beta[G_l(w)]\leqslant n$ for all $l\geqslant 1$, where n is the degree of $p_n(z)$ in $\hat{\pi}_n$. ▮

This brings us to the Proof of (1.13) of Theorem 1. As in [2, p. 19], we form, for each $\eta>0$, the product polynomial

$$H_j(z;\eta):=(\eta^{u_j-1}+\eta^{u_j-2}z+\cdots+z^{u_j-1})G_j(z^{u_j}) \quad \text{for all } j\geqslant 1, \tag{2.34}$$

where $G_j(w)$ satisfies (2.28). Because $G_j(w)$ has all positive coefficients from (2.28i), the polynomial $H_j(z;\eta)$ of (2.34) has all positive coefficients, so that the Eneström-Kakeya functional β of (1.2) can be applied to it. Setting $\eta_j:=(\beta[G_j])^{1/u_j}$, it follows, as in [2], that

$$\beta[H_j(z;\eta_j)]=(\beta[G_j])^{1/u_j}\leqslant n^{1/u_j}, \quad \text{for all } j\geqslant 1, \tag{2.35}$$

the last inequality following from (2.28iii). As n, the degree of our given $p_n(z)$, is fixed and as $u_j\to\infty$ as $j\to\infty$ from Lemma 1, the inequality of (2.35) then gives us that

$$\beta[H_j(z;\eta_j)]\leqslant 1+\frac{\ln n}{u_j}+O\left(\frac{1}{u_j^2}\right), \quad \text{as } j\to\infty. \tag{2.36}$$

Now, deg $(H_j(z;\eta_j)=u_j(\deg G_j)+u_j-1$, so that (2.36) implies (cf. (1.7)) that

$$\tau_{u_j[\deg G_j+1]-1-n} \leqslant 1 + \frac{\ln n}{u_j} + O\left(\frac{1}{u_j^2}\right), \quad \text{as } j \to \infty,$$

which further implies, from (1.8) and (2.28iv), that

$$\tau_{u_j(M+1)} \leqslant 1 + \frac{\ln n}{u_j} + O\left(\frac{1}{u_j^2}\right), \quad \text{as } j \to \infty. \tag{2.37}$$

Thus

$$u_j(M+1)\,[\tau_{u_j(M+1)} - 1] \leqslant (M+1)\ln n + O\left(\frac{1}{u_j}\right), \quad \text{as } j \to \infty,$$

so that

$$\overline{\lim_{j\to\infty}}\{u_j(M+1)\,[\tau_{u_j(M+1)} - 1]\} \leqslant (M+1)\ln n. \tag{2.38}$$

Next, consider any integer k satisfying

$$u_j(M+1) \leqslant k \leqslant u_{j+1}(M+1). \tag{2.39}$$

Because of the monotone decreasing nature of the τ_m's from (1.18), it follows that

$$k[\tau_k - 1] \leqslant u_{j+1}(M+1)\,[\tau_{u_j(M+1)} - 1], \tag{2.40}$$

for all k satisfying (2.39). But as $u_{j+1} \leqslant u_j + g$ from (2.13) of Lemma 1, then (2.40) yields

$$k[\tau_k - 1] \leqslant u_j(M+1)\,[\tau_{u_j(M+1)} - 1] + g(M+1)\,[\tau_{u_j(M+1)} - 1]. \tag{2.41}$$

As the last term of (2.41) tends to zero from (1.9) of Theorem B, we deduce with (2.38) that

$$\overline{\lim_{m\to\infty}}\, m[\tau_m - 1] \leqslant (M+1)\ln n,$$

which gives the desired result (1.13) of Theorem 1. ∎

We remark that our proof of Theorem 1 assumes only that $p_n(z) \in \hat{\pi}_n$ with $n \geqslant 1$. Obviously, if $p_n(z) \in \mathring{\pi}_n$, then from (1.11), there holds

$$\lim_{m\to\infty} m[\tau_m(p_n) - \rho(p_n)] = 0,$$

i.e. σ of (1.13) is necessarily zero in this case.

§ 3. Proof of (1.14) of Theorem 1

Consider

$$\tilde{p}_2(z) := (1+z)^2 = 1 + 2z + z^2, \tag{3.1}$$

so that $\rho(\tilde{p}_2) = 1$. Clearly, $\tilde{p}_2(z)$ is an element of both π_2^+ and $\hat{\pi}_2$, so that its associated nonnegative integer m_0 (cf. (1.7)) is given by $m_0 = 0$. Next, as $\tilde{p}_2(z)$ has a zero of multiplicity two at $z = -1$, then (1.10i) fails, so that $\tilde{p}_2 \in \hat{\pi}_2 \backslash \mathring{\pi}_2$. Moreover, as both zeros of $\tilde{p}_2(z)$ have argument $\pi = 2\pi\tilde{\theta}_j$, there are no irrational $\tilde{\theta}_j$'s, so that the hypotheses of Theorem 1 hold for $\tilde{p}_2$. Moreover, $\tau_m := \tau_m(\tilde{p}_2)$, defined in (1.7), necessarily satisfies (cf. (1.12))

$$\tau_m > 1, \quad \text{for all } m \geqslant 0. \tag{3.2}$$

For each nonnegative integer m, we next set

$$f_m(c) := (2m+1)c^{2m+3} - (2m+3)c^{2m+1} - 2. \tag{3.3}$$

Lemma 3. *For each nonnegative integer m, f_m has a unique positive zero, c_m, which satisfies*

$$c_m > 1. \tag{3.4}$$

Moreover,

$$2 = c_0 > c_1 > \cdots > c_m > c_{m+1} > \cdots. \tag{3.5}$$

In addition,

$$c_m > \frac{2m+2}{2m+1} \text{ for every } m \geqslant 1. \tag{3.6}$$

Proof. By Descartes' Rule of Signs, f_m has a unique positive zero for each nonnegative integer m. Since $f_m(1) = -4$ and since $f_m(c) > 0$ for all sufficiently large c, then $c_m > 1$, establishing (3.4). Now, $f_0(2) = 0$, whence $c_0 = 2$. Next, with (3.3) and (3.4),

$$f_{m+1}(c_m) = (2m+3)c_m^{2m+1}(c_m^2 - 1)^2 > 0,$$

so that $c_m > c_{m+1}$ for all $m \geqslant 0$, which gives (3.5). Finally, with (3.3),

$$f_m\left(\frac{2m+2}{2m+1}\right) = \left(1 + \frac{1}{2m+1}\right)^{2m+1} \cdot \left(\frac{1}{2m+1}\right) - 2 < \frac{e}{2m+1} - 2 < 0,$$

for all $m \geqslant 1$, so that $c_m > \left(\frac{2m+2}{2m+1}\right)$, which establishes (3.6). ∎

Lemma 4. *For each nonnegative integer m,*

$$r_{2m} \leqslant c_m. \tag{3.7}$$

Proof. Consider $Q_{2m}(z) := \sum_{j=0}^{2m} q_j z^j$, where the q_j's are recursively defined by

$$q_{j-2} + (2 - c_m) q_{j-1} - (2c_m - 1) q_j = c_m q_{j+1}, \quad j = 0, 1, \cdots, 2m-1, \tag{3.8}$$

where $q_{-2} := 0 =: q_{-1}$, and where $q_0 := 1$. Because these q_j's in (3.8) satisfy a homogeneous linear difference equation, it can be verified that the q_j's can be explicitly expressed as

$$q_j = (c_m + 1)^{-2}\{(-1)^j[(j+1)c_m^2 + (j+2)c_m] + c_m^{-j}\}, \quad j = -2, -1, \cdots, 2m. \tag{3.9}$$

Now, rearranging the terms in (3.8) gives

$$q_{j-2} + 2q_{j-1} + q_j = c_m(q_{j-1} + 2q_j + q_{j+1}), \quad j = 0, 1, \cdots, 2m-1, \tag{3.10}$$

from which it recursively follows that (cf. (3.4))

$$q_{j-2} + 2q_{j-1} + q_j = c_m^{-j} > 0, \quad j = 0, 1, \cdots, 2m. \tag{3.11}$$

Obviously, as the expression on the right in (3.9) contains only positive terms when j is even, then $q_{2j} > 0$ for all $j = 0, 1, \cdots, m$. In particular,

$$q_{2m} > 0. \tag{3.12}$$

Moreover, with (3.9) and with the fact that $f_m(c_m) = 0$, it can be verified that

$$q_{2m-1} + 2q_{2m} = c_m q_{2m} > 0. \tag{3.13}$$

(In fact, it is (3.13) which, upon working backwards, leads to the definition of $f_m(c)$ in (3.3))

Next, multiplying (3.13) by c_m and using the case $j = 2m$ of (3.9), gives

$$c_m(q_{2m-1}+2q_{2m})=(c_m+1)^{-2}c_m^{-2m}\{(2m+1)c_m^{2m+4}+(2m+2)c_m^{2m+3}+c_m^2\}.$$

As $c_m>1$ from (3.4) and as $m\geq 0$, it follows that

$$c_m(q_{2m-1}+2q_{2m})>(c_m+1)^{-2}c_m^{-2m}\{1+2c_m+c_m^2\}=c_m^{-2m}.$$

Thus, with the case $j=2m$ of (3.11),

$$c_m(q_{2m-1}+2q_{2m})>q_{2m-2}+2q_{2m-1}+q_{2m}. \tag{3.14}$$

Next, the product $Q_{2m}(z)\cdot\tilde{p}_2(z)=\left(\sum_{j=0}^{2m} q_jz^j\right)\cdot(1+z)^2$ can be expressed as

$$Q_{2m}(z)\cdot\tilde{p}_2(z)=:\sum_{j=0}^{2m+2}\gamma_jz^j=\sum_{j=0}^{2m+2}(q_{j-2}+2q_{j-1}+q_j)z^j, \tag{3.15}$$

where $q_{-2}:=q_{-1}:=q_{2m+1}:=q_{2m+2}:=0$. Clearly, from (3.11)—(3.13), we see that $Q_{2m}\cdot\tilde{p}_2\in\pi_{2m+2}^+$. Next, from (3.10) and (3.13), we deduce that the ratios γ_j/γ_{j+1} of the coefficients of $Q_{2m}(z)\cdot\tilde{p}_2(z)$ in (3.15) satisfy

$$\frac{\gamma_j}{\gamma_{j+1}}=c_m,\quad \text{for } j=0,1,\cdots,2m-1, \text{ and } 2m+1,$$

while (3.14) gives that the remaining coefficient ratio satisfies

$$\frac{\gamma_{2m}}{\gamma_{2m+1}}<c_m,$$

so that, by definition (cf. (1.2)), $\beta[Q_{2m}\cdot\tilde{p}_2]=c_m$. Hence (cf. (1.7)), $\tau_{2m}\leq c_m$, which establishes (3.7). ∎

Lemma 5. *For each nonnegative integer m, let c be a fixed number satisfying $1<c<c_m$, and let the associated numbers $\{p_k\}_{k=1}^{2m+2}$ (depending on c) be defined recursively by*

$$-cp_{j-1}-(2c-1)p_j+(2-c)p_{j+1}+p_{j+2}=0,\quad j=0,1,\cdots,2m, \tag{3.16}$$

where $p_{-1}:=0=:p_{2m+2}$, and where $p_0:=1$. Then,

$$p_j>0,\quad \textit{for all } j=0,1,\cdots,2m+1. \tag{3.17}$$

Proof. Set

$$g_k(t):=-t^{2k+3}+(2k+3)t+(2k+2),\quad k=0,1,\cdots,m, \tag{3.18}$$

so that

$$g_k'(t)=-(2k+3)(t^{2k+2}-1)<0\quad \text{for any } t>1;\ k=0,1,\cdots,m.$$

Thus, as $1<c<c_m$ and as $c_{k+1}<c_k$ (cf. (3.5)), then

$$g_k(c_{k+1})>g_k(c_k), \text{ and } g_m(c)>g_m(c_m). \tag{3.19}$$

Now, direct use of (3.18) and the fact that $f_k(c_k)=0$ gives that

$$g_k(c_k)=\left(\frac{2k+3}{2k+1}\right)g_{k-1}(c_k),$$

so that with (3.19),

$$g_k(c_k)=\left(\frac{2k+3}{2k+1}\right)g_{k-1}(c_k)>\left(\frac{2k+3}{2k+1}\right)g_{k-1}(c_{k-1}),\quad k=1,2,\cdots,m.$$

Thus, with the second inequality of (3.19), the above inequalities can be used recursively to deduce

$$g_m(c)>g_m(c_m)>\left(\frac{2m+3}{2m+1}\right)g_{m-1}(c_{m-1})>\left(\frac{2m+3}{2m-1}\right)g_{m-2}(c_{m-2})>\cdots>\frac{(2m+3)}{3}g_0(c_0).$$

But as $c_0=2$ from (3.5) of Lemma 3 and as $g_0(c_0)=0$, then

$$g_m(c)>0, \quad m=0, 1, \cdots, \tag{3.20}$$

which will be used below.

Now, the explicit solution of the coefficients $\{p_k\}_{k=-1}^{2m+2}$ in (3.16) can be verified to be

$$p_j=\frac{1}{g_m(c)}\{(-1)^j[2m+2-j-(1+j)c^{2m+3}]+(2m+3)c^{j+1}\},$$
$$j=-1, 0, \cdots, 2m+2, \tag{3.21}$$

where $p_{-1}:=0:=p_{2m+2}$, and where $p_0:=1$. In the case when j is odd, say $j=2l+1$, (3.21) gives

$$p_{2l+1}=\frac{1}{g_m(c)}\{(2l+2)c^{2m+3}+(2m+3)c^{2l+2}-(2m+1-2l)\}, \quad l=0, 1, \cdots, m.$$

As $c>1$, the quantity in braces above is bounded below by $4l+4$, so that with (3.20),

$$p_{2l+1}>0 \text{ for } l=0, 1, \cdots, m. \tag{3.22}$$

In the case when j is even, say $j=2l$, the assumption $c>1$ and (3.21) give (cf. (3.3))

$$\begin{aligned} g_m(c)(p_{2l}-p_{2l+2}) &= 2c^{2m+3}-(2m+3)c^{2l+1}(c^2-1)+2 \\ &> 2c^{2m+3}-(2m+3)c^{2m+1}(c^2-1)+2=-f_m(c), \end{aligned}$$

for any $l=0, 1, \cdots, m$. But as $1<c<c_m$, we see from Lemma 3 that $f_m(c)>0$, so that

$$p_0>p_2>\cdots>p_{2m}>0=p_{2m+2}. \tag{3.23}$$

Thus, (3.22) and (3.23) give the desired result of (3.17). ∎

Lemma 6. *For all nonnegative integers m,*

$$\tau_{2m}=c_m. \tag{3.24}$$

Proof. From (3.7) of Lemma 2, $\tau_{2m}\leqslant c_m$ for any nonnegative integer m. Suppose, on the contrary, that there is a nonnegative integer m, for which $\tau_{2m}<c_m$, so that (cf. (3.2)) $1<\tau_{2m}<c_m$. For any choice of c with $1<\tau_{2m}<c<c_m$, there exists a $Q_{2m}(z)$ in π_{2m} with $Q_{2m}\tilde{p}_2\in\pi_{2m+2}^{+}$ such that $\beta[Q_{2m}\tilde{p}_2]<c$. Writing $Q_{2m}(z)=\sum_{l=0}^{2m} q_l z^l$, $\beta[Q_{2m}\tilde{p}_2]<c$ implies that

$$q_{j-2}+2q_{j-1}+q_j<c(q_{j-1}+2q_j+q_{j+1}), \quad j=0, 1, \cdots, 2m+1,$$

or equivalently,

$$q_{j-2}+(2-c)q_{j-1}-(2c-1)q_j-cq_{j+1}<0, \quad j=0, 1, \cdots, 2m+1, \tag{3.25}$$

where $q_{-2}:=q_{-1}:=0=:q_{2m+1}=:q_{2m+2}$. With this value of c, let the positive numbers $\{p_k\}_{k=0}^{2m+1}$ (depending on c) be defined from (3.16) of Lemma 5. Multiplying (3.25) by p_j and summing on j gives

$$\sum_{j=0}^{2m+1} p_j[q_{j-2}+(2-c)q_{j-1}-(2c-1)q_j-cq_{j+1}]<0.$$

However, the left side of the above expression is

$$\sum_{j=0}^{2m+1} q_j[-cp_{j-1}-(2c-1)p_j+(2-c)p_{j+1}+p_{j+2}]=0$$

from (3.16) of Lemma 5. This contradiction establishes the desired result of (3.24) that $\tau_{2m}=c_m$. ∎

It now follows, for $\tilde{p}_2(z):=(1+z)^2$ in $\hat{\pi}_2\backslash\hat{\pi}_2$, that the associated optimal generalized Eneström–Kakeya functionals $\tau_{2m}(\tilde{p}_2)$ satisfy (cf. (3.24) and (3.6))

$$\tau_{2m}(\tilde{p}_2)=c_m>1+\frac{1}{2m+1}, \quad \text{for all } m\geq 1. \tag{3.26}$$

However, a sharp asymptotic behavior of c_m can be obtained. Let $\lambda>1$ be the unique positive zero of $(\lambda-1)e^\lambda-1$, i.e.,

$$\lambda \doteq 1.278464543. \tag{3.27}$$

It can then be verified from (3.3) that

$$c_m=1+\frac{\lambda}{2m+1}+\frac{\lambda^2-2\lambda}{2(2m+1)^2}+O\left(\frac{1}{(2m+1)^3}\right), \quad \text{as } m\to\infty. \tag{3.28}$$

With (3.28), we now establish

Lemma 7. *We have (cf.* (3.27))

$$\lim_{m\to\infty} m(\tau_m(\tilde{p}_2)-1)=\lambda. \tag{3.29}$$

Proof. From (3.24) and (3.28), it follows that $\lim_{m\to\infty}(2m+1)(\tau_{2m}-1)=\lambda$. But as $(2m+1)(\tau_{2m}-1)=2m(\tau_{2m}-1)+(\tau_{2m}-1)$, and as $\tau_{2m}=c_m\to 1$ as $m\to\infty$ from (3.28), then

$$\lim_{m\to\infty} 2m(\tau_{2m}-1)=\lambda. \tag{3.30}$$

Similarly, we know (cf. (1.8)) that $\tau_{2m+2}\leq\tau_{2m+1}\leq\tau_{2m}$ for all $m\geq 0$, whence

$$(2m+1)(\tau_{2m+2}-1)\leq(2m+1)(\tau_{2m+1}-1)\leq(2m+1)(\tau_{2m}-1),$$

or equivalently,

$$(2m+2)(\tau_{2m+2}-1)-(\tau_{2m+2}-1)\leq(2m+1)(\tau_{2m+1}-1)\leq 2m(\tau_{2m}-1)+(\tau_{2m}-1).$$

Since $\tau_{2m+2}-1$ and $\tau_{2m}-1$ both tend to zero as $m\to\infty$, the above inequalities, with (3.30), imply that

$$\lim_{m\to\infty}(2m+1)(\tau_{2m+1}-1)=\lambda,$$

which, with (3.30), gives the desired result of (3.29). ∎

We state without proof that, for this polynomial $\tilde{p}_2(z):=(1+z)^2$, the following sharper result can be shown:

$$\tau_{2m+1}(\tilde{p}_2)=\tau_{2m}(\tilde{p}_2), \quad \text{for all } m\geq 0,$$

so that

$$\tau_0(\tilde{p}_2)=\tau_1(\tilde{p}_2)>\tau_2(\tilde{p}_2)=\tau_3(\tilde{p}_2)>\cdots.$$

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The sharpness of this inequality in (1.4) had already been studied in 1913 by Hurwitz[4]. For a recent (corrected) form of Hurwitz's original contribution which gives the precise conditions on $p_n(z)$ in π_n^+ so that equality holds in (1.4), see [1].

To go beyond the case of polynomials having only positive real coefficients, as treated in (1.4), consider more generally any complex polynomial $p_n(z)$ in π_n with $n\geqslant 1$, and suppose that a multiplier polynomial $Q_m(z)$ in π_m can be found such that $Q_m(z)\cdot p_n(z)$ is in π_{m+n}^+. Then, applying the Eneström–Kakeya Theorem to the product $Q_m(z)p_n(z)$ gives (cf. (1.4)) $\beta[Q_m p_n]\geqslant\rho(Q_m p_n)$, but as $\rho(Q_m p_n)\geqslant\rho(p_n)$, then

$$\beta[Q_m p_n]\geqslant\rho(p_n). \tag{1.5}$$

This upper bound $\beta[Q_m p_n]$ for $\rho(p_n)$ is called a generalized Eneström–Kakeya functional for $p_n(z)$, when such a multiplier polynomial exists. On setting

$$\hat{\pi}_n := \{p_n(z)\in\pi_n:\ p_n(z) \text{ has no zeros on the ray } [0,+\infty)\}, \tag{1.6}$$
$$\text{for any } n\geqslant 1,$$

it was shown in Anderson, Saff and Varga [2, Prop. 1] that the existence of such a multiplier polynomial $Q_m(z)\in\pi_m$ for $p_n(z)$ for which $Q_m(z)\cdot p_n(z)\in\pi_{m+n}^+$, is equivalent with $p_n\in\hat{\pi}_n$. Moreover, for $p_n\in\hat{\pi}_n$, it easily follows (cf. [2]) that there exists a least nonnegative integer m_0 (depending on p_n) such that the set

$$\omega_m(p_n) := \{Q_m(z)\in\pi_m:\ Q_m(z)p_n(z)\in\pi_{m+n}^+\}$$

is nonempty for each $m\geqslant m_0$. Now,

$$\tau_m=\tau_m(p_n) := \inf\{\beta[Q_m p_n]:\ Q_m\in\omega_m(p_n)\},\quad m\geqslant m_0, \tag{1.7}$$

gives the optimal (least) upper bound estimate of $\rho(p_n)$ of this generalized Eneström–Kakeya functional, when restricted to polynomial multipliers $Q_m(z)$ of degree m. Moreover, with (1.5) and (1.7), it is evident that the $\tau_m(p_n)$'s are monotone decreasing:

$$\tau_{m_0}(p_n)\geqslant\tau_{m_0+1}(p_n)\geqslant\tau_{m_0+2}(p_n)\geqslant\cdots\geqslant\rho(p_n). \tag{1.8}$$

Because of the inequalities of (1.8), it is natural to ask if the sequence $\{\tau_m(p_n)\}_{m=m_0}^\infty$ tends to $\rho(p_n)$, as $m\to\infty$. An affirmative answer to this question, established in [2, Theorem 1], is stated as

Theorem B. *For any $p_n(z)$ in $\hat{\pi}_n$ with $n\geqslant 1$,*

$$\lim_{m\to\infty}\tau_m(p_n)=\rho(p_n). \tag{1.9}$$

Another question that can be asked is to characterize those elements $p_n(z)$ in $\hat{\pi}_n$ (with $n\geqslant 1$) for which there exists some positive integer

$$m_1=m_1(p_n)$$

such that

$$\tau_m(p_n)=\rho(p_n)$$

for all $m\geqslant m_1(p_n)$. To answer this question, it is convenient to define the subset $\mathring{\pi}_n$ of $\hat{\pi}_n$ (for each $n\geqslant 1$) by

$p_n(z)\in\mathring{\pi}_n$ iff (1.10)

- i) all zeros of $p_n(z)$ of modulus $\rho(p_n)$ are simple;
- ii) if $\{\xi_j\}_j^r=1$ denotes the set of all zeros of $p_n(z)$ on the circle $|z|=\rho(p_n)$, then $\arg\xi_j$ is a (nonzero) rational multiple of 2π, i.e., there is a positive integer D and positive integers n_j such that $\arg\xi_j=2\pi n_j/D$ with $0<n_j<D$ for all $1\leqslant j\leqslant r$;
- iii) for every zero ξ of $p_n(z)$ with $|\xi|<\rho(p_n)$, then $\xi^D\notin[0,+\infty)$.

Then (cf. [2, Theorem 2]), we have

Theorem C. *Given $p_n(z)$ in $\hat{\pi}_n$ with $n\geqslant 1$, there exists a positive integer $m_1=m_1(p_n)$ such that*

$$\tau_m(p_n)=\rho(p_n) \text{ for all } m\geqslant m_1(p_n) \tag{1.11}$$

iff $p_n(z)$ is an element of $\mathring{\pi}_n$.

We remark that the condition (1.10iii) above, used in defining $\mathring{\pi}_n$, strengthens the analogous condition in [2, eq. (1.14iii)].

With the above results, we come to the main problem of this paper. For each $p_n(z)$ in $\hat{\pi}_n\backslash\mathring{\pi}_n$ with $n\geqslant 1$, it necessarily follows from Theorem C that

$$\tau_m(p_n)>\rho(p_n) \text{ for all } m\geqslant m_0(p_n), \tag{1.12}$$

while from (1.9),

$$\lim_{m\to\infty}\tau_m(p_n)=\rho(p_n).$$

What then is the rate of convergence of $\tau_m(p_n)$ to $\rho(p_n)$, as $m\to\infty$, for each $p_n(z)$ in $\hat{\pi}_n\backslash\mathring{\pi}_n$? Our main result, Theorem 1 below, gives an upper bound for this rate of convergence.

Theorem 1. *For each $p_n(z)$ in $\hat{\pi}_n\backslash\mathring{\pi}_n$ with $n\geqslant 1$, there exists a nonnegative constant σ for which*

$$\varlimsup_{m\to\infty} m[\tau_m(p_n)-\rho(p_n)]=\sigma. \tag{1.13}$$

This result is best possible in the sense that there is a polynomial, namely $\tilde{p}_2(z):=(1+z)^2$ with $\rho(\tilde{p}_2)=1$, which satisfies the above hypotheses, for which

$$\lim_{m\to\infty} m[\tau_m(\tilde{p}_2)-1]=\lambda, \tag{1.14}$$

where $\lambda\doteqdot 1.27846$ is the unique positive zero of $(\lambda-1)e^\lambda-1=0$.

It may be of independent interest that the proof of (1.13) makes use of number-theoretic results. The proof of (1.13) will be given in Section 2, while that of (1.14) will be given in Section 3.

We remark that Theorem 1 automatically applies to any $p_n(z)$ in $\hat{\pi}_n\backslash\mathring{\pi}_n$ whose zeros $\xi_k=|\xi_k|e^{2\pi i\theta_k}$ are such that all θ_k's are rational numbers. This is the case for the particular polynomial $\tilde{p}_2(z):=(1+z)^2$ of Theorem 1.

For any $p_n(z)$ satisfying the hypotheses of Theorem 1 and for any $\varepsilon>0$, it follows from (1.12) and (1.13) that

$$\rho(p_n)<\tau_m(p_n)<\rho(p_n)+\frac{\sigma+\varepsilon}{m},$$